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On a pseudoconvex domain spread over a complex projective space induced from a complex Banach space with a Schauder basis

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Introduction.

Oka [18] solved the Levi problem, which is the problem to ask if a pseudoconvex domain is a domain of holomorphy, in a domain spread over C^n. At the same time, Bremermann [1] and Norguet [16] solved this problem in C^n. Their results were extended to a domain spread over the complex projective space $P_n(C)$ of dimension n by Fujita [4], Kiselman [9] and Takeuchi [22].

In the last fifteen years, the Levi problem has been discussed in various infinite dimensional spaces. Gruman [5] and Gruman and Kiselman [6] solved this problem in a complex Banach space E with a Schauder basis, and Hervier [7] extended this result to a domain spread over E. Dineen [2] and Gruman [5] solved this problem in an infinite dimensional vector space E with the finite open topology, and Kajiwara [8] extended this result to a domain of the complex projective space induced from E.

The aim of this paper is to prove the following two theorems having their sources in the Levi problem and in the imbedding theorem of a Stein manifold.

THEOREM 1. *Let E be a complex Banach space with a Schauder basis, and $P(E)$ the complex projective space induced from E. Let (Ω, ϕ) be a domain spread over the complex projective space $P(E)$. Suppose that Ω is not homeomorphic to $P(E)$ through ϕ. Then the following conditions are equivalent:*

(1) *Ω is pseudoconvex.*

(2) *For every finite dimensional linear subspace F of E and the projective space $P(F)$ induced from F, the inverse image $\phi^{-1}(P(F))$ of $P(F)$ by ϕ is a Stein manifold.*

(3) *Ω is a domain of holomorphy.*

(4) *Ω is a domain of existence.*

THEOREM 2. *Let H be a separable complex Hilbert space, $\{e_j\}_{j=1}^{\infty}$ an ortho-*

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normal basis of H, *and* $\boldsymbol{P}(H)$ *the complex projective space induced from* H. *Let* (Ω, ϕ) *be a pseudoconvex domain spread over* $\boldsymbol{P}(H)$. *Suppose that* Ω *is not homeomorphic to* $\boldsymbol{P}(H)$ *through* ϕ. *We denote by* H_n *the linear span of the set* $\{e_1, e_2, \cdots, e_n\}$ *and denote by* $\boldsymbol{P}(H_n)$ *the complex projective space induced from* H_n. *Then there exists an injective holomorphic mapping* f *of* Ω *into* H *such that for every positive integer* n *the restriction mapping* $f|\phi^{-1}(\boldsymbol{P}(H_n))$ *of* f *on* $\phi^{-1}(\boldsymbol{P}(H_n))$ *is a regular and proper holomorphic mapping of* $\phi^{-1}(\boldsymbol{P}(H_n))$ *into* H.

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1. Banach complex manifolds and domains spread over Banach complex manifolds.

Let E and F be complex Banach spaces, and U an open subset of E. A mapping $f: U \rightarrow F$ is said to be *holomorphic in* U if f is continuous in U and if, for any $(a, b) \in U \times (E-\{0\})$ and for any continuous linear functional $\alpha \in F'$, the composite mapping $\lambda \rightarrow \alpha \circ f(a+\lambda b)$ $(\lambda \in \boldsymbol{C})$ is holomorphic where it is defined. A function $p: U \rightarrow [-\infty, +\infty)$ is said to be *plurisubharmonic* if p is upper-semicontinuous in U and if, for any point (a, b) of $U \times (E-\{0\})$, the function $\lambda \rightarrow p(a+\lambda b)$ $(\lambda \in \boldsymbol{C})$ is subharmonic where it is defined.

A Hausdorff space M is called *a complex manifold modeled on a complex Banach space* E if there exists a family $\mathfrak{F} = \{(U_i, \phi_i); i \in I\}$ of pairs (U_i, ϕ_i) of open sets U_i of M and homeomorphisms ϕ_i of open sets U_i onto open sets of E satisfying the following conditions:

(1) For any elements i, j of I with $U_i \cap U_j \neq \emptyset$, the mapping $\phi_i \circ \phi_j^{-1}$: $\phi_j(U_i \cap U_j) \rightarrow \phi_i(U_i \cap U_j)$ between open sets in E is holomorphic.

(2) $\bigcup_{i \in I} U_i = M$.

$\mathfrak{F}$ is called *the atlas of* M. An element of $\mathfrak{F}$ is called *a chart of* M.

Let M and N be complex manifolds with atlases $\{(U_i, \phi_i); i \in I\}$ and $\{(U'_\alpha, \phi'_\alpha); \alpha \in A\}$ respectively. Then a mapping $f: M \rightarrow N$ is said to be *holomorphic* if, for any $i \in I$ and $\alpha \in A$ with $f(U_i) \cap U'_\alpha \neq \emptyset$, the mapping $\phi'_\alpha \circ f \circ \phi_i^{-1}$ is holomorphic. Particularly, if $N = \boldsymbol{C}$, f is called *a holomorphic function.* We denote by $H(M)$ the family of all holomorphic functions in M. A function $p: M \rightarrow [-\infty, +\infty)$ is said to be *plurisubharmonic* if, for any $i \in I$, the function $f \circ \phi_i^{-1}$ is plurisubharmonic.

We consider subsets Δ_1 and Δ_2 in $\boldsymbol{C}^2$ defined by

$$\Delta_1 = \{(z_1, z_2) \in \boldsymbol{C}^2;\ |z_1| = 1,\ z_2 \in [0, 1]\} \cup \{|z_1| \leqq 1,\ z_2 = 0\}, \tag{1.1}$$

$$\Delta_2 = \{|z_1| \leqq 1,\ z_2 \in [0, 1]\}. \tag{1.2}$$

A complex manifold M is said to *satisfy the Kontinuitätssatz* if any holomorphic mapping of a neighborhood of Δ_1 into M is extended holomorphically to Δ_2.

Let M be a complex manifold. If there exists a local biholomorphic mapping ϕ of a complex manifold Ω into M, (Ω, ϕ) is called *a region spread over* M. Moreover, if Ω is connected, (Ω, ϕ) is called *a domain spread over* M.

Let (Ω, ϕ) and (Ω', ϕ') be regions spread over M. If a holomorphic mapping λ of Ω into Ω' satisfies $\phi=\phi'\circ\lambda$, λ is called *a mapping of* (Ω, ϕ) *into* (Ω', ϕ'). Let (Ω, ϕ) be a region spread over a complex manifold M and let $\mathfrak{F}\subset H(\Omega)$. If (Ω', ϕ') is a region spread over M then a mapping λ of (Ω, ϕ) into (Ω', ϕ') is said to be *an* $\mathfrak{F}$*-extension of* Ω if for each $f\in\mathfrak{F}$ there exists a unique $f'\in H(\Omega')$ such that $f'\circ\lambda=f$. A mapping λ of (Ω, ϕ) into (Ω', ϕ') is said to be *a holomorphic extension of* Ω if λ is an $H(\Omega)$-extension of Ω. Ω is said to be *an* $\mathfrak{F}$*-domain of holomorphy* if each $\mathfrak{F}$-extension of Ω is an isomorphism. Ω is said to be *a domain of holomorphy* if Ω is an $H(\Omega)$-domain of holomorphy. Ω is said to be *a domain of existence* if there exists $f\in H(\Omega)$ such that Ω is an $\{f\}$-domain of holomorphy.

Let E be a complex Banach space with a norm $\|\cdot\|$ and let (Ω, ϕ) be a region spread over E. For a point z of E and for a positive number ε, we define the open ball $B(z, \varepsilon)$ by

$$(1.3) \qquad B(z, \varepsilon)=\{w\in E \ ; \ \|w-z\|<\varepsilon\}\,.$$

For any point x of Ω, there exists a positive number $\varepsilon(x)$ such that, for any positive number ε with $\varepsilon<\varepsilon(x)$, there exists uniquely an open neighborhood $\Delta(x, \varepsilon)$ of x which is mapped by ϕ homeomorphically onto the open ball $B(\phi(x), \varepsilon)$. The open neighborhood $\Delta(x, \varepsilon)$ is called *the open ball in* Ω *with center* x *and with radius* ε. We define *the boundary distance function* $d_\Omega(x)$ *on* Ω by

$$(1.4) \qquad d_\Omega(x)=\sup\{x\ ;\ \textit{the open ball } \Delta(x, \varepsilon) \textit{ exists}\}\,.$$

Let a and b be points of Ω. By *a line segment* $[a, b]$ in Ω we mean a set in Ω containing the points a and b and homeomorphic under ϕ to the line segment $[\phi(a), \phi(b)]$ in E. By *a polygonal line* $[x_0, x_1, \cdots, x_n]$ in Ω we mean a finite union of line segments of the form $[x_{j-1}, x_j]$ with $j=1, \cdots, n$.

REMARK 1.1. Let x and y be two points which belong to a connected component of Ω. Since there exists a polygonal line $[x_0, x_1, \cdots, x_n]$ with $x_0=x$ and with $x_n=y$, there exists a finite dimensional linear subspace F of E such that the set $\{x, y\}$ is contained in a connected component of the inverse image $\phi^{-1}(F)$ of F by ϕ.

2. Complex projective spaces induced from complex Banach spaces.

In this section we first give some properties of a complex projective space induced from a complex Banach space. Then we give the definition of pseudoconvexity of a domain spread over the complex projective space, and prove some lemmas with respect to pseudoconvexity.

Let E be a complex Banach space with the norm $\|\cdot\|$. Let z and z' be points in $E-\{0\}$. z and z' are said to be *equivalent* if there exists a complex number $\lambda\in\boldsymbol{C}-\{0\}$ such that $z'=\lambda z$. The quotient space $\boldsymbol{P}(E)$ of $E-\{0\}$ by this equivalence relation is called *the complex projective space induced from* E. We denote by Q the quotient map of $E-\{0\}$ onto $\boldsymbol{P}(E)$. For any $\xi\in E-\{0\}$, we denote by $[\xi]$ the equivalence class of ξ. Then we have $Q(\xi)=[\xi]$.

Let E' be the complex Banach space of continuous linear functionals on E. We set

$$S=\{(f, a)\in E'\times E;\ f(a)\neq 0\}. \tag{2.1}$$

For each $f\in E'-\{0\}$, we consider a hyperplane $E(f)$ of E and an open subset $U(f)$ of $\boldsymbol{P}(E)$ defined by

$$E(f)=\{\xi\in E;\ f(\xi)=0\}, \tag{2.2}$$

$$U(f)=\{[\xi]\in\boldsymbol{P}(E);\ f(\xi)\neq 0\} \tag{2.3}$$

respectively. For every $(f, a)\in S$, we define a homeomorphism $\phi_{(f,a)}$ of $U(f)$ onto $E(f)$ by

$$\phi_{(f,a)}([\xi])=(1/f(\xi))\xi-(1/f(a))a$$

for every $[\xi]\in U(f)$. The family $\{U(f), \phi_{(f,a)}\}_{(f,a)\in S}$ defines the complex structure of the projective space $\boldsymbol{P}(E)$.

Let $S(E)$ be the unit sphere in E. Then the topological space $\boldsymbol{P}(E)$ is a quotient space of $S(E)$. The topology of $S(E)$ as a subspace of E induces the topology on the quotient space $\boldsymbol{P}(E)$. $S(E)$ is a principal fibre bundle over $\boldsymbol{P}(E)$ with circle group. Since $S(E)$ is a subspace of the metric space E, the metric on $S(E)$ induces a metric $d(\ ,\ )$ on $\boldsymbol{P}(E)$ by

$$d(p, p')=\inf\{\|z-z'\|;\ z\in Q^{-1}(p)\cap S(E),\ z'\in Q^{-1}(p')\cap S(E)\} \tag{2.4}$$

for any points p and p' of $\boldsymbol{P}(E)$. Since E is complete and $S(E)$ is closed, $S(E)$ is a complete metric space. From the compactness of the fibre of $S(E)$, it follows that $\boldsymbol{P}(E)$ is also complete.

Let (Ω, ϕ) be a domain spread over the complex projective space $\boldsymbol{P}(E)$ induced from E. $E-\{0\}$ is the total space of the holomorphic principal bundle over $\boldsymbol{P}(E)$ with the complex multiplicative group $\boldsymbol{C}^*$. We consider the fibre product X of Ω and $E-\{0\}$ given by

(2.5) $$X=\{(z,\ w)\in\Omega\times(E-\{0\});\ \phi(z)=Q(w)\}.$$

We denote by $\tilde{\phi}$ and $\tilde{Q}$ projections of the fibre product X into $E-\{0\}$ and into Ω respectively. Then $(X,\ \tilde{\phi})$ is a domain spread over E.

For any $(z,\ w)\in X$ and for any $\lambda\in\boldsymbol{C}^{*}$, we set

(2.6) $$\lambda\cdot(z,\ w)=(z,\ \lambda w).$$

Then points $\lambda\cdot(z,\ w)$ of $\Omega\times(E-\{0\})$ belong to X for all $(z,\ w)\in X$ and for all $\lambda\in\boldsymbol{C}^{*}$. The mapping $(\lambda,\ x)\rightarrow\lambda\cdot x$ is a holomorphic mapping of $\boldsymbol{C}^{*}\times X$ onto X. Then Ω is the quotient space of X by this $\boldsymbol{C}^{*}$-action and $\tilde{Q}$ is the quotient map of X onto Ω. X is the total space of a holomorphic principal bundle over Ω with the complex multiplicative group $\boldsymbol{C}^{*}$. We have the following commutative diagram:

(2.7) $$\begin{tikzcd} X \arrow[r, "\tilde{Q}"] \arrow[d, "\tilde{\phi}"] & \Omega \arrow[d, "\phi"] \\ E-\{0\} \arrow[r, "Q"] & \boldsymbol{P}(E). \end{tikzcd}$$

Let f be a holomorphic function in X. We set

(2.8) $$\tilde{f}(x)=(1/2\pi)\int_{0}^{2\pi}f(e^{i\theta}\cdot x)d\theta$$

for every $x\in X$. Then $\tilde{f}$ is a holomorphic function in X and we have

(2.9) $$\tilde{f}(e^{i\eta}\cdot x)=\tilde{f}(x)$$

for every $\eta\in[0,\ 2\pi)$ and for every $x\in X$. By the identity theorem of a complex variable holomorphic function theory, we have

(2.10) $$\tilde{f}(\lambda\cdot x)=\tilde{f}(x)$$

for every $\lambda\in\boldsymbol{C}^{*}$. Therefore $\tilde{f}$ is constant on $\tilde{Q}^{-1}(z)$ for every $z\in\Omega$. We define a holomorphic function f^{*} in Ω by

(2.11) $$f^{*}(z)=\tilde{f}(\tilde{Q}^{-1}(z))$$

for every $z\in\Omega$. We have

(2.12) $$(g\circ\tilde{Q})^{*}=g$$

for every $g\in H(\Omega)$. Hence we obtain the following lemma.

LEMMA 2.1. *For any $f\in H(X)$, a holomorphic function $\tilde{f}$ in X defined by* (2.8) *is constant on $\tilde{Q}^{-1}(z)$ for every $z\in\Omega$. Thus we can define a holomorphic function f^{*} in Ω by* (2.11).

Let F be a closed linear subspace of E. We denote by X_F and by Ω_F regions spread over F and spread over the complex projective space $\boldsymbol{P}(F)$ induced from F, respectively, defined by

$$X_F = \tilde{\phi}^{-1}(F-\{0\}), \tag{2.13}$$

$$\Omega_F = \phi^{-1}(\boldsymbol{P}(F)). \tag{2.14}$$

X_F is a holomorphic principal bundle over Ω_F with the complex multiplicative group $\boldsymbol{C}^*$. We have the following commutative diagram induced from the commutative diagram (2.7):

$$\begin{tikzcd} X_F \arrow[r, "\tilde{Q}|X_F"] \arrow[d, "\tilde{\phi}|X_F"] & \Omega_F \arrow[d, "\phi|\Omega_F"] \\ F-\{0\} \arrow[r, "Q|(F-\{0\})"'] & \boldsymbol{P}(F). \end{tikzcd} \tag{2.15}$$

Let (Ω, ϕ) be a region spread over a complex projective space $\boldsymbol{P}(E)$ induced from a complex Banach space E. Then the region (Ω, ϕ) is said to be *pseudo-convex* if, for every $f \in E'-\{0\}$ and for the open set $U(f)$, defined by (2, 3), of $\boldsymbol{P}(E)$, the open set $\phi^{-1}(U(f))$ of Ω satisfies the Kontinuitätssatz.

LEMMA 2.2. *Let E be a complex Banach space and (Ω, ϕ) be a domain spread over the complex projective space $\boldsymbol{P}(E)$. Suppose that Ω is not homeomorphic to $\boldsymbol{P}(E)$ through ϕ. Then for any finite dimensional linear subspace F of E and for any connected component V_F of Ω_F, there exist a finite dimensional linear subspace G of E and a connected component V_G of Ω_G satisfying the following conditions:*

(1) *V_F is a closed complex submanifold of V_G.*

(2) *V_G is not homeomorphic to $\boldsymbol{P}(G)$ through $\phi|V_G$.*

PROOF. By Remark 1.1 and by the commutative diagram (2.15), there exist a finite dimensional linear subspace F_0 of E and a connected component V_{F_0} of Ω_{F_0} such that V_{F_0} is not homeomorphic to $\boldsymbol{P}(F_0)$ through $\phi|V_{F_0}$. We take a point z of V_F and a point w of V_{F_0}. By Remark 1.1 and by the commutative diagram (2.15), there exists a finite dimensional subspace F_1 such that a connected component V_{F_1} of Ω_{F_1} contains the set $\{z, w\}$. Let G be the complex vector space spanned by all elements of the union $F \cup F_0 \cup F_1$. Then $\boldsymbol{P}(F)$ and $\boldsymbol{P}(F_0)$ are closed complex submanifolds of $\boldsymbol{P}(G)$. We denote by V_G the connected component of Ω_G containing the set $\{z, w\}$. Since $(V_G, \phi|V_G)$ is a domain spread over $\boldsymbol{P}(G)$, both V_F and V_{F_0} are closed complex submanifolds of V_G. Then V_G satisfies the required conditions (1) and (2). This completes the proof.

LEMMA 2.3. *Suppose that Ω is not homeomorphic to $\boldsymbol{P}(E)$ through ϕ and that Ω is pseudoconvex. Then, for any finite dimensional linear subspace F of E, Ω_F is a Stein manifold. X satisfies the Kontinuitätssatz.*

PROOF. Let F be a finite dimensional linear subspace of E. Let V_F be any component of Ω_F. By Lemma 2.2 there exists a finite dimensional subspace G of E and a component V_G of Ω_G satisfying the conditions (1) and (2) in Lemma 2.2. Since Ω is pseudoconvex, V_G is also pseudoconvex. By Fujita [4], Kiselman [9] and Takeuchi [22], the pseudoconvex domain V_G spread over $\boldsymbol{P}(G)$ is a Stein manifold. Since V_F is a closed complex submanifold of the Stein manifold V_G, V_F is a Stein manifold. Thus Ω_F is a Stein manifold. X_F is the total space of a holomorphic principal bundle over the Stein manifold Ω_F with the complex multiplicative group $\boldsymbol{C}^*$. Therefore X_F is a Stein manifold by Matsushima and Morimoto [12]. Since $(X, \tilde{\phi})$ is a domain spread over E, X satisfies the Kontinuitätssatz by Noverraz [17]. This completes the proof.

LEMMA 2.4. *With the assumption of Lemma 2.2 the following conditions are equivalent:*

(1) *Ω is pseudoconvex.*

(2) *Ω_F is a Stein manifold for every finite dimensional linear subspace F of E.*

PROOF. It follows from Lemma 2.3 that (1) implies (2).

We will show that (2) implies (1). Let f be an element of $E'-\{0\}$. By the assumption, for every finite dimensional linear subspace F of E with $\dim_C F \geqq 2$ and $F \not\subset \{f=0\}$, Ω_F is a Stein manifold. We set $H=\phi^{-1}(\{[\xi] \in \boldsymbol{P}(F);\ f(\xi)=0\})$. Since H is a hypersurface of Ω_F and $\Omega_F \cap \phi^{-1}(U(f))=\Omega_F \setminus H$, $\Omega_F \cap \phi^{-1}(U(f))$ is a Stein manifold. $\phi^{-1}(U(f))$ and $\Omega_F \cap \phi^{-1}(U(f))$ are identified with regions spread over the Banach space $\{f=0\}$ and spread over the finite dimensional subspace $\{f=0\} \cap F$ of $\{f=0\}$ respectively. Therefore by Noverraz [17] the domain $\phi^{-1}(U(f))$ satisfies the Kontinuitätssatz. Thus Ω is pseudoconvex. This completes the proof.

3. Some properties of the fibre product X.

In this section we will research some properties of the fibre product X, defined in the preceding section, of Ω and $E-\{0\}$ for a complex Banach space E with a Schauder basis and for a pseudoconvex domain (Ω, ϕ) spread over the complex projective space $\boldsymbol{P}(E)$.

Let E be a complex Banach space with the norm $\|\cdot\|$ and a Schauder basis

$\{e_j\}_{j=1}^{\infty}$. Let (Ω, ϕ) be a pseudoconvex domain, which is not homeomorphic to $\boldsymbol{P}(E)$ through ϕ, spread over the complex projective space $\boldsymbol{P}(E)$.

Since Ω is pseudoconvex, by Lemma 2.3 X satisfies the Kontinuitätssatz. By Noverraz [**17**], we have the following Lemma 3.1.

LEMMA 3.1. *$-\log d_X$ is a continuous plurisubharmonic function in X where d_X is the boundary distance function on X. For any finite dimensional linear subspace F of E, $\tilde{\phi}^{-1}(F)$ is a Stein manifold.*

We can choose a Schauder basis $\{e_j\}_{j=1}^{\infty}$ of E such that the intersection of the image of $\tilde{\phi}$ and the linear space $\{\lambda e_1; \lambda \in \boldsymbol{C}\}$ is nonempty. For every $\xi \in E$, ξ can be represented in a unique way

$$\xi = \sum_{n=1}^{\infty} \xi_n e_n . \tag{3.1}$$

We denote by E_n the linear span of the set $\{e_1, e_2, \cdots, e_n\}$, and by u_n the mapping of E onto E_n defined by

$$u_n(\xi) = \sum_{j=1}^{n} \xi_j e_j . \tag{3.2}$$

We denote by μ_n a continuous linear functional of E defined by

$$\mu_n(\xi) = \xi_n \tag{3.3}$$

for every $\xi = \sum_{j=1}^{\infty} \xi_j e_j$.

LEMMA 3.2. *There exist a norm $|||\cdot|||$ of E and positive constants c_1 and c_2 satisfying the following conditions:*

(1) $c_1|\xi| \leqq |||\xi||| \leqq c_2|\xi|$ *for every* $\xi \in E$.

(2) $|||u_n(\xi)||| \leqq |||\xi|||$ *for every positive integer* n.

The proof of Lemma 3.2 is in Singer [**21**]. The condition (1) of Lemma 3.2 implies that the Banach space $(E, |||\ |||)$ with the norm $|||\ |||$ is equivalent to the Banach space E with the original norm $\|\ \|$. Therefore we may assume that the norm of E satisfies the condition

$$\|u_n(\xi)\| \leqq \|\xi\| \tag{3.4}$$

for every positive integer n.

Let x_0 be a point of X with $\tilde{\phi}(x_0) \in E_1$. We may assume that the norm $\|\ \|$ of E is chosen such that $d_X(x_0) \geqq 1$. For every n we set

$$X_n = \tilde{\phi}^{-1}(E_n), \tag{3.5}$$

$$A_n = \{x \in X ; \sup_{m \geqq n} \|u_m \circ \tilde{\phi}(x) - \tilde{\phi}(x)\| < d_X(x)\}, \tag{3.6}$$

$$v_n(x) = (\tilde{\phi} | \Delta(x, d_X(x)))^{-1} \circ u_n \circ \tilde{\phi}(x) \tag{3.7}$$

for every $x\in A_n$. Then $\sup_{m\geq n}\|u_m\circ\tilde{\phi}(x)-\tilde{\phi}(x)\|$ is continuous on X, and A_n is an open subset of X. v_n is a holomorphic mapping of A_n into X_n for every n.

Let (Y,ψ) be a region spread over a complex Banach space F. Then we use the notation $d_Y(A)=\inf\{d_Y(x)\,;\,x\in A\}$ where A is a subset of Y.

The proof of the following lemma is in Lemma 54.5 of Mujica [**13**].

LEMMA 3.3. *There exist two increasing sequences* $\{B_n\}_{n=1}^{\infty}$ *and* $\{C_n\}_{n=1}^{\infty}$ *of open sets* B_n *and* C_n *of* X *such that*

(a) $\{x_0\}\subset C_n\subset B_n\subset A_n$ *for every* $n\geqq 1$, $X=\bigcup_{n=1}^{\infty}B_n=\bigcup_{n=1}^{\infty}C_n$.

(b) $d_{A_n}(B_n)\geqq 2^{-n}$ *and* $B_m\cap X_n$ *is relatively compact in* $A_m\cap X_n$ *for every* m, $n\geqq 1$.

(c) $d_{C_{m+1}}(C_m)\geqq 2^{-m-1}$ *and* $v_n(C_m)\subset B_m\cap X_n$ *for every* $m\geqq 1$ *and every* $n\geqq m$.

For every $x\in X$, we define the sets $V(x)$ and $S(x)$ by

$$V(x)=\{\lambda\cdot x\,;\,\lambda\in \boldsymbol{C}^*\},\tag{3.8}$$

$$S(x)=\{e^{i\theta}\cdot x\,;\,0\leqq\theta\leqq 2\pi\}.\tag{3.9}$$

Let K be a compact subset of a Stein manifold S. We use the notation

$$K(S)=\{x\in S\,;\,|f(x)|\leqq\sup_{y\in K}|f(y)|\quad \textit{for all } f\in H(S)\}\tag{3.10}$$

The set $K(S)$ is called *the holomorphically convex hull of* K in the Stein manifold S. If $K(S)=K$, K is said to be *Runge in* S. Let S_1 be a Stein manifold and S_2 be a Stein open subset of S_1. S_2 is said to be *Runge relative to* S_1 if, for any compact subset K of S_2, $K(S_1)$ is a compact subset in S_2.

We denote by K_n the holomorphically convex hull of the topological closure of the set $B_n\cap X_{n+1}$ in the Stein manifold X_{n+1}. Since X_{n+1} is a Stein manifold, K_n is a compact subset of X_{n+1} and Runge in X_{n+1}. On the other hand $\sup_{m\geq n}\|u_m\circ\tilde{\phi}(x)-\tilde{\phi}(x)\|$ is continuous in X, and $\sup_{m\geq n}\log\|u_m\tilde{\phi}(x)-\tilde{\phi}(x)\|$ $-\log d_X(x)$ is a continuous plurisubharmonic function of X into $[-\infty,\infty)$. Therefore by Narasimhan [**15**], $A_n\cap X_{n+1}$ is Runge relative to X_{n+1} and K_n is compact in $A_n\cap X_{n+1}$.

LEMMA 3.4. *Let* $\{c_n\}_{n=1}^{\infty}$ *be a sequence of points of* X *such that* $c_n\in X_n$, $c_n\notin X_{n-1}$ *and* $V(c_n)\subset X\setminus K_n$. *Then, for any sequence* $\{\lambda_n\}_{n=1}^{\infty}$ *of positive numbers, there exists a sequence* $\{f_n\}_{n=1}^{\infty}$ *of holomorphic functions* f_n *in* X_n *such that*

$$f_{n+1}|X_n=f_n\,,\tag{3.11}$$

$$|f_{n+1}(x)-f_n\circ v_n(x)|<1/2^n\tag{3.12}$$

for any $x\in K_n$, *and*

$$\mathrm{Re}\,f_n(x)\geqq\lambda_n\tag{3.13}$$

for any $x \in S(c_n)$ *where* $\mathrm{Re} f_n$ *represents the real part of* f_n.

PROOF. We will show this lemma by induction with respect to n. We set $f_1(x)=\lambda_1$ for every $x \in X$. Then f_1 satisfies (3.13). We assume that there exist holomorphic functions f_k in X_k $(1 \leqq k \leqq n)$ with (3.11), (3.12) and (3.13). We set

$$g(x) = f_n \circ v_n(x) \tag{3.14}$$

for every $x \in X_{n+1} \cap A_n$. Closed subsets $K_n \cup X_n$ and $(X_{n+1} \setminus A_n)$ are mutually disjoint because K_n is a compact subset of $X_{n+1} \cap A_n$. Therefore there exists a C^∞-function η in X_{n+1} such that $\eta=1$ on a neighborhood of $K_n \cup X_n$, and that $\eta=0$ on a neighborhood of $(X_{n+1} \setminus A_n)$.

We consider a $\bar\partial$-equation on X_{n+1}:

$$\bar\partial v = (\mu_{n+1} \circ \tilde\phi(x))^{-1} g \bar\partial \eta \tag{3.15}$$

where μ_j are defined in (3.3). Since X_{n+1} is a Stein manifold, and since the right hand side of (3.15) is $\bar\partial$-closed, there exists a C^∞-function v on X_{n+1} satisfying (3.15). We set

$$h(x) = \eta(x) g(x) - (\mu_{n+1} \circ \tilde\phi(x)) v(x) \tag{3.16}$$

for every $x \in X_{n+1}$. Then h is holomorphic in X_{n+1} and satisfies $h | X_n = f_n$. Since v is holomorphic in a neighborhood of a Runge compact subset K_n of X_{n+1}, by Oka-Weil theorem there exists a holomorphic function w in X_{n+1} such that

$$|v(x) - w(x)| < 1/(2^{n+1} M) \tag{3.17}$$

for every $x \in K_n$ where $M = \sup\{|\mu_{n+1} \circ \tilde\phi(x)|\,;\, x \in K_n\}$. We set

$$F(x) = h(x) + (\mu_{n+1} \circ \tilde\phi(x)) w(x) \tag{3.18}$$

for every $x \in X_{n+1}$. Then we have

$$|F(x) - f_n \circ v_n(x)| < 1/2^{n+1} \tag{3.19}$$

for every $x \in K_n$.

We set

$$T = S(c_{n+1}) \cup K_n\,, \tag{3.20}$$

$$V_{n+1} = V(c_{n+1})\,. \tag{3.21}$$

We denote by $\hat T$ the holomorphically convex hull of T in X_{n+1}. Since X_{n+1} is Stein, $\hat T$ is compact in X_{n+1}.

We will show that $\hat T \subset V_{n+1} \cup K_n$. Let x be a point of $X_{n+1} \setminus (V_{n+1} \cup K_n)$. Since X_{n+1} is a Stein manifold, by Oka-Cartan theorem there exists a holomorphic function s in X_{n+1} with $s=0$ on V_{n+1} and with $s(x)=1$. Since K_n is a Runge compact subset of X_{n+1}, there exists a holomorphic function t in X_{n+1}, such that $|t(x)|>1$ and $\|t\|_{K_n}<1/(\|s\|_{K_n}+1)$ where $\|s\|_{K_n}$ and $\|t\|_{K_n}$ represent supremums of functions $|s(\cdot)|$ and $|t(\cdot)|$, respectively, on the compact set K_n. Then we have $|s(x)t(x)|>1$ and $\sup\{|s(y)t(y)|\,;\, y\in T\}<1$. Therefore x cannot belong to $\hat{T}$. Thus we have $\hat{T}\subset V_{n+1}\cup K_n$.

Since by the assumption $V_{n+1}\cap K_n=\emptyset$, it follows that $(\hat{T}\cap V_{n+1})\cap K_n=\emptyset$ and $\hat{T}=(\hat{T}\cap V_{n+1})\cup K_n$.

Since $\hat{T}$ is a Runge compact subset of X_{n+1}, there exist Stein neighborhoods Δ_1 and Δ_2 of $(\hat{T}\cap V_{n+1})$ and of K_n, respectively, in X_{n+1} with $\Delta_1\cap\Delta_2=\emptyset$. We set $L=\sup\{|F(x)|\,;\, x\in S(c_{n+1})\}$. We define a holomorphic function α in a Stein manifold $\Delta_1\cap V_{n+1}$ by

$$\alpha(\lambda\cdot c_{n+1})=(L+\lambda_{n+1}+1)/\lambda\mu_{n+1}\circ\tilde{\phi}(c_{n+1}) \tag{3.22}$$

for every $\lambda\cdot c_{n+1}\in\Delta_1\cap V_{n+1}$ $(\lambda\in\boldsymbol{C}-\{0\})$. Since $\Delta_1\cap V_{n+1}$ is a closed complex submanifold of Δ_1, by Oka-Cartan theorem there exists a holomorphic function A in Δ_1 such that $A|V_{n+1}\cap\Delta_1=\alpha$. We define a holomorphic function B on $\Delta_1\cup\Delta_2$ by $B|\Delta_1=A$ and $B|\Delta_2=0$. Since $\Delta_1\cup\Delta_2$ is a neighborhood of the Runge compact subset $\hat{T}$ in X_{n+1}, there exists a holomorphic function G on X_{n+1} such that

$$|G(x)-B(x)|<1/\{2^{n+1}(L'+1)\} \tag{3.23}$$

for every $x\in\hat{T}$ where $L'=\sup\{|\mu_{n+1}\circ\tilde{\phi}(x)|\,;\, x\in S(c_{n+1})\cup K_n\}$. We set $f_{n+1}(x)=F(x)+(\mu_{n+1}\circ\tilde{\phi}(x))G(x)$ for every $x\in X_{n+1}$. By (3.19) and (3.23) we have

$$|f_{n+1}(x)-f_n\circ v_n(x)|<1/2^n \tag{3.24}$$

for every $x\in K_n$. By (3.22) and (3.23) we have

$$\mathrm{Re}\, f_{n+1}(e^{i\theta}\cdot c_{n+1})\geqq\lambda_{n+1} \tag{3.25}$$

for every $\theta\in\boldsymbol{R}$. Since $f_{n+1}|X_n=f_n$, this completes the proof.

LEMMA 3.5. *Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty}\varepsilon_n<\infty$ and $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions f_n in X_n such that $f_{n+1}|X_n=f_n$ and $|f_{n+1}(x)-f_n\circ v_n(x)|<\varepsilon_n$ for every $x\in K_n$. Then there exists a holomorphic function f in X such that $f|X_n=f_n$.*

PROOF. Since, by Lemma 3.3, $v_{n+j}(C_{n+j-1})\subset B_{n+j-1}\cap X_{n+j}\subset K_{n+j-1}$ and $C_n\subset C_{n+j-1}$, we have $|f_{n+j}\circ v_{n+j}(x)-f_{n+j-1}\circ v_{n+j-1}(x)|=|f_{n+j}(v_{n+j}(x))-f_{n+j-1}\circ v_{n+j-1}(v_{n+j}(x))|<\varepsilon_{n+j-1}$ for any positive integers n and j and for any $x\in C_n$. Thus for any m, n we have

$$\begin{aligned}|f_{n+m}\circ v_{n+m}(x)-f_n\circ v_n(x)| &\leqq \sum_{j=1}^{m}|f_{n+j}\circ v_{n+j}(x)-f_{n+j-1}\circ v_{n+j-1}(x)|\\ &\leqq \sum_{j=1}^{m}\varepsilon_{n+j-1}\leqq\sum_{j=1}^{\infty}\varepsilon_j\end{aligned}$$

for every $x\in C_n$. Therefore the sequence $\{f_n\circ v_n\}_{n=1}^{\infty}$ converges uniformly on each C_n to a function $f\in H(X)$. Then f satisfies $f|X_n=f_n$. This completes the proof.

We can obtain the following two lemmas by the application of Lemma 3.4 and Lemma 3.5.

LEMMA 3.6. *With the conditions of Lemma* 3.4, *there exists a holomorphic function* f *in* X *such that* $\mathrm{Re}\,f(x)\geqq\lambda_n$ *for every* n *and for every* $x\in S(c_n)$.

LEMMA 3.7. *Let* F *be any finite dimensional complex linear subspace of* E. *Then the restriction mapping of* $H(X)$ *into* $H(\tilde{\phi}^{-1}(F))$ *is surjective.*

4. Proofs of Theorem 1 and Theorem 2.

In order to prove Theorem 1 and Theorem 2, we will prepare some lemmas. Throughout this section E means a complex Banach space with a Schauder basis $\{e_n\}_{n=1}^{\infty}$ and (Ω, ϕ) means a domain, which is not homeomorphic to the projective space $\boldsymbol{P}(E)$ through ϕ, spread over $\boldsymbol{P}(E)$.

LEMMA 4.1. *If* Ω *is a domain of holomorphy,* Ω *is pseudoconvex.*

PROOF. For any continuous linear functional f of E and the open set $U(f)=\{[\xi]\in\boldsymbol{P}(E)\,;\ f(\xi)\neq0\}$, we have only to show that the domain $\phi^{-1}(U(f))$ satisfies the Kontinuitätssatz. Since there exists a biholomorphic mapping p of $U(f)$ onto the complex Banach space $L=\{\xi\in E\,;\ f(\xi)=0\}$, the domain $(\phi^{-1}(U(f)),\ p\circ(\phi|\phi^{-1}(U(f))))$ is a domain spread over L. Since Ω is a domain of holomorphy and since, for any sequence $\{x_n\}_{n=1}^{\infty}$ of $\phi^{-1}(U(f))$ converging to a point of $\Omega\setminus\phi^{-1}(U(f))$, the set $\{p\circ\phi(x_n)\}$ is an unbounded subset of L, $\phi^{-1}(U(f))$ is also a domain of holomorphy. By Noverraz [**17**], $\phi^{-1}(U(f))$ satisfies the Kontinuitätssatz. This completes the proof.

With the conditions and notations in Section 3, we set

$$S(K_n)=\{e^{i\theta}\cdot x\,;\ \theta\in[0,2\pi],\ x\in K_n\} \tag{4.1}$$

for each n. $S(K_n)$ is compact in $X_{n+1}\cap A_n$. We denote by $\widehat{S(K_n)}$ the holomorphically convex hull of $S(K_n)$ in X_{n+1}. Since $A_n\cap X_{n+1}$ is Runge relative to X_{n+1}, $\widehat{S(K_n)}$ is a compact subset of $A_n\cap X_{n+1}$. We set $e^{i\theta}\cdot C_n=\{e^{i\theta}\cdot x\,;\ x\in C_n\}$. For any $\theta\in\boldsymbol{R}$, we have

$$e^{i\theta}\cdot C_n \cap X_{n+1} \subset S(K_n) \subset \widehat{S(K_n)}, \tag{4.2}$$

$$v_n(e^{i\theta}\cdot C_n) \subset S(K_n) \subset \widehat{S(K_n)}. \tag{4.3}$$

Hereafter we assume that Ω is pseudoconvex in a series of lemmas.

LEMMA 4.2. *Then for any holomorphic function f in X_n there exists a sequence $\{f_{n+k}\}_{k=0}^{\infty}$ of holomorphic functions in X_{n+k} satisfying the following conditions:*

(1) $f_n = f$,

(2) $f_{n+k} | X_{n+k-1} = f_{n+k-1}$,

(3) $|f_{n+k}(x) - f_{n+k-1}\circ v_{n+k-1}(x)| < 1/2^{n+k}$ *for every* $x \in \widehat{S(K_n)}$.

PROOF. We can prove this lemma by the same way as the proof of Lemma 3.4.

REMARK 4.3. By the same way as the proof of Lemma 3.5, we can prove that there exists a holomorphic function F in X such that $F|X_{n+k}=f_{n+k}$, $F(x) =\lim_{k\to\infty} f_{n+k}\circ v_{n+k}(x)$ for every $x\in X$. By (4.2) and (4.3), we have

$$\begin{aligned} |F(x)| &= \lim_{m\to\infty} |f_m\circ v_m(x)| \\ &\leqq \limsup_{m\to\infty} \left\{ \sum_{k=N}^{m} |f_k\circ v_k(x) - f_{k-1}\circ v_{k-1}(x)| + |f_N\circ v_N(x)| \right\} \\ &\leqq 2^{-N} + \sup\{ |f_N\circ v_N(y)| \,;\, y\in S(C_N)\} < \infty \end{aligned} \tag{4.4}$$

for every $N\geqq n$ and for every $x\in S(C_N)$ where $S(C_N)$ is the set $\{e^{i\theta}\cdot z\,;\,(\theta, z)\in \boldsymbol{R}\times C_N\}$. Thus we have $\sup\{|F(x)|\,;\,x\in S(C_N)\}<\infty$ for every $N\geqq 1$.

We denote by D_m an open subset of Ω defined by $D_m=\tilde{Q}(C_m)$ for every $m\geqq 1$.

LEMMA 4.4. *For any holomorphic function f in $\phi^{-1}(\boldsymbol{P}(E_n))$ there exists a holomorphic function F in Ω such that $F|\phi^{-1}(\boldsymbol{P}(E_n))=f$ and $\sup\{|F(x)|\,;\,x\in D_m\}$ $<\infty$ for every $m\geqq 1$.*

PROOF. We consider a holomorphic function g in X_n defined by $g=f\circ(\tilde{Q}|X_n)$. By Lemma 4.2 and by Remark 4.3, there exists a holomorphic function G in X such that $G|X_n=g$ and $\sup\{|G(x)|\,;\,x\in S(C_m)\}<\infty$ for every $m\geqq 1$. We set

$$\tilde{G}(x) = (1/2\pi)\int_0^{2\pi} G(e^{i\theta}\cdot x)d\theta$$

for every $x\in X$. Then $\tilde{G}$ is a holomorphic function in X and constant on $\tilde{Q}^{-1}(z)$ for every $z\in\Omega$. We define a holomorphic function F by $F(z)=\tilde{G}\circ\tilde{Q}^{-1}(z)$ for every $z\in\Omega$. Then we have $F|\phi^{-1}(\boldsymbol{P}(E_n))=f$ and $\sup\{|F(x)|\,;\,z\in D_m\}\leqq \sup\{|G(x)|\,;\,z\in S(C_m)\}<\infty$ for every $m\geqq 1$. This completes the proof.

LEMMA 4.5. *For any different points z and w in Ω, there exists a holomorphic function f in Ω such that $f(z)\neq f(w)$ and that* $\sup\{|f(p)|\,;\,p\in D_m\}<\infty$ *for every* $m\geqq 1$.

PROOF. There exist two different points x and y in X such that $\tilde{Q}(x)=z$ and $\tilde{Q}(y)=w$. There exists a positive integer N such that the set $\{x, y, v_N(x), v_N(y)\}$ is contained in C_N and that $\tilde{Q}(v_N(x))\neq\tilde{Q}(v_N(y))$. Then the compact sets $S(x)$, $S(y)$, $S(v_N(x))$ and $S(v_N(y))$, defined in (3.9), are contained in $S(C_N)$. We consider closed submanifolds $V(v_N(x))$ and $V(v_N(y))$, defined in (3.8), of the Stein mainfold X_N. By Oka-Cartan theorem, there exists a holomorphic function g in X_N satisfying $g|V(v_N(x))=2$ and $g|V(v_N(y))=0$. By Lemma 4.2, there exists a sequence $\{g_m\}_{m=N}^{\infty}$ of holomorphic functions g_m in X_{N+m} such that $g_m|X_{m-1}=g_{m-1}$, $g_N=g$ and $|g_m\circ v_m(t)-g_{m-1}\circ v_{m-1}(t)|<1/2^m$ for every $m>N$ and every $t\in S(C_{m-1})$. Let G be a holomorphic function defined by $G(t)=\lim_{m\to\infty}g_m\circ v_m(t)$ for every $t\in X$. Then we have $|G(t)-g\circ v_N(t)|\leqq 1/2^N$ for every $t\in S(C_N)$. Thus we have $\operatorname{Re}G(e^{i\theta}\cdot x)\geqq\operatorname{Re}g\circ v_N(e^{i\theta}\cdot x)-1/2^N\geqq 3/2$ and $\operatorname{Re}G(e^{i\theta}\cdot y)\leqq\operatorname{Re}g\cdot v_N(e^{i\theta}\cdot y)+1/2^N\leqq 1/2$. By Remark 4.3, the holomorphic function G in X satisfies $\sup\{|G(t)|\,;\,t\in S(C_m)\}<\infty$ for every $m\geqq 1$. We set

$$\tilde{G}(t)=(1/2\pi)\int_0^{2\pi}G(e^{i\theta}\cdot t)d\theta$$

for every $t\in X$. Then $\tilde{G}$ is a holomorphic function in X and constant on $\tilde{Q}^{-1}(\zeta)$ for every $\zeta\in\Omega$. We set $f(\zeta)=\tilde{G}\circ\tilde{Q}^{-1}(\zeta)$ for every $\zeta\in\Omega$. Then f is a holomorphic function and satisfies $\operatorname{Re}f(w)\leqq 1/2<3/2\leqq\operatorname{Re}f(z)$. Moreover we have $\sup\{|f(\zeta)|\,;\,\zeta\in D_m\}\leqq\sup\{|G(t)|\,;\,t\in S(C_m)\}<\infty$. f satisfies the requirement of this lemma. This completes the proof.

We set $\mathfrak{D}=\{D_n\}_{n=1}^{\infty}$ and set $|f|_n=\sup\{|f(x)|\,;\,x\in D_n\}$ for every $f\in H(\Omega)$ and every $n\geqq 1$. We denote by $A(\mathfrak{D})$ the Fréchet space defined by

$$A(\mathfrak{D})=\{f\in H(\Omega)\,;\ |f|_n<\infty\quad\text{for every } n\}.$$

LEMMA 4.6. *For each countable set P of Ω there exists a function $g\in A(\mathfrak{D})$ such that $g(x)\neq g(y)$ for all $(x, y)\in P\times P\setminus\Delta$ where Δ is the diagonal set of $P\times P$.*

PROOF. By Lemma 4.5, the set $S_{xy}=\{g\in A(\mathfrak{D})\,;\ g(x)\neq g(y)\}$ is nonempty for each $(x, y)\in P\times P\setminus\Delta$. The set S_{xy} is open in $A(\mathfrak{D})$. We claim that S_{xy} is dense in $A(\mathfrak{D})$. Let f be an element of $A(\mathfrak{D})$ with $f\notin S_{xy}$. We choose $g\in S_{xy}$ and set $g_n=f+(1/n)g$. Then we have $g_n\in S_{xy}$ for every n and $g_n\to f$ in $A(\mathfrak{D})$. Since $A(\mathfrak{D})$ is a Baire space, the set $S=\bigcap\{S_{xy}\,;\,(x, y)\in P\times P\setminus\Delta\}$ is dense in $A(\mathfrak{D})$, and in particular nonempty. This completes the proof.

PROOF OF THEOREM 1. It follows from Lemma 2.4 that (1) and (2) are equivalent. It follows from Lemma 4.1 that (3) implies (1). It is clear that (4) implies (3).

Now we will show that (1) implies (4). Let E_n be the linear span of the set $\{e_1, \cdots, e_n\}$. We may assume that $Q(e_1)\in\phi(\Omega)$. Since $\boldsymbol{P}(E)$ is separable, there exists a countable dense subset D of $\boldsymbol{P}(E)$. We set $P=\phi^{-1}(D)$. Then P is a countable dense subset of Ω. By Lemma 4.6, there exists a holomorphic function $g\in A(\mathfrak{D})$ such that $g(x)\neq g(y)$ for every $(x, y)\in P\times P\diagdown\varDelta$. Let d be the distance of $\boldsymbol{P}(E)$ defined by (2.4). We denote by Ω_n the region, defined by $\Omega_n=\phi^{-1}(\boldsymbol{P}(E_n))$, spread over $\boldsymbol{P}(E_n)$ for every n. We denote by d_n the boundary distance function of the region $(\Omega_n, \phi|\Omega_n)$ with respect to $d|\boldsymbol{P}(E_n)$. For each $x\in\Omega_n$ we denote by $B_n(x)$ the open neighborhood, which is homeomorphically mapped by $\phi|\Omega_n$ onto the set $\{\zeta\in\boldsymbol{P}(E_n);\ d(\phi(x), \zeta)\leqq d_n(x)\}$, of x in Ω_n. We set $L_n=\tilde{Q}(K_n)$ for each n where K_n is defined in Section 3. Each L_n is a compact subset of Ω_n and $\bigcup_{n=1}^{\infty}L_n=\bigcup_{n=1}^{\infty}\Omega_n$. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of points a_n in Ω_n such that $\{a_n\}_{n=1}^{\infty}$ is dense in Ω. We can find a sequence $\{b_n\}_{n=1}^{\infty}$ in Ω such that $b_n\in B_n(a_n)\diagdown L_n$ and $b_n\in\Omega_n\diagdown\Omega_{n-1}$. There exists a sequence $\{c_n\}_{n=1}^{\infty}$ in X such that $\tilde{Q}(c_n)=b_n$. Then we have $V(c_n)\cap K_n=\emptyset$. By Lemma 3.6, there exists a holomorphic function f in X such that $\operatorname{Re}f(x)\geqq n+|g(b_n)|$ for every n and for every $x\in S(c_n)$. We set

$$\tilde{f}(x)=(1/2\pi)\int_0^{2\pi}f(e^{i\theta}\cdot x)d\theta$$

for every $x\in X$. Then $\tilde{f}$ is a holomorphic function in X and constant on the fibre $\tilde{Q}^{-1}(z)$ for every $z\in\Omega$. We set $f^*(z)=\tilde{f}(\tilde{Q}^{-1}(z))$ for every $z\in\Omega$. Then f^* is holomorphic in Ω and satisfies $\operatorname{Re}f^*(b_n)\geqq n+|g(b_n)|$. Since the set of quotient $(f^*(x)-f^*(y))/(g(x)-g(y))$ with $(x, y)\in P\times P\diagdown\varDelta$ is countable, there exists $\theta\in(0, 1)$ such that $f^*(x)-f^*(y)\neq\theta(g(x)-g(y))$ for every $(x, y)\in P\times P\diagdown\varDelta$. If we set $h=f^*-\theta g$, then $h\in H(\Omega)$, $h(x)\neq h(y)$ for every $(x, y)\in P\times P\diagdown\varDelta$ and

$$\operatorname{Re}h(b_n)\geqq n \tag{4.5}$$

for every $n\geqq1$. We will show that Ω is the domain of existence of h. Let λ: $\Omega\to\tilde{\Omega}$ be an $\{h\}$-extension of Ω, and let $\tilde{h}\in H(\tilde{\Omega})$ with $\tilde{h}\circ\lambda=h$. To prove that λ is injective, let $a, b\in\Omega$ with $\lambda(a)=\lambda(b)$. There exist an open neighborhood $U(a)$ of a and an open neighborhood $U(b)$ of b such that $\lambda(U(a))=\lambda(U(b))$ and that $\lambda|U(a)$, $\lambda|U(b)$, $\phi|U(a)$ and $\phi|U(b)$ are isomorphisms. Then we have $\lambda(x)=\lambda(y)$, if $(x, y)\in U(a)\times U(b)$ and $\phi(x)=\phi(y)$. Thus we have $h(x)=\tilde{h}\circ\lambda(x)=\tilde{h}\circ\lambda(y)=h(y)$, if $(x, y)\in U(a)\times U(b)$ and $\phi(x)=\phi(y)$. We set $W=\phi(U(a))$. Then we have $W=\phi(U(a))=\phi(U(b))$ and W is an open subset of $\boldsymbol{P}(E)$. $W\cap D$ is nonempty. Thus there exist $x_0\in U(a)$ and $y_0\in U(b)$ such that $\phi(x_0)=\phi(y_0)\in W\cap D$. Then $h(x_0)=h(y_0)$. Since $(x_0, y_0)\in P\times P\diagdown\varDelta$, this is a contradiction. Therefore λ is injective. To prove that λ is surjective, we assume that $\tilde{\Omega}\neq\lambda(\Omega)$. Then there exists a point b_0 of $(\tilde{\Omega}\diagdown\lambda(\Omega))\cap\overline{\lambda(\Omega)}$ where $\overline{\lambda(\Omega)}$ is the topological closure of $\lambda(\Omega)$

in $\tilde{\Omega}$. Then there exists a subsequence $\{b_{n_k}\}$ of $\{b_n\}_{n=1}^{\infty}$ such that $\{\lambda(b_{n_k})\}$ converges to b_0. Then we have

$$|\tilde{h}(\lambda(b_{n_k}))| \geqq \operatorname{Re} \tilde{h}\circ\lambda(b_{n_k}) = \operatorname{Re} h(b_{n_k}) \geqq n_k .$$

This implies that $\tilde{h}$ is unbounded in a neighborhood of b_0. This is a contradiction. Thus λ is surjective. Therefore λ is an isomorphism. This implies that Ω is a domain of existence of h. This completes the proof.

PROOF OF THEOREM 2. Let Δ be the diagonal set of the product space $\Omega\times\Omega$. Let (z, w) be any point of $\Omega\times\Omega\setminus\Delta$. By Lemma 4.5, there exists a holomorphic function $g_{(z,w)}\in A(\mathfrak{D})$ such that $g_{(z,w)}(z)\neq g_{(z,w)}(w)$. There exists an open neighborhood $U((z, w))$ of (z, w) in $\Omega\times\Omega\setminus\Delta$ such that $g_{(z,w)}(\zeta_1)\neq g_{(z,w)}(\zeta_2)$ for every $(\zeta_1, \zeta_2)\in U((z, w))$. Since $\bigcup\{U((z, w))\,;\,(z, w)\in\Omega\times\Omega\setminus\Delta\}=\Omega\times\Omega\setminus\Delta$ and the open set $\Omega\times\Omega\setminus\Delta$ satisfies the Lindelöf property, there exists a sequence $\{(z_j, w_j)\}_{j=1}^{\infty}$ of elements of $\Omega\times\Omega\setminus\Delta$ such that $\bigcup_{j=1}^{\infty}U((z_j, w_j))=\Omega\times\Omega\setminus\Delta$. We set $g_n=g_{(z_n,w_n)}$ and $M_n=\sup\{|g_n(\zeta)|\,;\,\zeta\in D_n\}$ for every positive integer n. Each M_n is a finite positive number. We define an injective holomorphic mapping g of Ω into l^2 by

$$g = ((1/M_1)g_1, (1/2M_2)g_2, \cdots, (1/nM_n)g_n, \cdots).$$

Since $\phi^{-1}(\boldsymbol{P}(H_{n+1}))$ is a Stein manifold of dimension n for every n, by Narasimhan [**14**] and by Remmert [**20**] there exists $(2n+1)$-holomorphic functions $h_{n,j}(1\leqq j\leqq 2n+1)$ such that $h_n=(h_{n,1}, h_{n,2}, \cdots, h_{n,2n+1})$ is a regular, injective and proper holomorphic mapping of $\phi^{-1}(\boldsymbol{P}(H_{n+1}))$ into $\boldsymbol{C}^{2n+1}$. By Lemma 4.4, there exists a holomorphic mapping $\tilde{h}_n$ of Ω into $\boldsymbol{C}^{2n+1}$ such that $\tilde{h}_n|\phi^{-1}(\boldsymbol{P}(H_{n+1}))=h_n$ and $\sup\{\|\tilde{h}_n(x)\|_{2n+1}\,;\,x\in D_m\}<\infty$ for every $m\geqq 1$ where $\|\cdot\|_{2n+1}$ is the Euclidean norm of $\boldsymbol{C}^{2n+1}$. We set $k_n=\sup\{\|\tilde{h}_n(x)\|_{2n+1}\,;\,x\in D_n\}$ for every n. We define a holomorphic mapping h of Ω into l^2 by

$$h = ((1/k_1)\tilde{h}_1, (1/2k_2)\tilde{h}_2, \cdots, (1/nk_n)\tilde{h}_n, \cdots).$$

Then $h|\phi^{-1}(\boldsymbol{P}(H_n))$ is a regular, injective, proper holomorphic mapping of $\phi^{-1}(\boldsymbol{P}(H_n))$ into l^2. There exists an isomorphism α of $l^2\times l^2$ onto H. We define a holomorphic mapping f of Ω into H by $f(z)=\alpha(g(z), h(z))$ for every z. Then f satisfies the requirement of this theorem. This completes the proof.

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