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On the unit groups of Burnside rings

Dedicated to the memory of Professor Akira Hattori

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1. Introduction.

Let G be a finite group. The set $A^{+}(G)$ of the G-isomorphism classes of finite right G-sets makes a commutative semi-ring with respect to disjoint union $+$ and Cartesian product $\times$. Its Grothendieck ring is called the *Burnside ring* of G and is denoted by $A(G)$. A finite (right) G-set is the disjoint union of its orbits and each orbit is G-isomorphic to a homogeneous G-set $H\backslash G:=\{Hg \mid g\in G\}$. Two G-sets $H\backslash G$ and $K\backslash G$ are isomorphic if and only if $H=_{G}K$, that is, H is G-conjugate to K. Thus this ring is additively a free abelian group on $\{[H\backslash G] \mid (H)\in Cl(G)\}$, where $Cl(G)$ is the conjugacy classes (H) of subgroups H of G.

A *super class function* is a map of the set of subgroups of G to $\boldsymbol{Z}$ which is constant on each conjugacy class of subgroups. Let $\tilde{A}(G):=\boldsymbol{Z}^{Cl(G)}$ be the ring of integral valued super class functions. For any subgroup S of G, the map $[X]\mapsto|X^{S}|$, the number of fixed-points, extends to a ring homomorphism φ_{S}: $A(G)\rightarrow\boldsymbol{Z}$, and so we have a ring homomorphism

$$(1)\qquad \varphi:=\prod_{(S)}\varphi_{S}:\ A(G)\longrightarrow\tilde{A}(G):=\boldsymbol{Z}^{Cl(G)};\quad [X]\longmapsto(|X^{S}|).$$

It is well-known that this map is injective. Thus we can identify any element x of $A(G)$ with the super class function $\varphi(x)$, and so we simply write

$$x(S):=\varphi(x)(S)=\varphi_{S}(x)$$

for a subgroup S of G. Hence we can view the unit group $A(G)^{*}$ as a subgroup of $\{\pm 1\}^{Cl(G)}$.

Now, tom Dieck proved by a geometric method that for any $\boldsymbol{R}G$-module V the function

$$u(V):\quad S\longmapsto \operatorname{sgn}\dim V^{S}$$

belongs to the Burnside ring $A(G)$, where $\operatorname{sgn} m:=(-1)^{m}$ ([**Di79**, Proposition 5.5.9]). The first purpose of this paper is to prove this fact by a purely algebraic method. We shall prove the following somewhat generalized theorem in Section 2.

THEOREM A. *Let G be a finite group and let V be a $\boldsymbol{C}G$-module with real valued character. Then the function*

$$u(V):\quad S\longmapsto \operatorname{sgn}\dim_{C}V^{S}$$

is a member of the Burnside ring $A(G)$.

Since $u(V)^2=1$ by the injectivity of φ and $u(U\oplus V)=u(U)\cdot u(V)$, we have a group homomorphism into the unit group:

$$u_G:\quad \bar{R}(G)\longrightarrow A(G)^*,$$

where $\bar{R}(G)$ is the ring of real valued virtual characters of G. We call this map u_G a *tom Dieck homomorphism.*

In Section 3, we shall define various maps between Burnside rings (and their unit groups), and prove that the assignment $\boldsymbol{A}^*\colon H\mapsto A(H)^*$ together with restrictions, multiplicative inductions and conjugations forms a so-called G-functor (=a Mackey functor from $\mathbf{Set}_f^G$) and furthermore that the tom Dieck homomorphisms give a morphism between G-functors.

In Section 4, we prove a transfer theorem about the unit groups. From the theory of G-functors and Burnside rings we can obtain many information about $A(G)^*$. Since $\boldsymbol{A}^*$ is a G-functor, $A(G)^*$ is a module over the Burnside ring $A(G)$ (and also over $A(G)_{(2)}$, the localization at 2). We denote this action by

$$A(G)^*\times A(G)_{(2)}\longrightarrow A(G)\,;\ (u,\,a)\longmapsto u\uparrow a\,.$$

Furthermore, $u\uparrow a$ is given as an extention of U^A, the set of all maps of A to U with diagonal G-action. See Section 3. In general, for a prime p there are primitive idempotents $e_{G,Q}^{p}$ of $A(G)_{(p)}$ associated with conjugacy classes of p-perfect subgroups Q of G, where Q is called to be *p-perfect* provided Q has no normal subgroup of index p. Thus we obtain a direct product decomposition of the unit group $A(G)^*$. Applying "the stable element theorem" for G-functors to each direct factor, the following theorem is obtained:

THEOREM B. (i) *There is a direct product decomposition*

$$A(G)^*=\prod_{(G)}A(G)^*\uparrow e_{G,Q}^{2}\,.$$

(ii) *Let Q be a 2-perfect subgroup of G and let P be a subgroup of $N:=N_G(Q)$ such that P/Q is a Sylow 2-subgroup of N/Q. Then there are group isomorphisms:*

$$A(G)^*\uparrow e_{G,Q}^{2}\cong A(N)^*\uparrow e_{N,Q}^{2}\cong (A(P)^*)^N\uparrow e_{P,Q}^{2}\,,$$

where $(A(P)^*)^N$ *is the subgroup of* $A(P)^*$ *consisting of all elements* x *such that*

$$\mathrm{res}^{P}_{P^n \cap P} \mathrm{con}^{n}_{P}(x) = \mathrm{res}^{P}_{P^n \cap P}(x)$$

for any element n *of* N.

In Section 5, we consider another transfer theorem, that is, "the excision theorem". In his paper [**Ar82**], S. Araki proved by a geometric method the existence of an interesting isomorphism

$$e^{p}_{N,Q} A(N)_{(p)} \cong e^{p}_{N/Q,1} A(N/Q)_{(p)},$$

where N is a finite group with a p-perfect normal subgroup Q. An algebraic proof was given by T. Yoshida and T. Miyata independently and their proofs are found in [**Yo85**]. There is a similar isomorphism for character ring of a finite group and this result is considered to be a kind of "the excision theorem". In case of the unit groups of the Burnside rings, a transfer theorem of this type holds in a weak form as follows:

THEOREM C. *Let* N *be a finite group with* 2*-perfect normal subgroup* Q *and put* $W := N/Q$. *Let* $\bar{P}$ *be a Sylow* 2*-subgroup of* W. *Then* $A(N)^* \uparrow e^{2}_{N,Q}$ *is isomorphic to the subgroup of* $(A(\bar{P})^*)^W$ *consisting of elements* $\bar{v}$ *such that* $\bar{v}(S/Q) = 1$ *if* S *has a proper normal subgroup of odd index.*

As an easy corollary we obtain the excision theorem:

COROLLARY C1. *Let* N *be a finite group with* 2*-perfect normal subgroup* Q. *Let* $\Psi(Q)$ *be the intersection of all normal subgroups of* Q *of prime index, and put* $\tilde{N} := N/\Psi(Q)$ *and* $\tilde{Q} := W/\Psi(Q)$. *Then there is an isomorphism of groups:*

$$A(N)^* \uparrow e^{2}_{N,Q} \cong A(\tilde{N})^* \uparrow e^{2}_{\tilde{N},\tilde{Q}}.$$

In Section 6, we add some results on the unit group of the Burnside rings of finite groups. Many of them are motivated by Matsuda's papers [**Ma82**], [**MM83**].

Finally we study in Section 7 the unit groups for finite groups with abelian Sylow 2-subgroups. Let T be an abelian 2-group and let $\bar{T} := T/\Phi(T)$, where $\Phi(T)$ is the Frattini subgroup. Then it is known that the tom Dieck homomorphism $u_T \colon \bar{R}(T) \to A(T)^*$ is surjective and it gives an isomorphism

$$\bar{u}_T \colon \ \boldsymbol{F}_2[\bar{T}^{\wedge}] \xrightarrow{\cong} A(T)^*,$$

where $\boldsymbol{F}_2[\bar{T}^{\wedge}]$ is the group ring of the character group of $\bar{T}$ over the field $\boldsymbol{F}_2$ of order 2.

When G is a finite group with abelian Sylow 2-subgroup, the problem about the structure of $A(G)^*$ is by Theorems B and C reduced to the case where G

has a normal subgroup Q with $\Psi(Q)=1$ and G/Q has a normal Sylow 2-subgroup. In this case the following theorem holds:

THEOREM D. *Let G be a finite group with an abelian normal subgroup Q of odd order and an abelian Sylow 2-subgroup T. Put $\bar{T}:=T/\Phi(T)$ and $L:=N_G(T)$. Then the following hold:*

(i) *If $Q=1$, then $A(G)^*\uparrow e^2_{G,Q}$ is isomorphic to the additive group $(\boldsymbol{F}_2[\bar{T}^\frown])^L(\cong \boldsymbol{F}_2[\bar{T}^\frown/L])$ of L-fixed points.*

(ii) *If $Q\neq 1$ and $C_Q(T)\neq 1$, then $A(G)^*\uparrow e^2_{G,Q}$ is of order 1.*

(iii) *Assume that $Q\neq 1$ and $C_Q(T)=1$. Define*

$$\mathcal{M} := \{\Phi(T)C \mid T/C \text{ is cyclic and } C_Q(C)\neq 1\},$$
$$K := \bigcap\{M\in\mathcal{M}\}, \qquad \bar{K} := K/\Phi(T).$$

Then $A(G)^*\uparrow e^2_{G,Q}$ is isomorphic to $\boldsymbol{F}_2[\bar{K}^\frown]^L$.

Thus if we know enough about subgroups of a finite group with abelian Sylow 2-subgroup, we can calculate the order of the unit group of the Burnside ring. We carry out such calculation for the following simple groups:

$$L_2(2^n),\ L_2(q)\quad (q\equiv 3, 5 \pmod 8)),\ J_1.$$

NOTATION AND TERMINOLOGY. We always denote by G a finine group. The G-conjugacy class of a subgroup H of G is denoted by (H), and the set of all such classes is denoted by $Cl(H)$. The notation $H\leqq G$ means that H is a subgroup of G. We put $WH:=N_G(H)/H$. For subgroups A and B of G, we mean by $A=_G B$ (resp. $A\leqq_G B$) that A and B are conjugate in G (resp. A is G-conjugate to a subgroup of B). We put $A^g:=g^{-1}Ag$ for g in G. When G acts on a set X, we denote by X^G the set of elements fixed by G. The ordinary character ring of G is denoted by $R(G)$. The inner product of class functions χ and θ is defined as usual:

$$\langle\chi,\theta\rangle := \frac{1}{|G|}\sum_{g\in G}\chi(g)\overline{\theta(g)}.$$

Let p be a prime. A p-element (resp. p-group) is an element (resp. group) whose order is a power of p. A p'-element (resp. p'-group) is an element (resp. group) whose order is coprime to the prime p. The subgroup generated by all 2-elements (resp. $2'$-elements) of G is denoted by $O^{2'}(G)$ (resp. $O^2(G)$). Then $O^{2'}(G)$ (resp. $O^2(G)$) is the smallest normal subgroup of G by which factor group is a $2'$-group (resp. 2-group). For subsets X and Y of G, let $[X, Y]$ be the subgroup generated by $x^{-1}y^{-1}xy$, $x\in X$, $y\in Y$.

The unit group of a ring R is denoted by R^*. The cardinality of a finite set X is denoted by $|X|$ or $\#X$. We put $\operatorname{sgn} m:=(-1)^m$ for an integer m. Other notation and terminology about finite groups are standard. Refer to Gorenstein's book [**Go68**]. About category theory refer to [**ML71**].

2. Proof of Theorem A.

In this section we prove Theorem A. As in the introduction, let G be a finite group and let $Cl(G)$ denote the set of conjugacy classes (S) of subgroups S of G. We put $WS:=N_G(S)/S$ for a subgroup S of G. We begin with the following fundamental lemma ([**Yo85**, Lemma 2.1], [**Di79**, Section 1.3]):

LEMMA 2.1. *There is an exact sequence of abelian groups:*

$$0 \longrightarrow A(G) \xrightarrow{\varphi} \boldsymbol{Z}^{Cl(G)} \xrightarrow{\psi} \prod_{(S)} (\boldsymbol{Z}/|WS|\boldsymbol{Z}) \longrightarrow 0,$$

where φ is the injective ring homomorphism defined by

$$\varphi([X])(S) := |X^S|$$

as in the introduction, and for a super class function x, the S-component of $\psi(x)$ is defined by

$$\psi(x)_S := \sum_{gS \in WS} x(\langle g \rangle S) \qquad (\mathrm{mod}\,|WS|).$$

PROOF. It is well-known that φ is an injective ring homomorphism of which cokernel has the same order as the target group of ψ. See, for example, [**Di79**, Propositions 1.2.2 and 1.2.3]. The surjectivity of ψ is clear, and so it remains only to show that $\psi\varphi=0$. To prove this, let X be a finite G-set and S a subgroup of G. Put $W:=WS$. The fixed point set X^S becomes a W-set by conjugation. Let π be the permutation character of W afforded by this W-set X^S, so that

$$\pi(gS) = |X^{\langle g \rangle S}| = \varphi([X])(\langle g \rangle S).$$

Thus we have that

$$\psi(\varphi([X]))_S \equiv |WS| \langle \pi, 1_{WS} \rangle \equiv 0 \quad (\mathrm{mod}\,|WS|),$$

where $\langle \ , \ \rangle$ stands for the inner product of characters, as required. The lemma is proved.

By this lemma, we often identify $A(G)$ as a subgroup of $\tilde{A}(G):=\boldsymbol{Z}^{Cl(G)}$. We now apply this lemma to prove Theorem A.

THEOREM A. *Let V be a $\boldsymbol{C}G$-module with real valued character. Then the function*

$$u(V): \ S \longmapsto \mathrm{sgn}\,\dim_{\boldsymbol{C}} V^S$$

is a member of the Burnside ring $A(G)$.

PROOF. Let χ be the real valued character afforded by the $\boldsymbol{C}G$-module V.

By Lemma 2.1, in order to prove the theorem, it will suffice to show that for each subgroup S of G,

$$(1)\qquad \sum_{gS\in WS} u(V)(\langle g\rangle S)\equiv 0 \pmod{|WS|}.$$

Let θ be the character of WS afforded by the $\boldsymbol{C}WS$-module V^S, so that by an easy representation theory, we have that

$$\theta(gS)=\frac{1}{|S|}\sum_{s\in S}\chi(gs),\qquad gS\in WS,$$

and so θ is also a real valued character. Thus in order to prove (1), we may assume that $S=1$. Set $u_\chi(g):=u(V)(\langle g\rangle)$. Then by the definition of $u(V)$, it has the value

$$(2)\qquad u_\chi(g)=\operatorname{sgn}\langle \chi_{|\langle g\rangle}, 1_{\langle g\rangle}\rangle.$$

Now when $S=1$, (1) becomes

$$(3)\qquad \sum_{g\in G} u_\chi(g)\equiv 0 \pmod{|G|}.$$

To prove (3), it will suffice to show that u_χ is a virtual character of G. In fact, we can show that

$$(4)\qquad u_\chi=(-1)^{\chi(1)}\det\chi,$$

where $\det\chi$ is the linear character of G defined by the composition

$$\det\chi:\quad G\longrightarrow GL(V)\xrightarrow{\det}\boldsymbol{C}^*.$$

See [**Yo78**]. In order to prove (4), we may assume that G is cyclic. Since $u_{\chi+\varphi}=u_\chi\cdot u_\varphi$ and $\det(\chi+\varphi)=\det\chi\cdot\det\varphi$, we may further assume that either χ is a real valued linear character or $\chi=\lambda+\bar\lambda$ for some nonreal linear character λ. In the first case, we have that

$$u_\chi(g)=\operatorname{sgn}\langle \chi_{|\langle g\rangle}, 1_{\langle g\rangle}\rangle=\begin{cases}-1 & \text{if } g\in\operatorname{Ker}\chi\\ +1 & \text{otherwise},\end{cases}$$

and so $u_\chi=-\chi$, as required. In the second case, $\chi=\lambda+\bar\lambda$, and so $\langle \chi_{|\langle g\rangle}, 1_{\langle g\rangle}\rangle=0$ or 2. Thus u_χ and $\det\chi$ are both equal to the trivial character. Hence (4) holds in either case. The theorem is proved.

3. The unit groups as G-functors.

In the introduction we defined the Burnside ring $A(G)$ of a finite group G and the ring $\tilde{A}(G)$ of super class functions. In this section, we will study various maps between Burnside rings and rings of super class functions which are induced from various functors. We induce some equalities about these maps by using elementary properties of adjoint functors.

a. Adjoint functors. For a finite group G, let $\mathbf{Set}_f^G$ denote the category of finite (right) G-sets and G-maps. We denote by $\mathrm{Map}_G(X, Y)$ the set of G-maps of X to Y. The disjoint unions (resp. the Cartesian products) of pairs of objects gives coproducts $X+Y$ (resp. products $X\times Y$) in this category. The Burnside ring $A(G)$ is, as an additive group, the Grothendieck group of this category with respect to $+$, and it has the multiplication induced by $\times$.

For two G-sets X and Y, let Y^X denote the G-set consisting of all mappings of X to Y with G-action defined by

$$\alpha^g(x) := \alpha(xg^{-1})g \qquad \text{for } \alpha\in Y^X,\ g\in G,\ x\in X.$$

Thus for each finite G-set A we have an exponential functor

$$(-)^A:\ \mathbf{Set}_f^G \longrightarrow \mathbf{Set}_f^G;\ \ X \longmapsto X^A.$$

This is a right adjoint of the functor $A\times(-)$: $X\mapsto A\times X$, that is, there exists a natural bijection

$$\mathrm{Map}_G(A\times X, Y) \cong \mathrm{Map}_G(X, Y^A).$$

Furthermore there is another functor

$$A^{(-)}:\ (\mathbf{Set}_f^G)^{\mathrm{op}} \longrightarrow \mathbf{Set}_f^G;\ \ X \longmapsto A^X,$$

where op stands for the dual of the category. This functor also has the right adjoint $A^{(-)\mathrm{op}}$.

Remember that a functor which has a right (resp. left) adjoint preserves colimits (resp. limits) and that a right (or left) adjoint functor is unique (up to natural equivalence) if it exists. These properties are frequently used in this section from now on. About adjoint functors and (co) limits, refer to MacLane's book [**ML71**]. For example, the functor $A\times(-)$ has a right adjoint, and so it preserves colimits, particularly coproduct, whence we obtain the distributive law about coproduct $+$ and product $\times$. We can furthermore derive other arithmetical law for G-sets similar as natural numbers from properties of adjoint functors.

Let H be a subgroup of G and N a normal subgroup of G. Then there are functors

$$\mathbf{Set}_f^H \;\underset{\xrightarrow[\mathrm{Jnd}]{}}{\overset{\xrightarrow{\mathrm{Ind}}}{\longleftarrow \mathrm{Res} \longrightarrow}}\; \mathbf{Set}_f^G \;\underset{\xrightarrow[\mathrm{Inv}]{}}{\overset{\xrightarrow{\mathrm{Orb}}}{\longleftarrow \mathrm{Inf} \longrightarrow}}\; \mathbf{Set}_f^{G/N}$$

which satisfy

$$\mathrm{Ind} \dashv \mathrm{Res} \dashv \mathrm{Jnd}, \qquad \mathrm{Orb} \dashv \mathrm{Inf} \dashv \mathrm{Inv},$$

where $E\dashv F$ means that E is a left adjoint functor of F. Furthermore for any element g of G, there is an equivalence of categories

$$\mathrm{Con}:\ \mathbf{Set}_f^H \longrightarrow \mathbf{Set}_f^{H^g},$$

where $H^g := g^{-1}Hg$. If we need to indicate subgroups and elements, we put necessary letters on suitable places, for example, Res_H^G, Res_H, Inf_N^G, Con_H^g, Inv_N. These functors have the values on objects as follows:

(a.1) $\mathrm{Ind}^G:\ X \longmapsto X\times_H G$ (the induced G-set),
(a.2) $\mathrm{Res}_H:\ Y \longmapsto Y_H$ (the restriction of action),
(a.3) $\mathrm{Jnd}^G:\ X \longmapsto \mathrm{Map}_H(G, X)$ (the multiplicative induction),
(a.4) $\mathrm{Con}^g:\ X \longmapsto Xg$ (the conjugation),
(a.5) $\mathrm{Orb}_N:\ Y \longmapsto Y/N$ (the N-orbits),
(a.6) $\mathrm{Inf}_N:\ Z \longmapsto Z$ (the inflation),
(a.7) $\mathrm{Inv}_N:\ Y \longmapsto Y^N$ (the N-fixed points),

where $X\times_H G$ is the quotient set of G-set $X\times G$ (G-action is defined by $(x, g)g' := (x, gg')$) with respect to $(x, g)\sim(xh, h^{-1}g)$, $h\in H$, and $\mathrm{Map}_H(G, X)$ is the set of mappings α of G to X such that $\alpha(gh)=\alpha(g)h$ with action of G defined by $\alpha^u:\ g\mapsto\alpha(ug)$.

Up to natural isomorphisms, the following hold:

(a.8) $\mathrm{Ind}_H^G:\ D\backslash H \longmapsto D\backslash G,$
(a.9) $\mathrm{Res}_H^G:\ E\backslash G \longmapsto \coprod_{g\in E\backslash G/H} (E^g\cap H)\backslash H,$
(a.10) $\mathrm{Ind}_H^G\circ\mathrm{Res}_H^G = (H\backslash G)\times(-),$
(a.11) $\mathrm{Res}_K^G\circ\mathrm{Ind}_H^G = \coprod_{g\in H\backslash G/K} \mathrm{Ind}^K\circ\mathrm{Res}_{H^g\cap K}\circ\mathrm{Con}_H^g,$

where g in (a.11) runs over a complete set of representatives of $H\backslash G/K$. Taking the right adjoints of the both sides of (a.10) and (a.11), we have the following:

(a.12) $\mathrm{Jnd}_H^G\circ\mathrm{Res}_H^G = (-)^{H\backslash G},$
(a.13) $\mathrm{Res}_K^G\circ\mathrm{Jnd}_H^G = \prod_{g\in H\backslash G/K} \mathrm{Jnd}^K\circ\mathrm{Res}_{H^g\cap K}\circ\mathrm{Con}_H^g.$

The equalities (a.11) and (a.13) are both called the *Mackey decomposition.*

b. Maps between Burnside rings. We next consider the mappings induced from the functors defined in the subsection a. Let $A^+(G)$ denote the monoid consisting of isomorphism classes of $\mathbf{Set}_f^G$ and $\tilde{A}(G)\,(:=\boldsymbol{Z}^{Cl(G)})$ the ring of integral-valued super class functions. Then $A(G)$ is the Grothendieck ring of $A^+(G)$. By the fundamental theorem of Burnside rings (Lemma 2.1), $A(G)$ is isomorphic to a subring of $\tilde{A}(G)$ of finite index through the fixed-point-map $\varphi=\prod_{(S)}\varphi_S$, where for any subgroup S of G,

$$\varphi_S:\ A(G)\longrightarrow \boldsymbol{Z};\ \ [X]\longmapsto |X^S|\ (=:\ [X](S)).$$

stated as in the introduction.

Now, let $A^+ \cong \boldsymbol{N}^m$ be a free abelian monoid of rank m and let $A(\cong \boldsymbol{Z}^m)$ be the Grothendieck group of A^+. (Here $\boldsymbol{N}$ is the set of *nonnegative* integers.) Let B be a free abelian group of finite rank. A map

$$g: \quad (x_1, \cdots, x_m) \longmapsto (g_1(x_1, \cdots, x_m), \cdots)$$

of A^+(or A) to B is called to be *polynomial* or *algebraic* if each component $g_i(x_1, \cdots, x_m)$ is presented by a polynomial of $x_1, \cdots, x_m$. This definition does not depend on the choice of bases.

(b.01) Any polynomial map $f^+: A^+ \to B$ can be uniquely extended to a polynomial map $f: A \to B$.

(b.02) The composition of two polynomial maps is also a polynomial map.

(b.03) Let $\tilde{A}$ be a free abelian group containing A as a subgroup of finite index. Then any polynomial map f from A has at most one extended polynomial map to $\tilde{A}$.

(b.04) Let $f: A \to B$ be a mapping between finitely generated free abelian groups and let $\tilde{B}$ be a free abelian group containing B as a subgroup of finite index. Then f is polynomial if and only if f is polynomial as a map from A to $\tilde{B}$.

We will apply these facts to construct some operations of Burnside rings. The detail for polynomial maps is found in [**Dr71**].

Since the functors Ind, Res, Con, Orb and Inf have right adjoint functors, they preserve coproducts, and so they induce additive homomorphisms ind, res, con, orb and inf between Burnside rings. Among them, the maps res, inf and inv are ring homomorphisms; besides con is a ring isomorphism. On the other hand, the multiplicative induction Jnd and the exponential functor $(-)^A$ are not additive; but they are polynomial, and so they induce product preserving polynomial maps jnd and $(-)^A$. Indeed, by the adjointness and the Mackey formula, we have that for each subgroup S of G,

$$\begin{aligned} \#(\mathrm{Jnd}_H^G(X))^S &= \#\mathrm{Map}_G(S\backslash G, \mathrm{Jnd}_H^G(X))) = \#\mathrm{Map}_H(\mathrm{Res}_H(S\backslash G), X) \\ &= \#\mathrm{Map}_H(\coprod_{g \in S\backslash G/H} (S^g \cap H)\backslash H, X) = \prod_{g \in S\backslash G/H} |X^{S^g \cap H}|, \end{aligned}$$

and so the *value* of $\mathrm{Jnd}_H^G(X)$ at each subgroup S is a polynomial of $[X](S^g \cap H)$, $g \in S\backslash G/H$; and thus $[X] \mapsto [\mathrm{Jnd}_H^G(X)]$ is a polynomial map from $A^+(H)$ to $A(G)$ which preserves multiplication; hence it can be uniquely extended to one from $A(G)$. For the details, see [**Dr71**] and [**Di79**, 5.13]. These maps, excepting for orb, can be extended to the rings of super class functions. Hence we obtained the maps as in the following figure.

$$\begin{array}{ccccc}
A(H) & \underset{\text{jnd}}{\overset{\text{ind}}{\rightleftarrows}} \!\!\xleftarrow{\text{res}} & A(G) & \underset{\text{inv}}{\overset{\text{orb}}{\rightleftarrows}}\!\!\xleftarrow{\text{inf}} & A(G/N) \\
\downarrow \varphi & & \downarrow \varphi & & \downarrow \varphi \\
\tilde{A}(H) & \underset{\text{jnd}}{\overset{\text{ind}}{\rightleftarrows}}\!\!\xleftarrow{\text{res}} & \tilde{A}(G) & \underset{\text{inv}}{\longrightarrow}\!\!\xleftarrow{\text{inf}} & \tilde{A}(G/N),
\end{array}$$

$$\text{con}: \quad A(H) \longrightarrow A(H^g), \quad \tilde{A}(H) \longrightarrow \tilde{A}(H^g),$$
$$(-)^A: \quad A(G) \longrightarrow A(G), \quad \tilde{A}(G) \longrightarrow \tilde{A}(G).$$

Next we list the values of images of these maps at subgroups. Let $x \in A(H)$, $y \in A(G)$, $z \in A(G/N)$; and let $S \leqq G$, $T \leqq H$, $R/N \leqq G/N$. Then the following hold:

(b.1) $$\operatorname{ind}_H^G(x)(S) = \frac{1}{|H|} \sum_{g \in G} x(S^g),$$

where the summation is taken over elements g such that $S^g \leqq H$;

(b.2) $$\operatorname{res}_H^G(y)(T) = y(T);$$
(b.3) $$\operatorname{jnd}_H^G(x)(S) = \prod_{g \in S \backslash G/H} x(S^g \cap H);$$
(b.4) $$\operatorname{inf}_N(z)(T) = y(T);$$
(b.5) $$\operatorname{inv}_N(y)(R/N) = y(R);$$
(b.6) $$\operatorname{con}^g(x)(S^g) = x(S).$$

c. Maps between unit groups of Burnside rings. Since res, jnd, inf, inv, con preserve multiplications in the Burnside rings (and in the rings of super class functions), we obtain group homomorphisms as follows:

$$\begin{array}{ccccc}
A(H)^* & \underset{\text{jnd}}{\overset{\text{res}}{\leftrightarrows}} & A(G)^* & \underset{\text{inv}}{\overset{\text{inf}}{\leftrightarrows}} & A(G/N)^* \\
\downarrow \varphi & & \downarrow \varphi & & \downarrow \varphi \\
\tilde{A}(H)^* & \underset{\text{jnd}}{\overset{\text{res}}{\leftrightarrows}} & \tilde{A}(G)^* & \underset{\text{inv}}{\overset{\text{inf}}{\leftrightarrows}} & \tilde{A}(G/N)^*,
\end{array}$$

$$\text{con}: \quad A(H)^* \longrightarrow A(H^g)^*, \quad \tilde{A}(H)^* \longrightarrow \tilde{A}(H^g)^*.$$

Finally the exponential operation $(X, A) \mapsto X^A$ induces the map

$$A(G)^* \times A(G) \longrightarrow A(G)^*; \quad (u, a) \longmapsto u^a \text{ (or } u \uparrow a)$$

which makes $A(G)^*$ an $A(G)$-module. Similarly $\tilde{A}(G)^*=\{\pm\}^{Cl(G)}$, the unit group of the ring of super class functions, is also an $A(G)$-module and the injection

$$\varphi:\quad A(G)^* \longrightarrow \tilde{A}(G)^* = \{\pm 1\}^{Cl(G)}$$

is an $A(G)$-homomorphism.

LEMMA 3.1. *The assignments $H\mapsto A(H)^*$ and $H\mapsto\tilde{A}(H)^*$ together with* jnd, res, con *form G-functors $\boldsymbol{A}^*$ and $\tilde{\boldsymbol{A}}^*$, that is, for subgroups $H, K, L\leqq G$ and elements $g, g'\in G$, the following hold:*

(G.1) $\quad \mathrm{jnd}_K^L\circ\mathrm{jnd}_H^K=\mathrm{jnd}_H^L,\quad \mathrm{jnd}_H^H=\mathrm{id}\qquad$ *if $H\leqq K\leqq L$;*

(G.2) $\quad \mathrm{res}_H^K\circ\mathrm{res}_K^L=\mathrm{res}_H^L,\quad \mathrm{res}_H^H=\mathrm{id}\qquad$ *if $H\leqq K\leqq L$;*

(G.3) $\quad \mathrm{con}^{g'}\circ\mathrm{con}^g=\mathrm{con}^{gg'},\quad \mathrm{con}_H^h=\mathrm{id}\qquad$ *if $h\in H$;*

(G.4) $\quad \mathrm{con}^g\circ\mathrm{jnd}_H^K=\mathrm{jnd}^{K^g}\circ\mathrm{con}_H^g,$

$\quad\quad \mathrm{con}^g\circ\mathrm{res}_H^K=\mathrm{res}_{H^g}\circ\mathrm{con}_H^g\qquad$ *if $H\leqq K$;*

(G.5) (Mackey decomposition) *If $H, K\leqq L$, then*

$$\mathrm{res}_K^L\circ\mathrm{jnd}_H^L=\prod_{g\in H\backslash L/K}\mathrm{jnd}^K\circ\mathrm{res}_{H^g\cap K}\circ\mathrm{con}_H^g,$$

where g runs over a complete set of representatives of $H\backslash L/K$.

PROOF. It is well-known that the identities corresponding to (G.1)-(G.5) for functors Ind, Res, Con hold (for example, (a.11) for Mackey decomposition). Taking their right adjoints, we have ones for Jnd, Res, Con. Turning to Burnside rings, we have (G.1)-(G.5) on unit groups. The identities for super class function follow from the uniqueness of extensions of the polynomial maps to super class functions. The lemma is proved.

LEMMA 3.2. *The bilinear mappings*

$$A(H)^*\times A(H)\longrightarrow A(H)^*;\quad (u, a)\longmapsto u\uparrow a,\qquad H\leqq G,$$

form a pairing $\boldsymbol{A}^*\times\boldsymbol{A}\to\boldsymbol{A}^*$, that is, for $H\leqq K\leqq G$, $g\in G$, $u\in A(H)^*$, $v\in A(K)^*$, $a\in A(H)$, $b\in A(K)$, the following hold:

(P.1) $\quad \mathrm{res}_H(v\uparrow b)=\mathrm{res}_H(v)\uparrow\mathrm{res}_H(b);$

(P.2) $\quad \mathrm{con}^g(u\uparrow a)=\mathrm{con}^g(u)\uparrow\mathrm{con}^g(a);$

(P.3) $\quad \mathrm{jnd}^K(u)\uparrow b=\mathrm{jnd}^K(u\uparrow\mathrm{res}_H(b));$

(P.4) $\quad v\uparrow\mathrm{ind}^K(a)=\mathrm{jnd}^K(\mathrm{res}_H(v)\uparrow a).$

In particular, for a fixed element u of $A(G)^*$, the family of homomorphisms (from additive groups to multiplicative groups)

$$(u\uparrow)_H:\quad A(H)\longrightarrow A(H)^*;\quad a\longmapsto(\mathrm{res}_H u)\uparrow a,\qquad H\leqq G$$

becomes a morphism of the G-functor $\boldsymbol{A}$ to the G-functor $\boldsymbol{A}^*$.

PROOF. (P.1) and (P.2) are clear. We shall prove (P.3). We may assume that $b=[B]$ for a finite K-set B. We must show that (P.3) holds for all $u \in A(H)$. Since the both side of (P.3) are polynomial maps of u which are the unique extensions of polynomial maps from the semi-ring $A^{+}(H)$ of isomorphisms of finite H-sets to $A(H)^{*}$, we may further assume that $u=[U]$ for the finite H-set. So it will suffice to show that there is a (natural) K-isomorphism:

$$(*) \qquad \mathrm{Jnd}^{K}(U) \uparrow B \cong \mathrm{Jnd}^{K}(U \uparrow \mathrm{Res}_{H}(B)).$$

But using the definition of adjoint functors and Frobenius reciprocity, we see that for any K-set Y, there is a natural isomorphism:

$$\mathrm{Map}_{K}(Y, \mathrm{Jnd}^{K}(U) \uparrow B) \cong \mathrm{Map}_{K}(Y, \mathrm{Jnd}^{K}(U \uparrow \mathrm{Res}_{H}(B)).$$

Thus Yoneda's lemma (or the injectivity of φ in Lemma 2.1) implies the existence of the required isomorphism $(*)$.

We add some useful formulas about maps between Burnside rings (and unit groups). The proofs are omitted because they are easily proved by using properties of adjoint functors similarly as in Lemma 3.1.

LEMMA 3.3. *Let H be a subgroup of G and N a normal subgroup of G. Let $a \in A(G)$, $b \in A(H)$, $\bar{a} \in A(G/N)$, $u \in A(G)^{*}$, $v \in A(H)^{*}$, $\bar{u} \in A(G/N)^{*}$. Then the following hold:*

(1) $\mathrm{jnd}^{G} \circ \mathrm{res}_{H}(a) = a^{[H \backslash G]}$;

(2) $\mathrm{orb}_{N} \circ \mathrm{inf}_{N}(\bar{a}) = \mathrm{inv}_{N} \circ \mathrm{inf}_{N}(\bar{a}) = \bar{a}$;

(3) $\mathrm{inf}_{N}(\bar{u} \uparrow \bar{a}) = \mathrm{inf}_{N}(\bar{u}) \uparrow \mathrm{inf}_{N}(\bar{a})$;

(4) $\mathrm{inv}_{N}(u \uparrow \mathrm{inf}_{N}(\bar{a})) = \mathrm{inv}_{N}(u) \uparrow \bar{a}$;

(5) $\mathrm{inv}_{N}(\mathrm{inf}_{N}(\bar{u}) \uparrow a) = \bar{u} \uparrow \mathrm{orb}_{N}(a)$.

LEMMA 3.4. *Let N be a normal subgroup of G and let H be a subgroup of G containing N. Then the following hold:*

(1) $\mathrm{orb}_{N}^{G} \circ \mathrm{ind}_{H}^{G} = \mathrm{ind}_{H/N}^{G/N} \circ \mathrm{orb}_{N}^{H}$;

(2) $\mathrm{orb}_{N}^{H} \circ \mathrm{res}_{H}^{G} = \mathrm{res}_{H/N}^{G/N} \circ \mathrm{orb}_{N}^{G}$;

(3) $\mathrm{ind}_{H}^{G} \circ \mathrm{inf}_{N}^{H} = \mathrm{inf}_{N}^{G} \circ \mathrm{ind}_{H/N}^{G/N}$;

(4) $\mathrm{res}_{H}^{G} \circ \mathrm{inf}_{N}^{G} = \mathrm{inf}_{N}^{H} \circ \mathrm{res}_{H/N}^{G/N}$;

(5) $\mathrm{jnd}_{H}^{G} \circ \mathrm{inf}_{N}^{H} = \mathrm{inf}_{N}^{G} \circ \mathrm{jnd}_{H/N}^{G/N}$;

(6) $\mathrm{inv}_{N}^{H} \circ \mathrm{res}_{H}^{G} = \mathrm{res}_{H/N}^{G/N} \circ \mathrm{inv}_{N}^{G}$;

(7) $\mathrm{inv}_{N}^{G} \circ \mathrm{jnd}_{H}^{G} = \mathrm{jnd}_{H/N}^{G/N} \circ \mathrm{inv}_{N}^{H}$.

(Here orb_{N}^{G}, inf_{N}^{G}, inv_{N}^{G} are the maps between $A(G)$ and $A(G/N)$.)

d. tom Dieck homomorphisms. Finally we state functorial properties of the tom Dieck homomorphism. We denote by $\bar{R}(G)$ the ring of real valued virtual characters of G as in Section 2. Similarly as the case of Burnside rings, there exist various maps between the ring of real valued character:

$$\bar{R}(H) \underset{\text{jnd}}{\overset{\text{ind}}{\rightrightarrows}} \xleftarrow{\text{res}} \bar{R}(G) \underset{\text{inv}}{\overset{\text{orb}}{\rightrightarrows}} \xleftarrow{\text{inf}} \bar{R}(G/N),$$

$$\text{con}: \bar{R}(H) \longrightarrow \bar{R}(H^g).$$

(Here, the multiplicative induction jnd is not additive in general. Furthermore, it follows from Maschke's theorem that orb=inv.) Thus we again have a G-functor $\bar{R}$: $H \mapsto \bar{R}(H)$ together with ind, res and con.

For a finite G-set X, let $\boldsymbol{C}X$ be the permutation $\boldsymbol{C}G$-module and let π_X be the character afforded by $\boldsymbol{C}X$, so that $\pi_X(g) = |X^{\langle g \rangle}|$. The assignment $X \mapsto \pi_X$ gives a ring homomorphism

$$\text{char}_G: \ A(G) \longrightarrow \bar{R}(G); \ \ [X] \longmapsto \pi_X.$$

This map is commute with ind, res, con, jnd, inf, orb. In particular, we have a morphism of G-functors char: $\boldsymbol{A} \to \bar{R}$.

LEMMA 3.5. *The tom Dieck homomorphisms form a morphism*

$$u = (u_H): \ \bar{R} \longrightarrow \boldsymbol{A}^*$$

of G-functors, that is, u_H's commute with induction, restriction and conjugation. They commute also with inf *and* inv.

PROOF. We only check that

$$u_G \circ \text{ind}_H^G = \text{jnd}_H^G \circ u_H.$$

The remainder is left for readers. We use the notation

$$\theta^G := \text{ind}_H^G(\theta), \quad \chi_H := \text{res}_H^G(\chi) \qquad \text{for } \theta \in R(H), \chi \in R(G).$$

For a real valued virtual character θ of H and a subgroup S of G, we have that

$$\begin{aligned} u_G(\text{ind}_H^G(\theta))(S) &= \text{sgn}\langle \theta^G{}_S, 1_S \rangle \\ &= \text{sgn}\langle \theta, 1_{S^H}^G \rangle && \text{(Frobenius)} \\ &= \text{sgn}\langle \theta, \sum_{g \in S \backslash G/H} 1^H_{S^g \cap H} \rangle && \text{(Mackey)} \\ &= \prod_{g \in S \backslash G/H} \text{sgn}\langle \theta_{S^g \cap H}, 1_{S^g \cap H} \rangle && \text{(Frobenius)} \\ &= \prod_{g \in S \backslash G/H} u_H(\theta)(S^g \cap H) \\ &= \text{jnd}_H^G(u_H(\theta))(S), \end{aligned}$$

as required.

LEMMA 3.6. *The following equality for maps holds:*

$$(-1)^{(-)} = u_G \circ \mathrm{char}_G \colon \ A(G) \longrightarrow \bar{R}(G) \longrightarrow A(G)^*.$$

PROOF. Let X be a finite G-set and put $\pi := \pi_X$. Then we have that for any subgroup S of G,

$$u_G \circ \mathrm{char}_G([X])(S) = \mathrm{sgn}\langle \pi_S, 1_S \rangle = (-1)^{|X/S|} = (-1)^{[X]}(S)$$

by Lemma 3.3(5). Thus the result follows from Lemma 2.1.

4. A transfer theorem for the unit groups.

a. Idempotents of Burnside rings. We begin by arguing general theory about the action of Burnside rings on G-functors ([**Yo80**], [**Yo83**]).

Let p be a prime. We put

$$\boldsymbol{Z}_{(p)} := \{a/b \in \boldsymbol{Q} \mid \ a \in \boldsymbol{Z},\ b \in \boldsymbol{Z} - p\boldsymbol{Z}\}, \qquad A(G)_{(p)} := \boldsymbol{Z}_{(p)} \otimes_{\boldsymbol{Z}} A(G).$$

For a finite group Q, the subgroup generated by all p'-elements of Q is denoted by $O^p(Q)$. A finite group Q is called *p-perfect* if $O^p(Q)=Q$, that is, if Q has no normal subgroup of index p. Let $P_p(G) \subseteq Cl(G)$ denote the classes of all p-perfect subgroups.

There is a one-to-one correspondence between primitive idempotents of $A(G)_{(p)}$ and $P_p(G)$ (cf. [**Di79**, 1.4]). Let $e^p_{G,Q}$ be the primitive idempotent of $A(G)_{(p)}$ corresponding to a p-perfect subgroup Q of G. Then as a super class function, it satisfies the following:

$$e^p_{G,Q}(S) = \begin{cases} 1 & \text{if } O^p(S) =_G Q, \\ 0 & \text{otherwise}. \end{cases} \tag{a.1}$$

The presentation of $e^p_{G,Q}$ by the standard basis is obtained by the Möbius inversion formula (cf. [**Yo83**]). Let μ be the Möbius function of the subgroup lattice of G, that is, for $H, K \leqq G$,

$$\mu(H, H) = 1; \quad \mu(H, K) = 0 \qquad \text{unless } H \leqq K;$$

$$\sum_{A \leq K} \mu(H, A) = \sum_{A \geq H} \mu(A, K) = \begin{cases} 1 & \text{if } H =_G K, \\ 0 & \text{otherwise}. \end{cases} \tag{a.2}$$

Define the function δ_G and $\lambda_{G,Q}$ by

$$\delta_G(H, K) = \begin{cases} 1 & \text{if } H =_G K \\ 0 & \text{otherwise}; \end{cases} \tag{a.3}$$

$$\lambda_{G,Q}(D) = \frac{1}{|N_G(D)|} \sum_{K \leq G} \mu(D, K) \delta_G(O^p(K), Q). \tag{a.4}$$

Then we obtain the following idempotent formula:

$$e^{p}_{G,Q}=\sum_{(D)\in Cl(G)}\lambda_{G,Q}(D)[D\backslash G], \qquad (Q)\in P_p(G). \tag{a.5}$$

In particular, the coefficient $\lambda_{G,Q}(D)$ is a p-local integer.

For a subgroup H of G, we have that

$$\mathrm{res}^G_H(e^p_{G,Q})=\sum_{(R)\in P_p(H)}\delta_G(Q,\ R)e^p_{H,R}. \tag{a.6}$$

Let P/Q be a Sylow p-subgroup of $WQ:=N_G(Q)/Q$. Then the idempotent formula gives the following:

$$e^p_{G,Q}\in \mathrm{ind}^G_P(A(P)_{(p)}). \tag{a.7}$$

$$\mathrm{res}^G_P(e^p_{G,Q})=e^p_{P,Q}. \tag{a.8}$$

$$\mathrm{res}^G_H(e^p_{G,Q})=0 \qquad \text{unless } Q\leqq_G H. \tag{a.9}$$

Now, let ($\boldsymbol{a}$, ind, res, con) be a G-functor over $\boldsymbol{Z}_{(p)}$, so that $\boldsymbol{a}$: $H\mapsto\boldsymbol{a}(H)$ assigns each subgroup H of G to a module $\boldsymbol{a}(H)$ and the families of linear maps ind^K_H, res^K_H, con^g_H satisfy the similar identities (G.1)-(G.5) as in Lemma 3.1. Then the Burnside ring $A(H)_{(p)}$ acts on the component $\boldsymbol{a}(H)$ by

$$[D\backslash H]\cdot x:=\mathrm{ind}^H_D\,\mathrm{res}^H_D(x), \qquad x\in\boldsymbol{a}(H), \quad D\leqq H\leqq G. \tag{a.10}$$

Furthermore, the G-functor $\boldsymbol{a}$ is an "$\boldsymbol{A}$-module", that is, the following hold:

$$\mathrm{res}^K_H(by)=\mathrm{res}^K_H(b)\,\mathrm{res}^K_H(y), \tag{a.11}$$

$$\mathrm{con}^g_H(ax)=\mathrm{con}^g_H(a)\,\mathrm{con}^g_H(x), \tag{a.12}$$

$$\mathrm{ind}^K_H(a)y=\mathrm{ind}^K_H(a\,\mathrm{res}^K_H(y)), \tag{a.13}$$

$$b\,\mathrm{ind}^K_H(x)=\mathrm{ind}^K_H(\mathrm{res}^K_H(b)x), \tag{a.14}$$

where $H\leqq K\leqq G$, $g\in G$, $a\in A(H)_{(p)}$, $b\in A(K)_{(p)}$, $x\in\boldsymbol{a}(H)$, $y\in\boldsymbol{a}(K)$. The last two identities are the Frobenius formulas.

Associated the orthogonal decomposition $1=\sum e^p_{G,Q}$ of 1, there is a direct decomposition

$$\boldsymbol{a}(G)=\bigoplus_{(Q)} e^p_{G,Q}\boldsymbol{a}(G), \tag{a.15}$$

where the summation are taken over $P_p(G)$, the set of conjugacy classes of p-perfect subgroups. So we are interested in the submodule $e^p_{G,Q}\boldsymbol{a}(G)$ which is the module over the local ring $e^p_{G,Q}A(G)_{(p)}$.

b. Stable element theorem.

LEMMA 4.1 (Stable element theorem). *Let* ($\boldsymbol{a}$, ind, res, con) *be a G-functor over* $\boldsymbol{Z}_{(p)}$. *Let Q be a p-perfect subgroup of G and let P be a subgroup of N* $:=N_G(Q)$ *such that P/Q is a Sylow p-subgroup. Then the map* res^G_P *induces an isomorphism*

$$e^p_{G,Q}\boldsymbol{a}(G) \cong e^p_{P,Q}\boldsymbol{a}(P)^G = e^p_{G,Q}\boldsymbol{a}(P) \cap \boldsymbol{a}(P)^G,$$

where $\boldsymbol{a}(P)^G$ *is the set of elements* x *of* $\boldsymbol{a}(P)$ *with*

$$\mathrm{res}_{P^g \cap P}\mathrm{con}^g_P(x) = \mathrm{res}_{P^g \cap P}(x) \qquad \textit{for all } g \in G.$$

PROOF. Define two maps as follows:

$$i:\ e^p_{P,Q}\boldsymbol{a}(P)^G \longrightarrow e^p_{G,Q}\boldsymbol{a}(G);\quad x \longmapsto \mathrm{ind}^G_P(x),$$
$$r:\ e^p_{G,Q}\boldsymbol{a}(G) \longrightarrow e^p_{P,Q}\boldsymbol{a}(P)^G;\quad y \longmapsto \mathrm{res}^G_P(y).$$

By (a.8) and Frobenius reciprocity, i and r are well-defined and $e^p_{P,Q} \in A(P)^G$. Thus it remains only to show that i, r are isomorphisms. Using Mackey decomposition and Frobenius reciprocity, we have that

$$i \circ r = [P \backslash G]\mathrm{id}, \qquad r \circ i = \mathrm{res}^G_P([P \backslash G])\mathrm{id}.$$

Thus in order to prove that i and r are both isomorphisms, it will suffice to show that $[P \backslash G]e^p_{G,Q}$ has an inverse in $e^p_{G,Q}A(G)_{(p)}$. Indeed if this claim is true, then $\mathrm{res}^G_P([P \backslash G])e^p_{P,Q}$ is also a unit of $e^p_{P,Q}A(P)_{(p)}$, and so not only $i \circ r$ but also $r \circ i$ are isomorphism. By Lemma 2.1 and (a.1), there is a linear map with finite cokernel:

$$\varphi' := (\varphi_S):\ e^p_{G,Q}A(G)_{(p)} \longrightarrow \prod_{(S)}{}' \boldsymbol{Z}_{(p)},$$

where the product are taken over classes $(S) \in Cl(G)$ with $Q \leqq S \leqq P$. Let S be such a subgroup. Then

$$\varphi'([P \backslash G]e^p_{G,Q}) = |(P \backslash G)^S| = |(P \backslash N)^S| = |P \backslash N| \not\equiv 0 \qquad (\mathrm{mod}\ p).$$

Thus the image of $[P \backslash G]e^p_{G,Q}$ by φ' is invertible. Since φ' gives an integral extension of rings, $[P \backslash G]e^p_{G,Q}$ is also invertible. The lemma is proved.

REMARK. Since $e^p_{G,Q}$ is contained in the image of ind^G_P: $A(P)_{(p)} \to A(G)_{(p)}$ (see (a.7)), the G-functor $H \mapsto \mathrm{res}^G_H(e^p_{G,Q})\boldsymbol{a}(H)$ is P-projective (cf. Green [**Gr71**], [**Di79**, Section 6.1]). So this lemma follows also from the general theory of G-functors and Mackey functors. The stable element theorem for cohomological G-functors which is an analogue in group cohomology theory was proved in [**Yo80**, Theorem 3.2]. The present lemma can be immediately proved from [**Sa82**, Lemma 4.3]. The Mackey functor version of the stable element theorem is found in [**Di79**, Proposition 6.1.6].

COROLLARY 4.2. *Under the same assumption as in Lemma* 4.1, *there is an isomorphism*

$$e^p_{G,Q}\boldsymbol{a}(G) \cong e^p_{N,Q}\boldsymbol{a}(N).$$

PROOF. Take an element x of $e^p_{P,Q}\boldsymbol{a}(P)^N$, so that $x = e^p_{P,Q}x \in \boldsymbol{a}(P)^N$. Let

g be any element of G. Then

$$(*) \qquad \mathrm{res}_{P^g\cap P}\,\mathrm{con}_P^g(x) = \mathrm{res}_{P^g\cap P}^P(x).$$

Indeed, if g is in N, then this holds clearly. If g is not in N, the $P^g\cap P$ contains no G-conjugate of Q because $O^p(P)=Q$. Thus by (a.9) both sides of $(*)$ vanish. Hence $(*)$ holds for any element g of G. This means that

$$e_{P,Q}^p\,\boldsymbol{a}(P)^G = e_{P,Q}^p\,\boldsymbol{a}(P)^N .$$

Now the statement follows immediately from Lemma 4.1.

REMARK. (1) This useful isomorphism is, as shown in Araki's paper [**Ar82**], given by the composition:

$$e_{G,Q}^p\,\boldsymbol{a}(G) \xrightarrow{\text{incl}} \boldsymbol{a}(G) \xrightarrow{\text{res}} \boldsymbol{a}(N) \xrightarrow{\text{proj}} e_{N,Q}^p\,\boldsymbol{a}(G).$$

And its inverse is

$$e_{N,Q}^p\,\boldsymbol{a}(N) \xrightarrow{\text{incl}} \boldsymbol{a}(N) \xrightarrow{\text{ind}} \boldsymbol{a}(G) \xrightarrow{\text{proj}} e_{G,Q}^p\,\boldsymbol{a}(G).$$

(2) In his unpublished work (written in Japanese), T. Yoshida had obtained an extension of the isomorphism in this corollary to an equivalence of the representation categories of G-functors. Furthermore, D. Tambara had also written this isomorphism in his private letter to the author.

c. The proof of Theorem B. Now we return to the unit groups of the Burnside rings of finite groups. By Lemma 3.1, ($\boldsymbol{A}^*$, jnd, res, con) is a G-functor. The $A(G)$-module structure (a.10) on $A(G)^*$ induced by this G-functor is coincident with one defined by the exponential map $(u, a)\mapsto u\uparrow a$ in Section 3.c because $\mathrm{jnd}_H^G\circ\mathrm{res}_H^G=(-1)^{[H\backslash G]}$. Since $u^2=1$ for any element u of $A(G)^*$, the action of $A(G)$ on $A(G)^*$ can be extended to $A(G)_{(2)}$, for which the annihilator contains $2A(G)_{(2)}$.

THEOREM B. (i) *There is a decomposition into a direct product*

$$A(G)^* = \prod_{(Q)} A(G)^*\uparrow e_{G,Q}^2,$$

where (Q) runs over $P_2(G)$, the classes of 2-perfect subgroups.

(ii) *Let Q be a 2-perfect subgroup of G and let P be a subgroup of $N:=N_G(Q)$ such that P/Q is a Sylow 2-subgroup of N/Q. Then there are group homomorphisms induced by restriction maps:*

$$A(G)^*\uparrow e_{G,Q}^2 \cong A(N)^*\uparrow e_{N,Q}^2 \cong A(P)^{*N}\uparrow e_{P,Q}^2 = A(P)^{*N}\cap(A(P)^*\uparrow e_{G,Q}^2).$$

PROOF. This follows directly from applying Lemma 4.1 and Corollary 4.2 to the G-functor ($\boldsymbol{A}^*$, jnd, res, con) over $\boldsymbol{Z}_{(2)}$.

5. Another transfer theorem.

In this section, we prove Theorem C which is a unit group version of Araki's transfer theorem ([**Ar82**, Corollary B]). It is considered as an analogue of the excision theorem for relative Grothendieck rings of finite groups. To prove this theorem, we need to study further the value of units at subgroups.

a. Value of units.

LEMMA 5.1 ([**MM83**, Lemma 2.8]). *Let u be an element of the unit group $A(G)^*$. Let T be a subgroup of G and S a normal subgroup of T of odd index. Then $u(S)=u(T)$.*

PROOF. By the Solvability of Groups of Odd Order, we may assume that $W:=T/S$ is a cyclic group of odd order. Then by Lemma 2.1, we have that

$$\sum_{tS\in W} u(\langle t\rangle S)\equiv 0 \qquad (\bmod |W|).$$

But since $u(\langle t\rangle S)=\pm 1$ and $|W|$ is an odd number, this congruence derives that $u(\langle t\rangle S)=u(S)$ for every $tS\in W$. Hence $u(T)=u(S)$. The lemma is proved.

For a 2-perfect subgroup Q of G and subgroups S, T of G, we define a 2-local integer $\nu_Q(S;\,T)$ by

$$\text{(a.1)} \qquad [S\backslash G]e_{G,Q}^{2}=\sum_{(T)\in Cl(G)}\nu_Q(S;\,T)[T\backslash G].$$

The value of $\nu_Q(S;\,T)$ is explicitly given by

$$\text{(a.2)} \qquad \nu_Q(S;\,T)=\frac{|T|}{|S|}\sum_{D,K\leq S}\mu(D,K)\delta_G(D,T)\delta_G(O^2(T),Q),$$

where μ is the Möbius function of the subgroup lattice of G and δ_G is the function defined in (a.3) is Section 4. This fact is proved by using the idempotent formula or by comparing the value of both sides of (a.1) at each subgroup. But because we do not need (a.2) in this paper, the detail of its proof is omitted.

LEMMA 5.2. *Let Q be a 2-perfect subgroup of G.*
(i) *If u is an element of $A(G)^*$, then*

$$(u\uparrow e_{G,Q}^{2})(S)=\prod_{(T)\in Cl(G)}u(T)^{\nu_Q(S;\,T)}, \qquad S\leqq G.$$

(ii) *Let u be an element of $A(G)^*\uparrow e_{G,Q}^{2}$.*
(a) *If $u(S)=-1$, then $Q\leqq_G S$;*
(b) *If $u\neq 1$, then $u(S)=-1$ for a subgroup S of G such that $O^2(S)=Q$.*

PROOF. Put $e:=e_{G,Q}^{2}$ and $\nu(S;\,T):=\nu_Q(S;\,T)$.

(i) Note that if y is an element of $A(G)$, then $y(S)=(y\uparrow[S\backslash G])(G)$. Thus

$$\begin{aligned}(u\uparrow e)(S)&=((u\uparrow e)\uparrow[S\backslash G])(G)=(u\uparrow(e[S\backslash G]))(G)\\&=\prod_{(T)}(u\uparrow[T\backslash G])(G)^{\nu(S;T)}=\prod_{(T)}u(T)^{\nu(S;T)},\end{aligned}$$

proving (i).

(ii) We first show the following general results:

$$(*)\qquad \nu(S;\,T)=0 \quad \text{unless } T\leqq_G S;$$

$$(**)\qquad \nu(S;\,S)=\begin{cases}1 & \text{if } O^2(S)=_G Q\\ 0 & \text{otherwise.}\end{cases}$$

In general, the product $[S\backslash G]\cdot[R\backslash G]$ is a sum of elements of the form $[S\cap R^g\backslash G]$. Thus if an element $[T\backslash G]$ appears in $[S\backslash G]\cdot e$, then $T\leqq_G S$. Hence $(*)$ holds. Next we consider the value of (a.1) at S. If $T\leqq_G S$, then

$$[T\backslash G](S)=|(T\backslash G)^S|=\begin{cases}|WS| & \text{if } T=_G S\\ 0 & \text{otherwise.}\end{cases}$$

Thus by (a.1) and $(*)$, we have that

$$[S\backslash G](S)\cdot e(S)=\nu(S;\,S)[S\backslash G](S),$$

and so by (a.1) in Section 4,

$$\nu(S;\,S)=e^2_{G,Q}(S)=\begin{cases}1 & \text{if } O^2(S)=_G Q\\ 0 & \text{otherwise,}\end{cases}$$

proving $(**)$. Now, to prove (a) and (b), let S be a minimal subgroup of G such that $u(S)=-1$. Since $u\uparrow e=u$, (i) and the minimality of S yield that $u(S)=u(S)^{\nu(S;S)}=-1$, and so $\nu(S;\,S)$ is odd. Thus by $(**)$, we conclude that $O^2(S)=_G Q$. The lemma is proved.

b. Proof of Theorem C. In this subsection, N denotes a finite group with 2-perfect normal subgroup Q. We put $W:=N/Q$. We denote by $O^{2'}(S)$ the smallest subgroup of a finite group S such that $S/O^{2'}(S)$ is a (solvable) group of odd order.

PROPOSITION 5.3. *The homomorphism $u\mapsto \mathrm{inv}_Q(u)$ gives an isomorphism of $A(N)^*\uparrow e^2_{N,Q}$ onto the subgroup*

$$\{\tilde v\in A(W)^*\uparrow e^2_{W,1}\mid\ \tilde v(S/Q)=1\ \textit{unless}\ Q\leqq O^{2'}(S)\}.$$

The inverse image v of $\tilde v$ is given by

$$v(S)=\begin{cases}\tilde v(S/Q) & \textit{if } Q\leqq S\\ 1 & \textit{otherwise.}\end{cases}$$

PROOF. We put

$$e := e^2_{N,Q} \in A(N)_{(2)} \quad \text{and} \quad \bar{e} := e^2_{W,1} \in A(W)_{(2)}.$$

Comparing the value at each subgroup, we have that

$$\mathrm{inv}_Q(e) = \bar{e} \quad \text{and} \quad e = \mathrm{inf}_Q(\bar{e})\cdot e.$$

We put

$$B := \{\bar{v} \in A(W)^* \uparrow \bar{e} \mid \bar{v}(S/Q)=1 \text{ unless } Q \leqq O^{2'}(S)\}.$$

We must show that

$$(1) \qquad i: \ A(N)^* \uparrow e \longrightarrow B; \quad u \longmapsto \mathrm{inv}_Q(u)$$

is an isomorphism. We begin by proving that this map is well-defined. Let $u = u \uparrow e$ be any element of $A(N)^* \uparrow e$. Then we have that

$$\begin{aligned} \mathrm{inv}(u) \uparrow \bar{e} &= \mathrm{inv}(u \uparrow \mathrm{inf}(\bar{e})) = \mathrm{inv}((u \uparrow e) \uparrow \mathrm{inf}(\bar{e})) = \mathrm{inv}(u \uparrow (e \cdot \mathrm{inf}(\bar{e}))) \\ &= \mathrm{inv}(u \uparrow e) = \mathrm{inv}(u), \end{aligned}$$

and so $\mathrm{inv}(u) \in A(W)^* \uparrow \bar{e}$. In order to prove that $\mathrm{inv}(u)$ is in B, let S be a subgroup of N such that S contains Q but $O^{2'}(S)$ does not contain Q. Then by Lemma 5.2(ii)(b), $u(O^{2'}(S))=1$. Thus by Lemma 5.1, we have that

$$u(S) = u(O^{2'}(S)) = 1.$$

This means that $\mathrm{inv}(u)$ is contained in B for each $u \in A(N)^* \uparrow e$. Hence the map i defined in (1) is well-defined.

We will show that i is isomorphism. First, i is injective. In fact, suppose $i(u)=1$ and $S \leqq N$. Then $u(S)=i(u)(S)=1$ if S contains Q; and by Lemma 5.2(ii)(a), $u(S)=1$ if S does not contain Q. Thus $u=1$, as required. Now it remains only to prove the surjectivity. Let $\bar{u}$ be an element of B, so that $\bar{u} \uparrow \bar{e} = \bar{u}$ and $\bar{u}(S/Q)=1$ if $O^{2'}(S)$ does not contain Q. Define a super class function u of N by

$$(2) \qquad u(S) = \begin{cases} \bar{u}(S/Q) & \text{if } Q \leqq S \\ 1 & \text{otherwise}. \end{cases}$$

Since $\mathrm{inv}_Q(u) = \bar{u}$, it will suffice to show that u is contained in $A(N)^* \uparrow e$. So we have to show the following:

(3) $\qquad u$ is an element of $A(N)^*$;

(4) $\qquad u \uparrow e = u$.

We prove (3). Let H be a subgroup of N and put $WH := N_N(H)/H$. To apply Lemma 2.1, we need to show that

(5) $$\sum_{gH\in WH} u(\langle g\rangle H) \equiv 0 \qquad (\mathrm{mod}\,|WH|).$$

When H contains Q, this follows from the definition of u and Lemma 2.1 for $\bar{u}\in A(W)$. When H does not contain Q, we have that $O^{2'}(\langle g\rangle H)$ also does not contain Q for every $gH\in WH$ (because otherwise $Q=O^2(Q)\leqq O^2(O^2(\langle g\rangle H))\leqq H$, a contradiction), and so each term of (5) equals 1, whence (5) again holds. Since the value of u at any subgroup is ± 1, we conclude that u is in $A(N)^*$, proving (3).

Next we prove (4). We have to show the following:

(6) $$(u\uparrow e)(S) = u(S) \qquad \text{for any } S\leqq N.$$

If S does not contain Q, then the both sides of (6) equal to 1 by the definition of u and Lemma 5.2(ii)(a). So assume that S contains Q. Put $\bar{S}:=S/Q$. Then by Lemma 5.2(i), we have that

$$(u\uparrow e)(S) = \prod_{(T)\in Cl(H)} u(T)^{\nu(S;T)} = \prod_{(T/Q)\in Cl(W)} \bar{u}(T/Q)^{\nu(S;T)},$$

where $\nu(S;T)$ is, as before, defined by

$$e\cdot[S\backslash N] = \sum_{(T)\in Cl(N)} \nu(S;T)[T\backslash N].$$

On the other hand, since $\bar{u}\uparrow\bar{e}=\bar{u}$, we have that

$$u(S) = \bar{u}(\bar{S}) = \prod_{(\bar{T})\in Cl(W)} \bar{u}(\bar{T})^{\bar{\nu}(\bar{S};\bar{T})},$$

where $\bar{\nu}(\bar{S};\bar{T})$ is defined by

$$\bar{e}\cdot[\bar{S}\backslash W] = \sum_{(\bar{T})\in Cl(W)} \bar{\nu}(\bar{S};\bar{T})[\bar{T}\backslash W].$$

Since $\bar{e}=\mathrm{inv}_Q(e)$ and $\mathrm{inv}_Q([T\backslash N])=[\bar{T}\backslash W]$ if $Q\leqq N$ and $\mathrm{inv}_Q([T\backslash N])=0$ otherwise, we have that

$$\begin{aligned}[\bar{S}\backslash W]\cdot\bar{e} &= \mathrm{inv}_Q([S\backslash W]\cdot e) = \sum_{(T)} \nu(S;T)\mathrm{inv}_Q([T\backslash N]) \\ &= \sum_{(T/Q)} \nu(S;T)[(T/Q)\backslash W],\end{aligned}$$

and so $\nu(\bar{S};T/Q)=\nu(S;T)$. Thus (6) holds. The proof of the proposition is completed.

THEOREM C. *Let N be a normal subgroup with 2-perfect normal subgroup Q. Put $W:=N/Q$. Let $\bar{P}=P/Q$ be a Sylow 2-subgroup of W. Then $A(N)^*\uparrow e^2_{N,Q}$ is isomorphic to the subgroup*

$$\{\bar{v}\in A(\bar{P})^{*W} \mid \bar{v}(S/Q)=1 \ \ \textit{if } S\neq O^{2'}(S)\}.$$

PROOF. First by Theorem B, we have an isomorphism

$$A(N)^*\uparrow e^2_{P,Q} \cong (A(P)^*\uparrow e^2_{P,Q})\cap A(P)^{*N} =: C.$$

Note that $e_{\bar{P}}^2=1$ and that for a 2-subgroup S/Q of W,

$$Q \leqq O^{2'}(S) \Longleftrightarrow S = O^{2'}(S).$$

Thus applying Proposition 5.3 to P, we have that

$$A(P)^* \uparrow e_{P,Q}^2 \cong \{\bar{v} \in A(\bar{P}) \mid \bar{v}(S/Q)=1 \quad \text{if } O^{2'}(S) \neq S\} =: B$$

by the map $v \mapsto \mathrm{inv}_Q(v)$. So we need to show that inv_Q maps C onto $\bar{B} \cap A(\bar{P})^{*W}$. Since inv_Q commutes with restriction and conjugation, we have that $\mathrm{inv}_Q(C)$ is contained in $A(\bar{P})^{*W}$. Conversely let $\bar{v}$ be any element of $\bar{B} \cap A(\bar{P})^{*W}$. Then the inverse image v of $\bar{v}$ in $A(P)^* \uparrow e_{P,Q}^2$ is given by

$$v(S) = \begin{cases} \bar{v}(S/Q) & \text{if } Q \leqq S, \\ 1 & \text{otherwise}. \end{cases}$$

It is easily checked that v is contained in $A(P)^N$. Thus inv_Q gives a surjection of C to $\bar{B} \cap A(\bar{P})^{*W}$. The theorem is proved.

COROLLARY C1. *Let N be a finite group with 2-perfect normal subgroup Q. Let $\Psi(Q)$ denote the intersection of all normal subgroups of Q of prime index. Put $\tilde{N} := N/\Psi(Q)$ and $\tilde{Q} := Q/\Psi(Q)$. Then there is an isomorphism:*

$$A(N)^* \uparrow e_{N,Q}^2 \cong A(\tilde{N})^* \uparrow e_{\tilde{N},\tilde{Q}}^2.$$

PROOF. By Theorem C, we have that the group on the left side of the statement is isomorphic to

$$\{\bar{v} \in A(\bar{P})^{*W} \mid \bar{v}(S/Q)=1 \text{ if } O^{2'}(S) \neq S\},$$

where $\bar{P} := P/Q$ is a Sylow 2-subgroup of $W := N/Q$. On the other hand, the right side is isomorphic to

$$\{\bar{v} \in A(\bar{P})^{*W} \mid \bar{v}(S/Q)=1 \text{ if } O^{2'}(\tilde{S}) \neq \tilde{S}\},$$

where $\tilde{S} := S/\Psi(Q)$ for $S \leqq P$. Thus in order to finish the proof, we must show that for any 2-subgroup S/Q,

$$O^{2'}(S) = S \quad \text{if and only if} \quad O^{2'}(\tilde{S}) = \tilde{S}.$$

But this is clear from the Solvability of Groups of Odd Order.

6. Additional results on unit groups.

In this section, we add some miscellaneous algebraic results which are not found or not stated clearly in five papers: Dress [**Dr71**], tom Dieck [**Di79**], Matsuda [**Ma82**], [**Ma86**] and Matsuda-Miyata [**MM83**]. Many examples about unit groups are found in Matsuda's papers.

a. Character rings. Let $R_{\boldsymbol{Q}}(G)$ be the subring of the character ring $R(G)$ generated by the characters afforded by $\boldsymbol{Q}G$-modules. Of course we sometimes regard it as the Grothendieck ring of $\boldsymbol{Q}G$-modules. For any finite G-set X, let π_X be the permutation character of G given by X, so that $\pi_X(g)$ is the number of fixed points by g. Thus we have a ring homomorphism

$$\mathrm{char}_G: \ A(G) \longrightarrow R_{\boldsymbol{Q}}(G); \ [X] \longmapsto \pi_X.$$

See Section 3.d. For an element x of $A(G)$ which we view as a super class function, we have that

$$\mathrm{char}_G(x)(g) = x(\langle g \rangle), \qquad g \in G.$$

Let G^2 be the subgroups of G generated by g^2 for all $g \in G$, so that G/G^2 is an elementary abelian 2-group, and let $\hat{G}_2$ be the character group of G/G^2.

LEMMA 6.1. *The map* char_G *induces a group homomorphism*

$$\mathrm{char}_G^*: \ A(G)^* \longrightarrow R_{\boldsymbol{Q}}(G)^* = \{\pm 1\} \times \hat{G}_2.$$

In particular, if u *is an element of* $A(G)^*$, *then* $u(1)\cdot \mathrm{char}_G(u)$ *is a linear character of* G.

PROOF. Since char_G is a ring homomorphism, it maps units to units. If λ is a unit of $R_{\boldsymbol{Q}}(G)$, then its value is always ± 1, and so by the orthogonal relation, λ is in $\{\pm 1\} \times \hat{G}_2$.

COROLLARY 6.2 (cf. [**Yo78**, Lemma 5.2]). *Let* χ *be a real valued virtual character and let* C *be a cyclic subgroup of* G^2. *Then*

$$\chi(1) \equiv \langle \chi_C, 1_C \rangle \pmod 2.$$

PROOF. Let u_G: $\bar{R}(G) \to A(G)^*$ be the tom Dieck homomorphism. Then $\lambda := u_G(\chi)(1)\cdot u_G(\chi)$ is a linear character and for a generator g of C, we have that

$$1 = \lambda(g) = \mathrm{sgn}\, \chi(1)\cdot \mathrm{sgn}\langle \chi_C, 1_C \rangle,$$

because $g \in G^2 \leqq \mathrm{Ker}\, \lambda$.

Let H be a subgroup of G. The multiplicative (tensor) induction (cf. [**CR81**, Section 13]) defines a non-additive but multiplicative and polynomial map

$$\mathrm{jnd}_H^G: \ R(H) \longrightarrow R(G).$$

If θ is a virtual character of H and g is an element of G, then

$$\mathrm{jnd}_H^G(\theta)(g) = \prod_{t \in H\backslash G/\langle g \rangle} \theta(t g^{r(t)} t^{-1}),$$

where $r(t) := |\langle g \rangle : H^t \cap \langle g \rangle|$. The multiplicative induction jnd_H^G on the group of linear characters of H is the dual of the group-theoretic transfer map

$$V_{G,H}: \quad G/[G, G] \longrightarrow H/[H, H].$$

This follows from [**Go68**, Theorem 7.3.3]. There is an analogue of Mackey decomposition for multiplicative induction and group theoretic transfer (cf. [**CR81**, Section 13, Exercise 1]; [**Yo78**]). Furthermore, we have that

$$\mathrm{jnd}_H^G(-1_G)(g) = (-1)^{|H\backslash G/\langle g\rangle|}.$$

The assignments $H \mapsto R(H)^*$, $R_{\boldsymbol{Q}}(H)^*$, $\hat{H}_2$ together with multiplicative induction, restriction and conjugation form G-functors.

LEMMA 6.3. *The maps* $\mathrm{char}_H^*: A(H)^* \to R_{\boldsymbol{Q}}(H)^*$ *commute with* jnd, res, con. *Thus they form a morphism between G-functors.*

PROOF. Direct verification.

LEMMA 6.4. *The composition*

$$\bar{R}(G) \xrightarrow{u_G} A(G)^* \xrightarrow{\mathrm{char}_G^*} \{\pm 1\} \times \hat{G}_2 \xrightarrow{\mathrm{proj}} \hat{G}_2,$$

where u_G is the tom Dieck homomorphism from the ring of real valued characters, is equal to the determinant map $\det: \chi \mapsto \det \chi$.

PROOF. Let χ be a real valued character of G. Then for any element g of G,

$$\mathrm{proj} \circ \mathrm{char}_G^* \circ u_G(\chi)(g) = \mathrm{sgn}\, \chi(1) \cdot \mathrm{sgn}\langle \chi_{\langle g\rangle}, 1_{\langle g\rangle}\rangle.$$

By the way similar as in the proof of Theorem A, we can easily prove that this is equal to $(\det \chi)(g)$.

REMARK. If we adopt tom Dieck's definition of the Burnside ring, we interpret this lemma as follows. Let V be an orthogonal $\boldsymbol{R}G$-module of dimensional n and SV the unit sphere of V. Then $u_G([V])$ is represented by $1-[SV] \in A(G)$. See [**Di79**, Proposition 5.5.9]. Furthermore $\mathrm{char}_G: A(G) \to R_{\boldsymbol{Q}}(G)$ is given by the equivariant Euler characteristic

$$\mathrm{char}_G: \quad [X] \longmapsto \sum_{i \geq 0} (-1)^i [H^i(X; \boldsymbol{Q})] \in R_{\boldsymbol{Q}}(G).$$

Thus we have that $\mathrm{char}_G \circ u_G([V]) = (-1)^n [\boldsymbol{Q}w]$, where G acts on $\boldsymbol{Q}w$ by $gw := w$ if the mapping degree of g on SV is $+1$ and $= -w$ otherwise, and so $\mathrm{char}_G \circ u_G([V]) = (-1)^n [\det V]$, where $\det V$ is the n-fold exterior power space.

b. The values of units and the order of $A(G)^*$.

PROPOSITION 6.5. *Let $u \in \tilde{A}(G) := \{\pm 1\}^{Cl(G)}$. Then u is contained in $A(G)^*$*

if and only if for each subgroup S of G, the map

$$gS \longmapsto u(\langle g\rangle S)/u(S), \qquad gS \in WS,$$

is a linear character of $WS := N_G(S)/S$.

PROOF. Assume first that u is in $A(G)^*$. By Lemma 6.1, we have that $u(S)^{-1}\mathrm{char}_{WS}(\mathrm{inv}_S(u))$ is a linear character of WS, of which value at gS is $u(\langle g\rangle S)/u(S)$, as required. Conversely, let u be a super class function of G such that $gS \mapsto u(\langle g\rangle S)/u(S)$ is a linear character of WS for each subgroup S of G. Then for subgroup S,

$$\sum_{gS \in WS} u(\langle g\rangle S) \equiv 0 \pmod{|WS|}.$$

In fact this summation is equal to 0 or $\pm|WS|$ by the orthogonal relation. Thus Lemma 2.1 gives that u is an element of $A(G)$. Since the value of u is ± 1, it is in the unit group.

COROLLARY 6.6. *Let u be an element of $A(G)^*$, and let S be a subgroup of G. Then $u(S)=u(\langle g\rangle S)$ for any element g of $N_G(S)^2$ $(:=\langle n^2 \mid n \in N_G(S)\rangle)$.*

PROOF. If g is an element of $N_G(S)^2$, then gS is contained in the kernel of any linear character with values ± 1.

LEMMA 6.7. (i) *The order $A(G)^*$ is equal to the number of idempotents e of $\boldsymbol{Q}\otimes_{\boldsymbol{Z}} A(G)$ such that $2e \in A(G)$.*

(ii) $\#\{(H) \in Cl(G) \mid H$ *is perfect*$\} \leqq \log_2|A(G)^*| \leqq \#\{(H) \in Cl(G) \mid O^{2'}(H) = H\}$.

(iii) *G is of odd order if and only if $A(G)^* = \{\pm 1\}$.*

(iv) *If G has a Sylow 2-subgroup of order 2, then the second inequality in* (ii) *is an equality.*

PROOF. (i) is well-known $u \mapsto (1-u)/2$, $e \mapsto 1-2e$. (ii) is proved by counting the number of primitive idempotents of $A(G)$ and $A(G)_{(2')} = A(G)[1/2]$. See [**Yo83**]. (iii) follows form (i). See also [**Dr69**]. (iv) is proved by the idempotent formula for $A(G)[1/2]$.

c. Normal subgroups. We collect some generalizations of Matsuda and Miyata's results.

PROPOSITION 6.8. *Let H be a normal subgroup of G of odd index. Then $A(G)^*$ is isomorphic to $(A(H)^*)^{G/H}$, the G/H-fixed point subgroup, via the restriction map.*

PROOF. The restriction map and the multiplicative induction map induce well-defined maps

$$r:\quad A(G)^* \longrightarrow (A(H)^*)^{G/H};\quad v \longmapsto \mathrm{res}_H^G(v),$$
$$j:\quad (A(H)^*)^{G/H} \longrightarrow A(G)^*;\quad u \longmapsto \mathrm{jnd}_H^G(u).$$

Since H is of odd index, by (b.3) in Section 3, we have that

$$j(u)(S) = \prod_{g\in S\backslash G/H} u(S^g \cap U) = u(S\cap H)^{|G:SH|} = u(S\cap H).$$

Thus if S is a subgroup of H, then $j(u)(S)=u(S)$, and so $r\circ j=\mathrm{id}$. Next let v be an element of $A(G)^*$. Then by Lemma 5.1, $v(S\cap H)=v(S)$. Thus we have that $j\circ r=\mathrm{id}$. The lemma is proved.

PROPOSITION 6.9 (cf. [**Ma82**, Theorem 4.1]). *Let $H_1, \cdots, H_r$ be a normal subgroups of G. For each subset I of $R:=\{1, 2, \cdots, r\}$, we put*

$$H_I := \langle H_i \mid i\in I\rangle,$$
$$\mathrm{Im}_I := \mathrm{Im}(\mathrm{inf}:\ A(G/H_I)^* \longrightarrow A(G)^*),$$
$$\mathrm{Ker}_I := \bigcap_{i\in I}\mathrm{Ker}(\mathrm{inv}:\ A(G)^* \longrightarrow A(G/H_I)^*),$$
$$A(G)_I^* := \mathrm{Im}_I \cap \mathrm{Ker}_{I'},\qquad \textit{where } I' := R-I.$$

Then $A(G)^*$ is the direct product of subgroups $A(G)_I^*$, $I\subseteq R$.

PROOF. Let E be the set of all endomorphisms of abelian group $A(G)^*$. We write the action of E on $A(G)^*$ by exponential form, that is, we write $u\uparrow f := f(u)$, $f\in E$, $u\in A(G)^*$. Then E becomes a ring by the ordinary way. To each normal subgroup H of G, there assigns an element f_H of E defined by

$$f_H:\quad A(G)^* \xrightarrow{\mathrm{inv}} A(G/H)^* \xrightarrow{\mathrm{inf}} A(G)^*.$$

For two normal subgroups H, K of G, we have that $f_H\circ f_K = f_K\circ f_H = f_{HK}$. Thus f_H's are pairwise commutative idempotents of E. Now for each i of R, let f_i be the idempotent of E assigned to H_i. Put

$$f_I := \prod_{i\in I} f_i \qquad f_I' := \prod_{i\in I}(1-f_i),$$
$$e_I := f_I f_{I'}' \qquad (I' := R-I).$$

Note that f_I is the idempotent corresponding to the normal subgroup H_I. Then we have that

$$\sum_{I\subseteq R} e_I = \mathrm{id},\qquad e_I e_I = e_I,\qquad e_I e_J = 0\quad (I\neq J).$$

This induces the decomposition of $A(G)^*$ to the direct product:

$$A(G)^* = \prod_{I\subseteq R} A(G)^* \uparrow e_I.$$

From the fact that $\mathrm{inv}_H\circ\mathrm{inf}_H^G$ is an identity on $A(G/H)$, we can easily prove that

$$A(G)^* \uparrow e_I = \mathrm{Im}_I \cap \mathrm{Ker}_{I'} .$$

Hence the proposition is proved.

COROLLARY 6.10. *Let H be a normal subgroup of G. Then*

$$A(G)^* \cong A(G/H)^* \times \{u \in A(G)^* \mid u(S)=1 \text{ if } S \geqq H\} .$$

PROOF. Use the following easy results:

$$\mathrm{Im}(\mathrm{inf}\colon A(G/H) \to A(G)) = \{x \in A(G) \mid x(S)=x(SH) \quad \text{for } S \leqq G\},$$
$$\mathrm{Ker}(\mathrm{inv}\colon A(G)^* \to A(G/H)^*) = \{u \in A(G)^* \mid u(S)=1 \quad \text{if } H \leqq S \leqq G\} .$$

7. Abelian Sylow 2-subgroups.

In this section, we study the unit group $A(G)^*$ for a finite group G with abelian Sylow 2-subgroups. First we prove the transfer theorem of Burnside type. Next we decide the order of the unit groups for some simple groups.

a. Proof of Theorem D. We begin with proving Matsuda's theorem (cf. [**Ma82**, Example 4.5]). Let T be a finite abelian group. Put $\bar{T} := T/T^2$, where $T^2 := \langle t^2 \mid t \in T \rangle$, and let $\bar{T}^{\wedge}$ be the character group of $\bar{T}$. We regard the character ring $R(\bar{T})$ as a subgroup of $R_{\boldsymbol{Q}}(T)$. Then the tom Dieck homomorphism u_T induces a map:

$$\text{(a.1)} \qquad \bar{u}_T\colon \ \boldsymbol{F}_2[\bar{T}^{\wedge}] \cong R(\bar{T})/2R(\bar{T}) \longrightarrow A(T)^* .$$

For an element λ of $\bar{T}^{\wedge}$ and a subgroup S of T, we have that

$$\text{(a.2)} \qquad \bar{u}_T(\lambda)(S) = \begin{cases} -1 & \text{if } S \subseteqq \mathrm{Ker}\,\lambda \\ +1 & \text{otherwise}. \end{cases}$$

LEMMA 7.1. *Let T be a finite abelian group. Then the above map $\bar{u}_T$: $\boldsymbol{F}_2[\bar{T}^{\wedge}] \to A(T)^*$ is an isomorphism. In particular, $A(T)^*$ is an elementary abelian 2-group of rank $|\bar{T}|$.*

PROOF. Let $\mathcal{M}$ be the set of subgroups of T of index at most 2, so that $|\mathcal{M}| = |\bar{T}|$. Consider the maps

$$\boldsymbol{F}_2[\bar{T}^{\wedge}] \xrightarrow{\bar{u}_T} A(T)^* \xrightarrow{\varphi} \{\pm 1\}^{\mathcal{M}},$$

where φ maps u to $(u(M))_{M \in \mathcal{M}}$. Since

$$\varphi\bar{u}_T(1_T) = -1, \qquad \varphi\bar{u}_T(\lambda)(M) = \begin{cases} -1 & \text{if } \mathrm{Ker}\,\lambda = M \\ +1 & \text{otherwise}, \end{cases}$$

we have that $\varphi\circ\bar{u}_T$ is an isomorphism. Let u be an element of $\operatorname{Ker}\varphi$. Suppose $u\neq 1$ and let M be a maximal subgroup of T such that $u(M)=-1$. Then by Proposition 6.5, the set

$$\{tM\in T/M \mid u(\langle t\rangle M)=u(M)\}$$

is a subgroup of T/M of index at most 2. But this set is a trivial subgroup of T/M by the maximality of M, and so $M\in\mathcal{M}$. This contradicts to the assumption that u is in $\operatorname{Ker}\varphi$. Thus φ is injective. Hence φ and $\bar{u}_T$ are both isomorphisms. The lemma is proved.

THEOREM D. *Let G be a finite group with an abelian normal subgroup Q of odd order and abelian Sylow 2-subgroup T. Put $\bar{T}:=T/\Phi(T)$ and $L:=N_G(T)$. Then the following hold:*

(i) *If $Q=1$, then $A(G)^*\uparrow e_{G,Q}^{2}$ is isomorphic to the additive group $(\boldsymbol{F}_2[\bar{T}^\wedge])^L \cong\boldsymbol{F}_2[\bar{T}^\wedge/L])$ of L-fixed points.*

(ii) *If $Q\neq 1$ and $C_Q(T)\neq 1$, then $A(G)^*\uparrow e_{G,Q}^{2}$ is of order 1.*

(iii) *Assume that $Q\neq 1$ and $C_Q(T)=1$. Define*

$$\mathcal{M} := \{\Phi(T)C \mid T/C \text{ is cyclic and } C_Q(C)\neq 1\},$$
$$K := \bigcap_{M\in\mathcal{M}} M, \qquad \bar{K} := K/\Phi(T).$$

Then $A(G)^*\uparrow e_{G,Q}^{2}$ is isomorphic to $\boldsymbol{F}_2[\bar{K}^\wedge]^L(\cong\boldsymbol{F}_2[\bar{K}^\wedge/L])$.

PROOF. By Theorem C, $A(G)^*\uparrow e_{G,Q}^{2}$ is isomorphic to

$$\{\bar{v}\in A(TQ/Q)^{*G/Q} \mid \bar{v}(S/Q)=1 \text{ if } O^{2'}(S)\neq S\}.$$

We identify $A(TQ/Q)$ and $A(T)$. Then by Sylow's theorem, this group is isomorphic to

$$\{v\in A(T)^{*G} \mid v(S)=1 \text{ if } O^{2'}(SQ)\neq SQ\}. \tag{1}$$

We next prove the following transfer theorem of Burnside type:

$$A(T)^{*G} = A(T)^{*L}, \qquad \text{where } L:=N_G(T). \tag{2}$$

To prove this, let v be an element of $A(T)^{*L}$, so that $v(S^h)=v(S)$ if $S, S^h\leqq T$ and $h\in L$. Assume that $S, S^g\leqq T$ for an element g of G. Then by Sylow's theorem, we have that $g\in N_G(S)L$. (See [**Go68**, Chapter 7, Theorem 1.1].) Thus $v(S^g)=v(S)$. This proves that $v\in A(T)^{*G}$. Hence (2) holds. Next we prove that for a subgroup S of T,

$$O^{2'}(SQ)\neq SQ \quad \text{if and only if} \quad C_Q(S)\neq 1. \tag{3}$$

For $S\leqq T$, define

$$[Q, S] := \langle a^{-1}s^{-1}as \mid a\in Q, s\in S\rangle,$$

so that $O^{2'}(SQ)=S\cdot[Q, S]$. Since Q is an abelian group of odd order on which the 2-group S acts, we have that

$$Q = C_Q(S)\times[Q, S].$$

(See [**Go68**, Chapter 5, Theorem 2.3].) Thus (3) is proved. Hence the group given in (1) is equal to

$$(4) \qquad B := \{v\in A(T)^{*L} \mid v(S)=1 \text{ if } C_A(S)\neq 1\}.$$

Now we consider the group B in (4). Assume first that $Q=1$. In this case, we have that $B=A(T)^{*L}$, and so by Lemma 7.1, it is isomorphic to $\boldsymbol{F}_2[\bar{T}^{\wedge}]$. The canonical mapping of $\bar{T}^{\wedge}$ to $\bar{T}^{\wedge}/L$, the set of L-orbits in $\bar{T}^{\wedge}$, induces an isomorphism $\boldsymbol{F}_2[\bar{T}^{\wedge}]^L\cong\boldsymbol{F}_2[\bar{T}^{\wedge}/L]$. Hence (i) is proved. Assume next that $C_Q(T)\neq 1$. Then clearly the group B is trivial, proving (ii).

Finally we consider the third case where $Q\neq 1$ and $C_Q(T)=1$. Define the families of subgroups as follows:

$$\mathcal{C} := \{C\leqq T\mid C_Q(C)\neq 1\}, \qquad \mathcal{D} := \{D\in\mathcal{C}\mid T/D \text{ is cyclic}\},$$
$$\mathcal{E} := \{E\leqq D\Phi(T)\mid D\in\mathcal{D}\} = \{E\leqq M\mid M\in\mathcal{M}\}.$$

Since a non-cyclic abelian group can not act regularly on an abelian group (cf. [**Go68**, Chapter 3, Theorem 3.3]), we have that any $C\in\mathcal{C}$ is contained in some $D\in\mathcal{D}$. By Corollary 6.6 or Lemma 7.1, we have that

$$v(S) = v(S\cdot\Phi(T)) \qquad \text{for } S\leqq T.$$

Furthermore, for any $v\in A(T)^*$,

$$(5) \quad v(C)=1 \text{ for all } C\in\mathcal{C} \quad \text{if and only if} \quad v(E)=1 \text{ for all } E\in\mathcal{E}.$$

Indeed, for any $E\leqq D\Phi(T)$ with $D\in\mathcal{D}$,

$$v(E) = v(E\cdot\Phi(T)) = v((E\cdot\Phi(T)\cap D)\cdot\Phi(T)) = v(E\cdot\Phi(T)\cap D) = 1,$$

proving (5). Now we put

$$R := \boldsymbol{F}_2[\bar{T}^{\wedge}] = R(\bar{T})/2R(\bar{T}).$$

By Lemma 7.1 and (5), the group B in (4) is isomorphic to $B'\cap T^L$, where

$$(6) \qquad B' := \{\chi\in R \mid \langle\chi_E, 1_E\rangle\equiv 0 \pmod 2 \text{ for } E\in\mathcal{E}\},$$

where χ_E is the restriction of χ to E and $\langle\ \ \rangle$ is the reduction of the ordinary inner product.

We will next show that

$$(7) \qquad i:\ \boldsymbol{F}_2[\bar{K}^{\wedge}] \longrightarrow B';\quad \theta\longmapsto \mathrm{ind}_K^T(\theta)$$

is an isomorphism. Let $\rho_{T/K}$ be the sum of all elements of $\bar{T}^\wedge$ of which kernel contains K. Then $i(\theta)=\rho_{T/K}\cdot\tilde{\theta}$, where $\tilde{\theta}$ is an extension of θ to T(since T is abelian, such an extension surely exists). This show that i is injective. In order to prove the surjectivity of i, take an element χ of B'. Then there is a subset Λ of $\bar{T}^\wedge$ such that

$$\chi=\sum_{\lambda\in\Lambda}\lambda \qquad \text{for some } \Lambda\subseteqq\bar{T}^\wedge.$$

We will prove the following:

$$\text{If } \lambda\in\Lambda,\ \mu\in\bar{T}^\wedge \text{ and } \operatorname{Ker}\mu\in\mathcal{M}, \text{ then } \lambda\mu\in\Lambda. \tag{8}$$

Put $M:=\operatorname{Ker}\mu$, $N:=\operatorname{Ker}\lambda$. Then $M, M\cap N\in\mathcal{E}$. Since $\chi\in B'$, Frobenius reciprocity yields that

$$\langle\chi_{M\cap N}, 1_{M\cap N}\rangle\equiv\langle\chi, 1_{M\cap N}^T\rangle\equiv\langle\chi, 1_T+\lambda+\mu+\lambda\mu\rangle \pmod 2,$$
$$\langle\chi_M, 1_M\rangle\equiv\langle\chi, 1_M^T\rangle\equiv\langle\chi, 1_T+\mu\rangle \pmod 2.$$

Thus $\langle\chi,\lambda\rangle\equiv\langle\chi,\lambda\mu\rangle\equiv 1 \pmod 2$, and so $\lambda\mu\in\Lambda$, proving (8). Now, let X be the subgroup of $\bar{T}^\wedge$ generated by all $\mu\bar{T}^\wedge$ with $\operatorname{Ker}\mu\in\mathcal{M}$, so that $X=(T/K)^\wedge\leqq\bar{T}^\wedge$. (We identified a linear character of T/K as one of T of which kernel contains K, as usual.) Then the statement (8) says that Λ is a union of cosets of X in $\bar{T}^\wedge$, and so χ is a summation of characters of the form $\rho_{T/K}\cdot\lambda\,(=i(\lambda_K))$, $\lambda\in\bar{T}^\wedge$. The surjectivity of (7) is also proved. Hence i in (7) is an isomorphism.

We can now finish the proof. The map i in (7) commutes with conjugation. Hence we have that

$$A(G)^*\uparrow e_{G,Q}^2\cong B'\cap\boldsymbol{F}_2[\bar{T}^\wedge]^L\cong\boldsymbol{F}_2[\bar{K}^\wedge].$$

The theorem is proved.

COROLLARY D1. *Let G be a finite group with elementary abelian Sylow 2-subgroup. Define*

$$\mathcal{Q}:=\{O^2(S)\mid O^{2'}(S)=S\leqq G\}.$$

Then the following hold:

(i) $$A(G)^*=\prod_{(Q)}A(G)^*\uparrow e_{G,Q}^2,$$

where (Q) runs over conjugacy classes such that $Q\in\mathcal{Q}$.

(ii) *Let $Q\in\mathcal{Q}$ and let $\bar{T}$ be a Sylow 2-subgroup of $\bar{N}:=N_G(Q)/Q$. Put $\bar{L}:=N_{\bar{N}}(\bar{T})$ and $\bar{C}:=C_{\bar{T}}(Q/[Q,Q])$. Then*

$$A(G)^*\uparrow e_{G,Q}^2\cong\boldsymbol{F}_2[\bar{C}^\wedge/\bar{L}].$$

PROOF. (i) Let Q be a 2-perfect subgroup of G. By Theorem C, if Q does not belong to $\mathcal{Q}$, then $A(G)^*\uparrow e_{G,Q}^2=1$. Thus the theorem follows directly from Theorem B.

(ii) By Theorem B and Corollary C1, we may assume that Q is an abelian normal subgroup of G. There is a subgroup S of G such that S has no proper normal subgroup of odd index and $O^2(S)=Q$. By Sylow's theorem, we may assume that S/Q is contained in $\bar{T}$. Since $Q=C_Q(\bar{T})\times[Q, \bar{T}]$ and S has no proper normal subgroup of odd order, we have that $C_Q(\bar{T})=1$. In the case where $Q=1$, the conclusion holds clearly. As in Theorem D, let

$$\mathcal{M} := \{M \leqq \bar{T} \mid \bar{T}/M \text{ is of order 2 and } C_Q(M)\neq 1\}, \qquad K := \bigcap_{M\in\mathcal{M}} M.$$

Then we must show that $K=\bar{C}:=C_{\bar{T}}(Q)$. Indeed, clearly each $M\in\mathcal{M}$ contains $\bar{C}$, and so K contains $\bar{C}$. The inverse follows form the easy and well-known fact that Q is generated by subgroups $C_Q(M)$ for subgroups M of $\bar{T}$ of index 2. The corollary is proved.

b. Examples. Finally we give some examples about groups with abelian Sylow 2-subgroups. Define families of subgroups as follows:

$$\mathcal{Q} := \{O^2(S) \mid S=O^{2'}(S)\leqq G\},$$
$$\mathcal{S} := \{S\leqq G \mid S/Q\in \mathrm{Syl}_2(WQ) \text{ for some } Q\in\mathcal{Q}\}.$$

Clearly $O^{2'}(S)=S$ and $O^2(S)\in\mathcal{Q}$ for any $S\in\mathcal{S}$, and for any $Q\in\mathcal{Q}$, there is $S\in\mathcal{S}$ such that $O^2(S)=Q$.

In this section, we use the following notation:

$Q = O^2(S) \in \mathcal{Q}$ with $S\in\mathcal{S}$, $L:=N_G(S)$,
$r(Q) := \log_2|A(G)^* \uparrow e^2_{G,Q}|$,
$d(m) := \#\{k\geqq 1 \mid k \text{ divides } m\}$.
p a prime.
C_m a cyclic group of order m.
E_p^m an elementary abelian p-group of order p^m.
D_{2m} a dihedral group of order $2m$.
$L_2(q)$ $(=PSL(2, \boldsymbol{F}_q))$ the projective special linear group.
A_n, S_n the symmetric and alternating group of degree n.

Note that $L_2(4)\cong L_2(5)\cong A_5$, $L_2(2)\cong S_3$, $L_2(3)\cong A_4$. For the groups $L_2(q)$ and the Janko group J_1, refer [**Wa69**], [**Go68**, 15.1].

EXAMPLE 1. Let $G:=L_2(q)$, $q=2^n\geqq 4$. Then one of the following holds:
(a) $Q=1$. $S\cong E_2^n$ and L is a Frobenius group of order $q(q-1)$. $r(Q)=2$.
(b) $Q\cong C_m$ for $m|(q-1)$, $m\neq 1$ and $L=S\cong D_{2(q-1)}\cdot r(Q)=1$.
(c) $Q\cong C_m$ for $m|(q+1)$, $m\neq 1$, and $L=S\cong D_{2(q+1)}\cdot r(Q)=1$.
(d) $Q=S=L\cong L_2(2^m)$, where $m|n$ and $m\geqq 2$. $r(Q)=1$.
(e) $Q=S=L\cong A_5$. $r(Q)=1$. This case occurs only when $q\equiv\pm 1 \pmod 5$

and $q>4$.

In every case, Q up to conjugate is uniquely determined by its order. Hence we conclude that

$$\begin{aligned}\log_2|A(G)^*| &= 2+(d(q-1)-1)+(d(q+1)-1)+(d(n)-1)+\alpha \\ &= d(q-1)+d(q+1)+d(n)-1+\alpha,\end{aligned}$$

where $\alpha=1$ if $q\equiv\pm1 \pmod 5$ and $q>4$, and $=0$ otherwise. (Degenerated case: $\log_2|A(L_2(2))^*|=3$.) The value 5 in the case $q=4$ equals to the result in [**Di79**, 1.7].

EXAMPLE 2. Let $G:=L_2(q)$, $q=p^n\equiv3 \pmod 8$, $q\geqq5$. Then one of the following holds:

(a) $Q=1$. $S\cong E_2^2$ and $L\cong A_4$. $r(Q)=2$.
(b) $Q\cong C_m$ for $m|(q-1)/2$, $m\neq1$. $S=L\cong D_{q-1}$. $r(Q)=1$.
(c) $Q\cong C_m$ for $m|(q+1)/4$, $m\neq1$. $S=L\cong C_2\times D_{q+1}$. $r(Q)=2$.
(d) $Q=S=L\cong L_2(p^m)$ where $m|n$ and $p^m\geqq5$. $r(Q)=1$.
(e) $Q=S=L\cong A_5$. $r(Q)=1$. This case occurs when $q\equiv\pm1 \pmod 5$.

Hence we conclude that

$$\begin{aligned}\log_2|A(G)^*| &= 2+(d((q-1)/2)-1)+2(d((q+1)/4)-1)+(d(n)-\delta_{p,3})+\alpha \\ &= d((q-1)/2)+2d((q+1)/4)+d(n)-1-\delta_{p,3}+\alpha,\end{aligned}$$

where $\alpha=1$ if $q\equiv\pm1 \pmod 5$, and $=0$ otherwise.

EXAMPLE 3. Let $G:=L_2(q)$, $q=p^n\equiv5 \pmod 8$. Then one of the following holds:

(a) $Q=1$. $S\cong E_2^2$ and $L\cong A_4$. $r(Q)=2$.
(b) $Q\cong C_m$ for $m|(q-1)/4$, $m\neq1$. $S=L\cong C_2\times D_{q-1}$. $r(Q)=2$.
(c) $Q\cong C_m$ for $m|(q+1)/2$, $m\neq1$. $S=L\cong D_{q+1}$. $r(Q)=1$.
(d) $Q\cong E_p^m$ for $1\leqq m\leqq n$. S is a Frobenius group of order $2p^m$. $r(Q)=1$.
(e) $Q=S=L\cong L_2(p^m)$ where $m|n$. $r(Q)=1$.
(f) $Q=S=L\cong A_5$. $r(Q)=1$. This case occurs when $q\equiv\pm1 \pmod 5$.

Hence we conclude that

$$\begin{aligned}\log_2|A(G)^*| &= 2+2(d((q-1)/4)-1)+(d((q+1)/2)-1)+\beta_p+d(n)+\alpha \\ &= 2d((q-1)/4)+d((q+1)/2)+d(n)-1+\beta_p+\alpha,\end{aligned}$$

where β_p is the number of G-conjugacy classes of nontrivial p-subgroups of G and α is defined as in Example 1. Again we have that $\log_2|A(L_2(5))^*|=5$.

EXAMPLE 4. Let $G=J_1$ (the Janko group of order $2^3\cdot3\cdot5\cdot7\cdot11\cdot19$). Then one of the following holds:

(a) $Q=1$. $S\cong E_2^3$ and L is a Frobenius group of order 168. $r(Q)=2$.
(b) $Q\cong C_m$, $m=3, 5, 15$. $N_G(Q)\cong D_6\times D_{10}$. $r(Q)=2, 2, 1$, respectively.

(c) $Q \cong C_m$, $m=7, 11, 19$. $S \cong D_{2m}$. $r(Q)=1$.
(d) $Q \cong A_5$. $S = L \cong C_2 \times A_5$. $r(Q)=2$.
(e) $Q = S = L \cong L_2(11)$. $r(Q)=1$. Thus we have that

$$\log_2 |A(G)^*| = 13.$$

Simple groups with abelian Sylow 2-subgroups are isomorphic to one of $L_2(q)$, where $q=2^n$ or $q \cong \pm 3 \pmod 8$, J_1 and ${}^2G_2(q)$ (Ree group, where $q=3^{2n+1}>3$ ([**Be70**], [**Wa69**])). It is not so easy to determine the order of the unit group for a Ree group because of the complexity of the structure of Sylow 3-subgroups.

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