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Actions of finite groups on finite von Neumann algebras and the relative entropy

Dedicated to Professor Osamu Takenouchi on his 60th birthday

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Introduction.

Let M be a finite von Neumann algebra with a faithful normal normalized trace τ and N be a von Neumann subalgebra of M. Then, the relative entropy $H(M|N)$ is naturally defined as an extended notion of the conditional entropy in commutative cases. This relative entropy is used in Connes-Stormer's work [4] as a technical tool for finite dimensional algebras M. Recently, O. Pimsner and S. Popa have deeply studied it ([12]). One of their main results is to make clear the relationship between $H(M|N)$ and Jones' index $[M:N]$ for a type II_1 factor M and its subfactor N and give the formula on $H(M|N)$ for this pair. Another one is to compute completely the value of $H(M|N)$ for an arbitrary subalgebra N of a finite dimensional algebra M.

The aim of this paper is to give the complete formula on $H(M|M^G)$ for an arbitrary action α of a finite group G on a finite von Neumann algebra M by the following method, where M^G is the fixed point subalgebra of M under the action α.

[A] *A general case may be reduced to the case that the action α is centrally ergodic, see Proposition* 2.1.

[B] *The case where α is centrally ergodic may be reduced to the case that M is a factor, see Proposition* 2.2.

[C] *When M is a factor, $H(M|M^\alpha)$ may be computed in association with the conjugacy invariants of actions introduced and deeply studied by V. Jones* [6], *see Theorem* 2.6.

Applying these formulas, we can show the fact that $H(M|M^\alpha)\leqq\log|G|$ holds in general and we can characterize such actions α that $H(M|M^\alpha)$ attains $\log|G|$,

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see Corollary 2.7 and Remark 2.8.

In order to carry out these computations, we need some investigations on the relative entropy, besides the several deep results of Pimsner-Popa ([12]). One obstacle to compute concretely the relative entropy $H(M|N)$ for a given subalgebra N of M is that the formula:

$$(*) \quad H(M|N)=H(M|L)+H(L|N)$$

is not assured in general, even if both of $H(M|L)$ and $H(L|N)$ are known to be computable for a subalgebra L such that $M\supset L\supset N$. The formula $(*)$ is shown to hold in the following two cases, as described in Proposition 1.7, which will play a crucial role to compute $H(M|M^\alpha)$ in the cases [B] and [C].

(i) *M is a factor and $L=\sum_{i=1}^n e_i M e_i$ for central projections e_i of N such that $\sum_{i=1}^n e_i=1$.*

(ii) *$L=\sum_{j=1}^m f_j M f_j$ for central projections f_j of M such that $\sum_{j=1}^m f_j=1$ and $E_N^L(f_j)=\tau(f_j)$.*

For the case [C] (M is a factor and $N=M^\alpha$), we apply (i) to our computations as follows. Since the center $Z(M^\alpha)$ of M^α is finite dimensional (cf. 2.1.3 in [6]), we may take e_i as minimal projections of $Z(M^\alpha)$. Thus, we have the formula $(*)$ with

$$H(M|L)=\sum_{i=1}^n \eta\tau(e_i) \quad \text{and} \quad H(L|N)=\sum_{i=1}^n \tau(e_i)H(M_{e_i}|M_{e_i}^\alpha).$$

If one knows the structure of the relative commutant $(M_{e_i}^\alpha)'\cap M_{e_i}$, one may compute $H(M_{e_i}|M_{e_i}^\alpha)$ by using 4.4 in [12] and so $H(M|M^\alpha)$. Therefore, we also have to make clear the structure of $(M^G)'\cap M$ and we show in Proposition 2.3,

$$(M^G)'\cap M=(M^K)'\cap M=v(K)''$$

where $K=\{k\in G;\ \alpha_k=\mathrm{Ad}v_k$ for some unitary v_k in $M\}$. This family of unitaries v_k implementing α_k $(k\in K)$ is interpreted as a μ-representation v of K for some multiplier μ of K. Associated with the canonical factor decomposition: $v\cong\sum_\chi v^\chi$ of v, we denote by f_χ the corresponding projection and by d_χ the dimension of $\chi\in(K,\hat{\mu})$. Then, Theorem 2.6 asserts that

$$[\mathrm{C}] \quad H(M|M^\alpha)=\log|G/K|+\sum_\chi \tau(f_\chi)\log(d_\chi^2/\tau(f_\chi)).$$

For the case [B] (α is centrally ergodic), applying (ii) and taking $f_1, f_2, \cdots, f_m$ as minimal projections of $Z(M)$, whose existence is assured by the centrally ergodicity of α, we get in Proposition 2.2,

$$[\mathrm{B}] \quad H(M|M^G)=H(M_{f_1}|(M_{f_1})^H)+\sum_{j=1}^m \eta\tau(f_j)$$

where $H=\{g\in G\,;\ \alpha_g(f_1)=f_1\}$.

For the general case [A], applying the reduction theory on the relative entropy (see [**10**]), we have in Proposition 2.1,

$$[\mathrm{A}]\quad H(M|M^G)=\int_\Gamma H(M(\gamma)|M(\gamma)^G)d\mu(\gamma)$$

where $Z(M)^G\cong L^\infty(\Gamma,\mu)$, $M\cong\int_\Gamma^\oplus M(\gamma)d\mu(\gamma)$, *and the reduced action* α^γ *of* G *on* $M(\gamma)$ *is centrally ergodic for* μ*-almost all* $\gamma\in\Gamma$.

NOTATIONS. We fix some notations frequently used in this paper. For a von Neumann algebra M, $M^+=\{$all positive elements of $M\}$, $Z(M)=M'\cap M=$center of M, $M^p=\{$all projections of $M\}$. For a set I, $|I|$ denotes the cardinal number of I. $\boldsymbol{C}$, $\boldsymbol{R}$ and $\boldsymbol{N}$ denote the set of all complex, real and natural numbers respectively.

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§1. Reduced relative entropy.

In this section, we introduce a reduced relative entropy which is a slight generalization of Pimsner-Popa's relative entropy [**12**], and we describe elementary properties and some technical results concerning it.

Throughout this paper, let M be a finite von Neumann algebra with a faithful normal normalized trace τ and N be a von Neumann subalgebra of M. Then, a function h_N^M on M^+ is defined by

$$h_N^M(x)=\tau\eta E_N^M(x)-\tau\eta(x)\quad \text{for } x\in M^+$$

where E_N^M is the unique τ-preserving conditional expectation of M onto N (see Umegaki [**14**]) and η is a continuous function defined by $\eta(t)=-t\log t$ $(t>0)$, $\eta(0)=0$.

We first list up some elementary properties of h_N^M, which are immediately obtained from the definition and 1°$\sim$11° in §3 of [**12**]. We denote h_N^M by h and E_N^M by E if there is no fear of confusion.

1° $h(x)\geqq 0$ *for* $x\in M^+$.

2° $h(x)$ *is strongly continuous on* M^+.

3° $h(\lambda x)=\lambda h(x)$ *for* $\lambda\in\boldsymbol{R}^+$, $x\in M^+$.

4° $h(p)=\tau\eta E(p)$ *for* $p\in M^p$.

5° $h(x+y)\leqq h(x)+h(y)$ *for* x, $y\in M^+$ *with* $xy=0$. *Under the additional condition* $E(x)E(y)=0$, *we have the equality.*

6° $h(px)=\tau(\eta E(p)x)$ *for* $p\in(N'\cap M)^p$, $x\in N$.

7° $h(px)=\tau(p)\tilde{h}(\tilde{x})$ *for* $p\in(Z(M)\cap Z(N))^p$, $x\in pMp$, *where* $\tilde{x}$ *is the image of* x *via the canonical isomorphism of* pMp *onto* M_p, *and* $\tilde{h}$ *is defined for* $M_p\supset N_p$ *with the normalized trace* τ_p, $\tau_p(\tilde{x})=\tau(pxp)/\tau(p)$.

Now, we define reduced relative entropy $H^y(M|N)$ associated with $y\in M^+$ as follows

DEFINITION 1.1. For $y\in M^+$, set

$$S^y(M)=\{\Delta=(x_i)_{i\in I}\,;\ x_i\in M^+,\ \sum_{i\in I}x_i\leqq y,\ \text{and } I \text{ is a finite set}\}.$$

Taking $\Delta=(x_i)_{i\in I}\in S^y(M)$, we set

$$H^y_\Delta(M|N)=\sum_{i\in I}h^M_N(x_i).$$

Then, the reduced relative entropy of M to N associated with $y\in M^+$ is defined by

$$H^y(M|N)=\sup\{H^y_\Delta(M|N)\,;\ \Delta\in S^y(M)\}.$$

When $y=1$, $H^1(M|N)$ is the ordinary relative entropy $H(M|N)$ studied by Pimsner-Popa [12]. We need the above notion in order to clarify some of the arguments by taking y as a projection.

We make some preparations for to describe elementary properties of $H^y(M|N)$. We denote by $\|\ \|_1$, L^1-norm of M, $\|x\|_1=\tau(|x|)$ for $x\in M$. We abbreviate $Z(M)\cap Z(N)$ by Z for fixed M and N. For $p\in Z^p$, $H(M_p|N_p)$ is the relative entropy associated with the normalized trace τ_p of M_p described in 7°. We set

$$T=\{y=\sum_{j\in J}\lambda_j p_j\,;\ \lambda_j\in \boldsymbol{R}^+,\ p_j\in Z^p \text{ such that } \sum_{j\in J}p_j=1,\ \text{and } J \text{ is a finite set}\},$$

and define essential entropy of M relative to N by

$$EH(M|N)=\sup\{H(M_p|N_p)\ ;\ p\neq 0\in Z^p\}.$$

PROPOSITION 1.2. *$H^y(M|N)$ has the following properties.*

(a) $H^{y_1}(M|N)\leqq H^{y_2}(M|N)$ *for* y_1, $y_2\in M^+$ *with* $y_1\leqq y_2$.

(b) $H^{\lambda y}(M|N)=\lambda H^y(M|N)$ *for* $\lambda\in\boldsymbol{R}^+$, $y\in M^+$.

(c) $H^p(M|N)=\tau(p)H(M_p|N_p)$ *for* $p\in Z^p$.

(d) $H^p(M|N)=\sum_{j\in J}H^{p_j}(M|N)$ *for* p, p_j $(j\in J)\in Z^p$ *with* $p=\sum_{j\in J}p_j$ *where* J *is a finite set.*

(e) $|H^{y_1}(M|N)-H^{y_2}(M|N)|\leqq\|y_1-y_2\|_1 EH(M|N)$ *for* y_1, $y_2\in T$.

PROOF. (a) is clear by the definition. (b) and (c) follow immediately from 3° and 7° respectively. (d) is obtained from the following observation. For $\Delta=(x_i)_{i\in I}\in S^p(M)$, set $\Delta_j=(p_jx_ip_j)_{i\in I}$. Then, $\Delta_j\in S^{p_j}(M)$ and $H^p_\Delta(M|N)=\sum_{i\in I}h(x_i)=\sum_{i\in I}\sum_{j\in J}h(p_jx_ip_j)=\sum_{j\in J}H^{p_j}_{\Delta_j}(M|N)$ by 5°. Conversely, for $\Delta_j=(x_{ij})_{i\in I_j}\in S^{p_j}(M)$, set $\Delta=(x_{ij})_{i\in I_j,j\in J}$. Then, $\Delta\in S^p(M)$ and $H^p_\Delta(M|N)=\sum_{j\in J}H^{p_j}_{\Delta_j}(M|N)$. Finally, we shall prove property (e). For $y_1=\sum_{i\in I}\lambda_ip_i$ and $y_2=\sum_{j\in J}\mu_jq_j$ in T, if we set $r_{ij}=p_iq_j$, then we have

$$\|y_1-y_2\|_1=\sum_{i,j}|\lambda_i-\mu_j|\tau(r_{ij})\quad\text{and}$$

$$H^{y_1}(M|N)-H^{y_2}(M|N)=\sum_{i,j}(\lambda_i-\mu_j)\tau(r_{ij})H(M_{r_{ij}}|N_{r_{ij}})\quad\text{[by (b), (c), (d)]}.$$

Hence, we have

$$|H^{y_1}(M|N)-H^{y_2}(M|N)|\leqq\|y_1-y_2\|_1EH(M|N).\qquad\text{[Q. E. D.]}$$

PROPOSITION 1.3. (i) [Approximation] *For any* $y\in Z^+$ *and any* $\varepsilon>0$, *there exists* y_1 *in* T *such that* $\|y-y_1\|_1<\varepsilon$ *and* $|H^y(M|N)-H^{y_1}(M|N)|<\varepsilon EH(M|N)$.

(ii) [Continuity] *If* $EH(M|N)<+\infty$, *then for any* $y\in Z^+$ *and any* $\varepsilon>0$ *there exists* $\delta>0$ *such that* $|H^y(M|N)-H^{y'}(M|N)|<\varepsilon$ *for* $y'\in Z^+$ *with* $\|y-y'\|_1<\delta$.

PROOF. (i) For $y\in Z^+$, there exist y_1 and y_2 in T such that $y_1\leqq y\leqq y_2$ and $\|y_1-y_2\|_1<\varepsilon$ by the spectral decomposition of y. Then,

$$|H^{y_1}(M|N)-H^{y_2}(M|N)|<\varepsilon EH(M|N)\quad\text{[by (e)]}$$

$$H^{y_1}(M|N)\leqq H^y(M|N)\leqq H^{y_2}(M|N)\quad\text{[by (a)]}.$$

Hence, we have the desired conclusion.

(ii) It is enough to assume that $EH(M|N)>0$. For $\varepsilon>0$, put $\delta=\varepsilon/(5EH(M|N))>0$. Then applying (i), there exist y_1 and y_2 in T satisfying that

$$\|y-y_1\|_1<\delta,\qquad |H^y(M|N)-H^{y_1}(M|N)|<\delta EH(M|N),$$

$$\|y'-y_2\|_1<\delta,\ \text{and}\ |H^{y'}(M|N)-H^{y_2}(M|N)|<\delta EH(M|N).$$

Hence, we have

$$\|y_1-y_2\|_1\leqq\|y_1-y\|_1+\|y-y'\|_1+\|y'-y_2\|_1<3\delta,$$

$$|H^{y_1}(M|N)-H^{y_2}(M|N)|<3\delta EH(M|N)\quad\text{[by (e)]}.$$

Therefore,

$$|H^y(M|N)-H^{y'}(M|N)|$$

$$\leqq|H^y(M|N)-H^{y_1}(M|N)|+|H^{y_1}(M|N)-H^{y_2}(M|N)|+|H^{y_2}(M|N)-H^{y'}(M|N)|$$

$$<5\delta EH(M|N)=\varepsilon.\qquad\text{[Q. E. D.]}$$

For $y\in M^+$ and $\varepsilon>0$, put

$$S_\varepsilon^y(M)=\{\Delta=(\lambda_i p_i)_{i\in I}\,;\ \lambda_i\in\boldsymbol{R}^+,\ p_i\in M^p \text{ such that } \sum_{i\in I}\lambda_i p_i\leqq y$$
$$\text{and } \|y-\sum_{i\in I}\lambda_i p_i\|_1<\varepsilon, \text{ where } I \text{ is a finite set}\}.$$

LEMMA 1.4. *For* $y\in M^+$,

$$H^y(M|N)=\sup\{H_\Delta^y(M|N)\,;\ \Delta\in S_1^y(M)\}.$$

More precisely, for any $\varepsilon>0$, *there exists* Δ *in* $S_\varepsilon^y(M)$, *such that*

$$H^y(M|N)\leqq H_\Delta^y(M|N)+\varepsilon.$$

The proof is similar to that in [**12**, Lemma 3.1].

For $y\in M^+$ and two positive numbers $\varepsilon>0$ and $\delta>0$, we set

$$S_{\varepsilon,\delta}^y(M)=\{\Delta=(\lambda_i p_i)_{i\in I}\in S_\varepsilon^y(M)\,;\ \tau(p_i)=\delta \text{ for each } i\in I\}.$$

LEMMA 1.5. *Let* M *be a continuous finite von Neumann algebra with a faithful normal normalized trace* τ *and* y *be a positive element of* M. *Then, for any* $\varepsilon>0$, *there is* $\delta_0>0$ *satisfying that, for an arbitrary* δ $(0<\delta\leqq\delta_0)$, *there exists* $\Delta\in S_{\varepsilon,\delta}^y(M)$ *such that* $H^y(M|N)\leqq H_\Delta^y(M|N)+\varepsilon$.

PROOF. By Lemma 1.4, for $\varepsilon>0$, there exists $\Delta_0=(\lambda_i p_i)_{i\in I}$ in $S_{\varepsilon/2}^y(M)$ such that $H^y(M|N)\leqq H_{\Delta_0}^y(M|N)+\varepsilon/2$. Set $y_0=\sum_{i\in I}\lambda_i p_i$ and $c=\sum_{i\in I}\lambda_i$. Then, we may assume that $\lambda_i>0$ and ε is small enough to satisfy that $t<\eta(t)$ and $\eta(t_1)\leqq\eta(t_2)$ for $t_1\leqq t_2$ on $[0, \varepsilon/2c]$. Let $\delta_0=\eta^{-1}(\varepsilon/2c)$. Then, $0<\delta\leqq\delta_0$ implies that $\delta<\eta(\delta)\leqq\eta(\delta_0)=\varepsilon/2c$. For each projection p_i $(i\in I)$, we can write $p_i=\sum_{j\in J_i}p_{ij}+r_i$, where p_{ij} $(j\in J_i)$ are projections of M with $\tau(p_{ij})=\delta$ and r_i is a projection of M with $\tau(r_i)<\delta$, because M is continuous. Set

$$\Delta_1=(\lambda_i p_{ij})_{j\in J_i,\, i\in I}\,,\quad y_1=\sum_{i,j}\lambda_i p_{ij}\,,\quad \Delta_2=(\lambda_i r_i)_{i\in I} \text{ and } y_2=\sum_i\lambda_i r_i\,.$$

Then, we see that, using 3°, 4° and 5°,

$$H_{\Delta_0}^y(M|N)\leqq H_{\Delta_1\cup\Delta_2}^y(M|N)=H_{\Delta_1}^y(M|N)+H_{\Delta_2}^y(M|N)\,,$$
$$H_{\Delta_2}^y(M|N)=\sum_i h(\lambda_i r_i)=\sum_i\lambda_i\tau\eta E(r_i)$$
$$\leqq\sum_i\lambda_i\eta\tau(r_i)<\sum_i\lambda_i\eta(\delta)<\varepsilon/2\,.$$

Therefore, we have $H^y(M|N)\leqq H_{\Delta_1}^y(M|N)+\varepsilon$. It is easy to check that Δ_1 lies in $S_{\varepsilon,\delta}^y(M)$. [Q. E. D.]

The next proposition plays an important role in concrete computations of the relative entropy $H(M|N)$ in the case that either M or N is not a factor.

PROPOSITION 1.6. *M denotes a finite von Neumann algebra with a faithful normal normalized trace τ and N denotes a von Neumann subalgebra of M.*

(i) *Let M be a factor. For projections e_i in $Z(N)$ $(i=1, 2, \cdots, n)$ such that $\sum_{i=1}^{n} e_i=1$, L denotes the von Neumann subalgebra $\sum_{i=1}^{n} e_iMe_i$ of M. Then, we have*

$$H(M|N)=H(M|L)+H(L|N) \text{ and } H(M|L)=\sum_{i=1}^{n}\eta\tau(e_i).$$

(ii) *Let N be a factor. For projections f_j in $Z(M)$ $(j=1, 2, \cdots, m)$ such that $\sum_{j=1}^{m} f_j=1$, L denotes the von Neumann subalgebra $\sum_{j=1}^{m} f_jNf_j$ of M. Then, we have*

$$H(M|N)=H(M|L)+H(L|N) \text{ and } H(L|N)=\sum_{j=1}^{m}\eta\tau(f_j).$$

PROOF of (i). When M is a finite type I factor, (i) follows from Pimsner-Popa's formula for finite dimensional algebras [**12**, Theorem 6.2]. We suppose that M is a type II_1 factor.

We first show the proposition in the case of $n=2$. It is enough to assume that $H(M|N)>0$. Take an arbitrary $\varepsilon>0$ and set $\varepsilon_1=(1/4)\min\{\varepsilon, c_1c_2\varepsilon/H(M|N)\}$, where $c_i=\tau(e_i)$ $(i=1, 2)$. Then, by Lemma 1.5, for $\varepsilon_1>0$, we can find $\delta>0$ and $\Delta_i=(\lambda_{ij}p_{ij})_{j\in J_i}$ in $S_{\varepsilon_1,\delta}^{e_i}(L)$ $(i=1, 2)$ such that

(0) $H^{e_i}(L|N)\leqq H_{\Delta_i}^{e_i}(L|N)+\varepsilon_1$.

Take and fix $(j, k)\in J_1\times J_2$. Since $\tau(p_{1j})=\tau(p_{2k})=\delta$ and M is a type II_1 factor, there exists a system of matrix units $(u_{st})_{s,t=1,2}$ in M such that $u_{11}=p_{1j}$ and $u_{22}=p_{2k}$. Set

$$q_{(j,k)}^{1}=\sum_{s,t=1}^{2}\sqrt{c_sc_t}\,u_{st} \quad\text{and}\quad q_{(j,k)}^{2}=\sum_{s,t=1}^{2}(-1)^{s+t}\sqrt{c_sc_t}\,u_{st}.$$

Then, it is easy to see the following properties.

(1) $q_{(j,k)}^{1}+q_{(j,k)}^{2}=2c_1p_{1j}+2c_2p_{2k}$.

(2) $e_1q_{(j,k)}^{l}e_1=c_1p_{1j}$ and $e_2q_{(j,k)}^{l}e_2=c_2p_{2k}$ for $l=1, 2$.

(3) $E_L^M(q_{(j,k)}^{l})=c_1p_{1j}+c_2p_{2k}$ for $l=1, 2$.

(4) $h_L^M(q_{(j,k)}^{l})=\tau(p_{1j})\eta(c_1)+\tau(p_{2k})\eta(c_2)=\delta(\eta(c_1)+\eta(c_2))$

(5) $\tau(q_{(j,k)}^{l})=\tau(E_L^M(q_{(j,k)}^{l}))=\delta$.

Take a partition Δ in M defined by

$$\Delta=(d_{(j,k)}^{l}q_{(j,k)}^{l})_{(j,k)\in J_1\times J_2,\ l=1,2},$$

where $d_{(j,k)}^{l}=\lambda_{1j}\lambda_{2k}\delta/(2c_1c_2)$.

Denote $\sum_{j\in J_i}\lambda_{ij}p_{ij}$ by y_i and $\tau(y_i)=\sum_j\lambda_{ij}\delta$ by b_i for $i=1, 2$. Then, $y_i\leqq e_i$ and $0\leqq c_i-b_i<\varepsilon_1$, so that

(6) $b_1b_2/c_1c_2 \geqq 1-(\varepsilon_1/c_1c_2)$.

Under these preparations, we get the followings.

(7) $\sum_{j,k,l} d^l_{(j,k)}q^l_{(j,k)} = (b_2/c_2)y_1+(b_1/c_1)y_2 \leqq e_1+e_2 = 1$.

Then, we see $\Delta\in S_1(M)$.

(8) $\tau(\sum_{j,k,l} d^l_{(j,k)}q^l_{(j,k)}) \geqq 1-(\varepsilon_1/c_1c_2)$.

Hence, we have $\sum_{j,k,l} d^l_{(j,k)}\delta\geqq 1-(\varepsilon_1/c_1c_2)$ [by (5)].

(9) $H_\Delta(M|L) = \sum_{j,k,l} h^M_N(d^l_{(j,k)}q^l_{(j,k)})$

$= \sum_{j,k,l} d^l_{(j,k)}\delta(\eta(c_1)+\eta(c_2))$ [by (4) and 3°]

$\geqq \{1-(\varepsilon_1/c_1c_2)\}(\eta(c_1)+\eta(c_2))$ [by (8)].

Hence, by the formula $H(M|L)\leqq\eta(c_1)+\eta(c_2)$ [**12**, Lemma 4.3] and the selection of ε_1, we have

(9′) $H_\Delta(M|L) \geqq H(M|L)-\varepsilon/4$,

(9″) $H(M|L) \geqq \eta(c_1)+\eta(c_2)-(K/4)\varepsilon$, where $K=(\eta(c_1)+\eta(c_2))/H(M|L)$.

Set $E(\Delta)=(E^M_L(d^l_{(j,k)}q^l_{(j,k)}))_{j,k,l}$. Then, easy calculations show by 3°, 5° and (3) that

(10) $H_{E(\Delta)}(L|N) = (b_2/c_2)H^{e_1}_{\Delta_1}(L|N)+(b_1/c_1)H^{e_2}_{\Delta_2}(L|N)$

$\geqq \{1-(\varepsilon_1/c_1c_2)\}\{H^{e_1}(L|N)+H^{e_2}(L|N)-2\varepsilon_1\}$ [by (0), (6)].

Hence, by the selection of ε and (d) in Proposition 1.2, we get

$$H_{E(\Delta)}(L|N) \geqq H(L|N)-(3/4)\varepsilon.$$

By the formula: $H_\Delta(M|N)=H_\Delta(M|L)+H_{E(\Delta)}(L|N)$, combining with (9′) and (10), we see that

(11) $H(M|N) \geqq H(M|L)+H(L|N)$.

The opposite inequality is always true so that we get the desired equality. Moreover, we note that (9″) implies that $H(M|L)\geqq\eta(c_1)+\eta(c_2)$ and so,

(12) $H(M|L) = \eta(c_1)+\eta(c_2)$.

We can prove the proposition in the case of $n\geqq 3$ by the induction on n.

The equality: $H(M|L)=\sum_{i=1}^n\eta\tau(e_i)$ has been obtained by Pimsner and Popa in [**12**, Lemma 4.3]. Our advantage is to have found a good choice of partitions of unity to establish $H(M|N)=H(M|L)+H(L|N)$ and $H(M|L)=\sum_{i=1}^{n}\eta\tau(e_i)$ at the same time. This idea of finding a suitable partition of unity is due to them, but, in order to carry out our work, we had to elaborate some significant improvements on their method to apply it to II_1 case.

PROOF of (ii). Since N is a factor

(13) $E_N^L(f_j)=\tau(f_j) \quad (j=1, 2, \cdots, m)$,

so that we get

(14) $h_N^L(f_j y)=\tau(\eta E_N^L(f_j)y)=\tau(y)\eta\tau(f_j)$.

Let $S'(L)$ be the set of partitions $(f_j y_{ij} f_j)_{i\in I_j, j=1,2,\cdots,m}$ of the unity in L such that for each j, $y_{ij}\in N^+$ and $\sum_{i\in I_j} y_{ij}=1$. Then,

(15) for $\Delta\in S'(L)$,

$$H_\Delta(L|N)=\sum_{i,j} h(f_j y_{ij})=\sum_j\sum_{i\in I_j}\tau(y_{ij})\eta\tau(f_j)=\sum_j\eta\tau(f_j).$$

Therefore, taking $i=1$ and $y_{ij}=1$,

(16) $H(L|N)\geqq\sum_j\eta\tau(f_j)$.

Conversely, for any $\Delta=(y_i)_{i\in I}\in S(L)$ with $\sum_{i\in I} y_i=1$, put $\Delta'=(f_j y_i)_{i\in I, j=1,2,\cdots,m}$. Then, $\Delta'\in S'(M)$ and $H_\Delta(L|N)\leqq H_{\Delta'}(L|N)$. Hence, by using (15), $H_\Delta(L|N)\leqq\sum_j\eta\tau(f_j)$. Therefore, $H(L|N)\leqq\sum_j\eta\tau(f_j)$. Combining this with (16), we get

(17) $H(L|N)=\sum_j\eta\tau(f_j)$.

For any $\varepsilon>0$, there exists $\Delta=(x_i)_{i\in I}\in S(M)$ such that

$$\sum_{i\in I} x_i=1 \quad \text{and} \quad H(M|L)\leqq H_\Delta(M|L)+\varepsilon.$$

Then, $E(\Delta)=(E_L^M(x_i))\in S'(L)$, and so by (15) and (17) we get

(18) $H_{E(\Delta)}(L|N)=H(L|N)$.

Therefore,

$$H(M|N)\geqq H_\Delta(M|N)=H_\Delta(M|L)+H_{E(\Delta)}(L|N)$$
$$\geqq H(M|L)-\varepsilon+H(L|N) \quad [\text{by (18)}].$$

Since $\varepsilon>0$ is arbitrary, $H(M|N)\geqq H(M|L)+H(L|N)$. Hence, we get

$$H(M|N)=H(M|L)+H(L|N). \qquad [\text{Q.E.D.}]$$

REMARK 1.7. The proof of (ii) assures that the statement in (ii) remains true if only the equality (13) holds, even though N is not a factor. Moreover, by (c) and (d) in Proposition 1.2, we know that, in (ii),

$$H(M|L)=\sum_{j=1}^{m}\tau(f_j)H(M_{f_j}|L_{f_j}).$$

§ 2. Computations of $H(M|M^{\alpha})$ and $H(M\rtimes_{\alpha}G|M)$.

Let M be a finite von Neumann algebra on a separable Hilbert space H with a faithful normal normalized trace τ and α be an action of a finite group G on M. We denote by M^{α}, or M^{G} if α is clear, the fixed point algebra of M under the action α.

The action α on M induces the action of G on the center $Z(M)$ of M and we note that $Z(M)^{G}=Z(M)\cap Z(M^{G})$. Then, by the reduction theory (see [5]), there exists a standard finite measure space (Γ, μ) such that

$$(Z(M)^{G}, \tau)\underset{\theta}{\cong}\{\text{diagonalizable operators}\}\cong L^{\infty}(\Gamma, \mu)$$

and (M, τ) is decomposed into a direct integral as

$$(M, \tau)\underset{\theta}{\cong}\int^{\oplus}(M(\gamma), \tau^{\gamma})d\mu(\gamma).$$

Moreover, for μ-almost all $\gamma\in\Gamma$, there exists an action α^{γ} of G on $M(\gamma)$ such that the field $\gamma\rightarrow\alpha^{\gamma}$ of actions is measurable and

$$\alpha\underset{\mathrm{Ad}\,\theta}{\cong}\int_{\Gamma}^{\oplus}\alpha^{\gamma}d\mu(\gamma).$$

In this case, we see that

$$M^{G}\underset{\theta}{\cong}\int_{\Gamma}^{\oplus}M(\gamma)^{G}d\mu(\gamma).$$

Thus, for μ-almost all $\gamma\in\Gamma$, the relative entropy $H(M(\gamma)|M(\gamma)^{G})$ is defined, associated with the normalized trace τ^{γ} of $M(\gamma)$. Then, we get the following.

PROPOSITION 2.1. *In the above situation, the actions α^{γ} are centrally ergodic for μ-almost all $\gamma\in\Gamma$ and*

$$H(M|M^{G})=\int_{\Gamma}H(M(\gamma)|M(\gamma)^{G})d\mu(\gamma).$$

The proof follows immediately from [10].

Owing to this proposition, we may assume that the action α of G on M is centrally ergodic, namely, $Z(M)^{G}=\boldsymbol{C}$. In this case, the center $Z(M)$ of M is finite dimensional because G is a finite group. Denote the minimal projections of $Z(M)$ by $f_1, f_2, \cdots, f_m$ and by H the stabilizer of G at the projection f_1 under the action α. Then we get an action β of H on the factor M_{f_1}, the reduced algebra of M by the projection f_1, by a suitable restriction of α. We note that $(M, G, \alpha)\cong\mathrm{Ind}_H^G(M_{f_1}, H, \beta)$ in Takesaki's sense ([13]) but the given trace τ of M is not necessarily invariant under the action α.

PROPOSITION 2.2. *Let α be a centrally ergodic action of a finite group G on a finite von Neumann algebra M with a faithful normal normalized trace τ. Then, using the above notations, we have*

$$H(M|M^G) = H(M_{f_1}|(M_{f_1})^H)+\sum_{j=1}^{m}\eta\tau(f_j).$$

PROOF. Denote M^G by N and $\sum_{j=1}^{m}f_jNf_j$ by L. Then, it is sufficient to check the following two properties, owing to (ii) in Proposition 1.6 (see also Remark 1.7).

(1) $E_N^L(f_j)=\tau(f_j)$ for each $j=1, 2, \cdots, m$.

(2) $H(M_{f_j}|N_{f_j})=H(M_{f_1}|(M_{f_1})^H)$ for each $j=1, 2, \cdots, m$.

These equalities can be checked by routine arguments.

Proposition 2.1 and 2.2 assure that the computation of the relative entropy $H(M|M^\alpha)$ for a finite von Neumann algebra M may be reduced to the case that M is a finite factor. For a given action α of a finite group G on M, we denote by $K(\alpha)$, or simply K if α is clear, the normal subgroup $\{g\in G;\ \alpha_g$ is an inner automorphism of $M\}$ of G. Actions of finite groups on the type II_1 factors are studied by V. Jones in [6]. We will give the computation of the relative entropy $H(M|M^\alpha)$, associated with Jones' conjugacy invariants of the action α. To do this, we need more precise information on the structure of M^α and $M\cap(M^\alpha)'$.

First, we reformulate some notions in [6] from a slightly different point of view. For an action α of a finite group G on a type II_1 factor M, the characteristic invariant $[\lambda, \mu]$ of α is defined in [6]. Its representative (λ, μ) is given as follows by choosing the section $(v_k)_{k\in K}$ where v_k's are unitaries in M such that $\alpha_k=\mathrm{Ad}v_k$ $(k\in K)$ and $v_e=1$.

$$v_{k_1k_2}=\mu(k_1, k_2)v_{k_1}v_{k_2} \quad (k_1, k_2\in K),$$

$$\alpha_g(v_k)=\lambda(g, k)v_{gkg^{-1}} \quad (g\in G,\ k\in K).$$

We note that μ is a $\boldsymbol{T}$-valued 2-cocycle (multiplier) of K, λ is a $\boldsymbol{T}$-valued map of $G\times K$ and they satisfy some relations (see [6, section 1.2]).

For this multiplier μ, we denote by $\mathrm{Rep}(K, \mu)$ the set of all μ-(multiplier) representations of K and denote by $\widehat{(K, \mu)}$ the unitary equivalence classes of irreducible μ-representations of K. For $\pi\in\mathrm{Rep}(K, \mu)$ and $g\in G$, set

$$(g\cdot\pi)(k)=\lambda(g, k)\pi(gkg^{-1}) \quad \text{for } k\in K.$$

Then, we see that $g\cdot\pi\in\mathrm{Rep}(K, \mu)$ and $\pi\to g\cdot\pi$ $(g\in G)$ is an action of G on $\mathrm{Rep}(K, \mu)$ which preserves each unitary equivalence class and $(g\cdot v)_k=\alpha_g(v_k)$. Thus, this action induces the action of G on $(K, \hat{\mu})$. For simplicity, set $X=(K, \hat{\mu})$ and denote by Ω the G-orbit space of X.

Let $v\cong\sum_{\chi\in X}\pi^{\chi}\otimes 1_{\chi}$ be the canonical factor decomposition of v as μ-representations of K. Then, projections f_{χ} $(\chi\in X)$ of M such that $\sum_{\chi\in X}f_{\chi}=1$ are defined, associated with this decomposition. We denote by N the von Neumann subalgebra generated by v_k $(k\in K)$ and by $S(\alpha)$ the set $\{\chi\in X;\ \tau(f_{\chi})\neq 0\}$. Then, it is clear that

(1) $Z(N)=\sum_{\chi\in X}\boldsymbol{C}f_{\chi}$ and $N=\sum_{\chi\in X}f_{\chi}Nf_{\chi}$ where $f_{\chi}Nf_{\chi}\cong M(d_{\chi}, \boldsymbol{C})$ $(d_{\chi}=\dim\pi^{\chi})$ for $\chi\in S(\alpha)$,

(2) N is α-invariant, $g\cdot v=\sum_{\chi\in X}g\cdot\pi^{\chi}\otimes 1_{\chi}$ and $\alpha_g(f_{\chi})=f_{g\cdot\chi}$,

(3) $M^K=N'\cap M=\sum_{\chi\in X}L_{\chi}$ where $L_{\chi}=f_{\chi}(N'\cap M)f_{\chi}$,

(4) M^K is α-invariant and the restriction of α on M^K to the group K is a trivial action.

For an orbit $\omega\in\Omega$, set $e_{\omega}=\sum_{\chi\in\omega}f_{\chi}$ and $|\omega|=$the number of $\chi\in\omega$. Then, for each $\chi, \chi'\in\omega$, $\tau(f_{\chi})=\tau(f_{\chi'})$ and $d_{\chi}=d_{\chi'}$, so that $\tau(e_{\omega})=|\omega|\tau(f_{\chi})$ $(\chi\in\omega)$ and we may set $d_{\omega}=d_{\chi}$ $(\chi\in\omega)$. By (2), we get $\alpha_g(e_{\omega})=e_{\omega}$ for $g\in G$ so that e_{ω} is in M^G. Thus, $e_{\omega}Me_{\omega}$ is α-invariant and this action of G on $e_{\omega}Me_{\omega}$ is also denoted by α.

Take and fix $\chi_1\in\omega$ and put $H=\{g\in G;\ g\cdot\chi_1=\chi_1\}$ and denote L_{χ_1} by L_1. Then, the action α induces the action $\bar{\alpha}$ of H/K on L_1 by (3) and (4). Under these situations, we get the followings.

PROPOSITION 2.3. *Let α be an action of a finite group G on a type II_1 factor M. Then,*

(i) *$\bar{\alpha}$ is an outer action of H/K on L_1.*

(ii) *There exists a canonical isomorphism θ from $M_{e_{\omega}}$ onto $M(|\omega|, L_1)\otimes M(d_{\omega}, \boldsymbol{C})$ which transforms $M^G_{e_{\omega}}$ onto the algebra $\{[\delta_{ij}\beta_j(x)];\ x\in L_1^H\}\otimes\boldsymbol{C}$ where β_j $(j=1, 2, \cdots, |\omega|)$ are some outer automorphisms of L_1, $M_{e_{\omega}}\cap(M^G_{e_{\omega}})'$ onto the algebra $\{[\delta_{ij}\lambda_j];\ \lambda_j\in\boldsymbol{C}\}\otimes M(d_{\omega}, \boldsymbol{C})$, and f_{χ} to (minimal projection)$\otimes 1$.*

(iii) $Z(M^G)=\sum_{\omega\in\Omega}\boldsymbol{C}e_{\omega}$.

(iv) $M\cap(M^G)'=M\cap(M^K)'=N$.

We will prove this proposition after Lemma 2.5. Here we note that M^K is a factor if and only if $S(\alpha)$ consists of one point and that M^G is a factor if and only if the action of G on $S(\alpha)$ is transitive. At first, we will investigate these cases.

LEMMA 2.4. *Assume M^K is a factor. Then,*

(i) $M\cap(M^G)'=M\cap(M^K)'$.

(ii) *If an automorphism β of M satisfies that, for some $x\neq 0$ in M, $\beta(y)x=xy$ for all $y\in M^G$, then there exist a unitary u in M and $g\in G$ such that $\beta_g=(\mathrm{Ad}u)\alpha_g$.*

PROOF. Since $S(\alpha)$ consists of one point by the assumption, the multiplier representation v of K is factorial. Then, $N=v(K)''$ is a finite type I factor because G is a finite group. Therefore, we get $M\cong M^K\otimes N$ by the fact $M^K=M\cap N'$.

(i) Since M^K and N are α-invariant, the action α induces the actions α^1 and α^2 on M^K and N respectively by restrictions and $\alpha_g\cong\alpha_g^1\otimes\alpha_g^2$ for $g\in G$. It follows from the fact N is a type I factor that the action α^2 is inner. Hence, the reduced action $\bar{\alpha}$ of G/K on M^K is seen to be outer so that $M^K\cap((M^K)^{\bar{\alpha}})'=\boldsymbol{C}$ by [**11**]. Noticing that $(M^K)^{\bar{\alpha}}=M^G$, we get $M^K\cap(M^G)'=\boldsymbol{C}$, which implies that $M\cap(M^G)'=N$.

(ii) is checked by slight modifications of the proof of Lemma 3.4 in [**3**] combined with the following duality property (*).

(*) Suppose $M\cap(M^G)'=\boldsymbol{C}$. If an automorphism β of M satisfies that $\beta(y)=y$ for all $y\in M^G$, then there exists $g\in G$ such that $\beta=\alpha_g$.

This property (*) is explained in [8] or [**9**]. [Q. E. D.]

Next, we consider the transitive case under some general situations. Here we recall the assumption that G is a finite group and M is a type II_1 factor.

Let X be a finite set $\{1, 2, \cdots, n\}$ such that G acts on X transitively. This action is denoted by $X\ni j\to g\cdot j\in X$ for $g\in G$. We denote by H the stabilizer of G at $1\in X$. Let f_j $(j\in X)$ be projections of M such that $\sum_{j\in X}f_j=1$. We denote by M_1 the reduced algebra M_{f_1} which is often identified with f_1Mf_1.

LEMMA 2.5. *Under the above situations, if $\alpha_g(f_j)=f_{g\cdot j}$ and M^α is contained in $\sum_{j=1}^n f_jMf_j$, then there exists an isomorphism θ of M onto $M(n, M_1)$ such that the isomorphism θ transforms M^α onto the subalgebra $\{[\delta_{ij}\beta_j(x)]\,;\ x\in M_1^H\}$ of $M(n, M_1)$ for some $\beta_j\in\mathrm{Aut}M_1$ $(j\in X)$. Moreover if $H\supset K(\alpha)$ and $(M_1^{K(\alpha)})'\cap M_1=F$ is a factor, θ transforms the relative commutant $(M^G)'\cap M$ onto the subalgebra $\{[\delta_{ij}\beta_j(y_j)]\,;\ y_j\in F\}$ of $M(n, M_1)$.*

PROOF. For each $j\in X$, there exsits $g\in G$ such that $g\cdot 1=j$ by transitivity of the action of G on X. Then, f_1 is equivalent to f_j because $\tau(f_j)=\tau(f_{g\cdot 1})=\tau(\alpha_g(f_1))=\tau(f_1)$ and M is a type II_1 factor. Hence, there exist partial isometries u_j in M such that

(1) $u_j^*u_j=f_1$ and $u_ju_j^*=f_j$.

Hence, we see that

(2) $\alpha_g(u_j)^*\alpha_g(u_j) = f_{g\cdot 1}$, $\alpha_g(u_j)\alpha_g(u_j)^* = f_{g\cdot j}$ for each $g\in G$,

and that there exists a canonical isomorphism θ of M onto $M(n, M_1)$ such that

(3) $\theta(\sum_{i,j} u_i x_{ij} u_j^*) = [x_{ij}] \in M(n, M_1)$.

Set

(4) $\beta_g(x) = u_{g\cdot 1}^*\alpha_g(x)u_{g\cdot 1}$ for $x\in f_1Mf_1$ and $g\in G$.

Then, it is easy to check the followings by direct calculations.

(5) $\alpha_g(x) = u_{g\cdot 1}\beta_g(x)u_{g\cdot 1}^*$ for $x\in f_1Mf_1$ and $g\in G$.

(6) $\beta_g \in \mathrm{Aut}M_1$ and $\beta_{g_1g_2} = \mathrm{Ad}v(g_1, g_2)\beta_{g_1}\beta_{g_2}$ for some unitary $v(g_1, g_2)$ in M_1.

(7) $\theta\alpha_g\theta^{-1} = (\mathrm{Ad}\lambda_g V(g))\tilde{\beta}_g$, where $\lambda_g=[\delta_{g\cdot i,j}]$, $\tilde{\beta}_g([x_{ij}])=[\beta_g(x_{ij})]$ and $V(g)=[\delta_{ij}v(g)_j]$ by the unitary $v(g)_j=u_{g\cdot j}^*\alpha_g(u_j)u_{g\cdot 1}$ in M_1. Thus, for $g\in G$, $\alpha_g\in\mathrm{Int}M$ if and only if $\beta_g\in\mathrm{Int}M_1$.

For each $j\in X$, choose an element $g_j\in G$ such that $g_j\cdot 1=j$ and so we get $G=\sum_{j=1}^n g_jH$. Set $\beta_j=\beta_{g_j}$ for $j\in X$ and denote by L the subalgebra $\{[\delta_{ij}\beta_j(x)]\,;\, x\in M_1^H\}$ of $M(n, M_1)$. Then, for $x\in M_1^H$,

(8) $\theta^{-1}([\delta_{ij}\beta_j(x)]) = \sum_{j=1}^n \alpha_{g_j}(x) = \dfrac{1}{|H|}\sum_{g\in G}\alpha_g(x)$ [by (3), (5)]

so that $\theta^{-1}([\delta_{ij}\beta_j(x)])\in M^G$. Conversely, take $y\in M^G$ and set $y_1=f_1yf_1$. Then, $y=\sum_j f_jyf_j$ by the assumption $M^G\subset\sum_{j=1}^n f_jMf_j$ and $\alpha_{g_j}(y_1)=\alpha_{g_j}(f_1)y\,\alpha_{g_j}(f_1)=f_jyf_j$. Hence,

(9) $\theta^{-1}([\delta_{ij}\beta_j(y_1)]) = \sum_{j=1}^n \alpha_{g_j}(y_1) = \sum_j f_jyf_j = y$.

Thus, we see that the isomorphism θ^{-1} transforms L onto M^G.

Each element $[x_{ij}]$ in $L'\cap M(n, M_1)$ satisfies that

(10) $x_{ij}\beta_j(y) = \beta_i(y)x_{ij}$ for any $y\in M_1^H$ and $i, j\in X$.

In the case that $i=j$, the equality (10) implies that

(11) $\beta_i^{-1}(x_{ii}) \in (M^H)'\cap M_1$

and in the case that $i\neq j$,

(12) $\beta_i^{-1}(x_{ij})\beta_i^{-1}\beta_j(y) = y\beta_i^{-1}(x_{ij})$ for any $y\in M_1^H$.

By the assumption, F must be a type I factor so that M^K is a factor. We note that the restriction to the subgroup H of the cocycle crossed action β of G on M_1 is an ordinary action and that $K(\beta)=K(\beta|_H)=K(\alpha)$ holds by (6) and (7). Thus, applying Lemma 2.4, we see (i) $(M_1^H)'\cap M_1=F$ and (ii) there exist a unitary element u in M_1 and $h \in H$ such that $\beta_i^{-1}\beta_j = (\mathrm{Ad}u)\beta_h$ if $x_{ij} \neq 0$. By (11) and (i), we get $x_{ii} = \beta_i(y_i)$ for some $y_i \in F$. In the case that $i \neq j$, x_{ij} must be 0. Indeed, suppose $x_{ij} \neq 0$. Then, by (ii), $\beta_h^{-1}\beta_i^{-1}\beta_j$ is an inner automorphism of M_1 so that $h^{-1}g_i^{-1}g_j \in K(\alpha)$ by (6). This implies that $g_j \in g_i H$ because $H \supset K(\alpha)$ and so $i=j$ which is a contradiction. Hence, we get the desired conclusions. [Q. E. D.]

PROOF OF PROPOSITION 2.3. For $(M_{e_\omega}, G, \alpha)$, take $\omega = X$ and apply Lemma 2.5. The assumptions of Lemma 2.5 are clearly satisfied. We note further the following. By a suitable perturbation of unitary elements of $N_{e_\omega}(=v(K)''_{e_\omega})$, we may choose partial isometries u_j in M_{e_ω} satisfying that $u_j^* \alpha_{g_j}(v_k) u_j = f_1 v_k f_1$ $(j=1, 2, \cdots, n)$ for all $k \in K$. Thus, we see that for each $j=1, 2, \cdots, n$ $\beta_j(x)=x$ for all $x \in N_{\chi_1} = (M_1^H)' \cap M_1 = M(d_\omega, \boldsymbol{C})$. These observations imply the statement (ii) and

(1) $M_{e_\omega}^G \cong L_1^H$,

(2) $(M_{e_\omega}^G)' \cap M_{e_\omega} = (M_{e_\omega}^K)' \cap M_{e_\omega}$.

The statement (i) may follow in general from 1.5.1 in [**2**] but has been already checked at (7) in our proof of Lemma 2.5. The statement (iii) is clear by (i) and the above (1) in the same way as described in section 2.1 of [**6**]. The statement (iv) follows from (iii) and the above (2), which we need but was not found in Jones' work [**6**]. [Q. E. D.]

Now, we have the following theorem.

THEOREM 2.6. *Let M be a finite factor and α be an action of a finite group G on M. Then, we have*

$$\begin{aligned} H(M \mid M^\alpha) &= \log|G/K| + \sum_{\omega \in \Omega} \tau(e_\omega) \log(d_\omega^2 |\omega| / \tau(e_\omega)) \\ &= \log|G/K| + \sum_{\chi \in X} \tau(f_\chi) \log(d_\chi^2 / \tau(f_\chi)). \end{aligned}$$

PROOF. Assume that M is a type II_1 factor. Then, for $\omega \in \Omega$, by (ii) of Proposition 2.3, we may take minimal projections h_k $(k=1, 2, \cdots, d_\omega|\omega|)$ of $((M^G)_{e_\omega})' \cap M_{e_\omega}$ such that $\tau_\omega(h_k) = (|\omega| d_\omega)^{-1}$ for the normalized trace τ_ω of M_{e_ω} and $(h_k M h_k, h_k M^G h_k) \cong (L_\chi, L_\chi^H)$ for some $\chi \in \omega$. Thus, by (i) of Proposition 2.3 and [**7**], we get

$$[M_{h_k} : (M^G)_{h_k}] = |H/K| \quad \text{for every } k,$$

where [:] is Jones' index. Hence, applying Theorem 4.4 in [**12**], we have

$$\begin{aligned} H(M_{e_\omega}|M^G_{e_\omega}) &= 2\sum_k \eta\tau_\omega(h_k)+\sum_k \tau_\omega(h_k)\log|H/K| \\ &= \sum \tau_\omega(h_k)\log(|H/K|/\tau_\omega(h_k)^2) \\ &= \log|\omega|\,|H/K|+\log|\omega|d_\omega^2 \quad [\text{by } \tau_\omega(h_k)=(|\omega|d_\omega)^{-1}] \\ &= \log|G/K|+\log|\omega|d_\omega^2. \quad [\text{by } |\omega|=|G/H|] \end{aligned}$$

Next, applying (i) of Proposition 1.7 together with (c), (d) of Proposition 1.2 and (iii) of Proposition 2.3, we have

$$\begin{aligned} H(M|M^G) &= \sum_\omega -\tau(e_\omega)\log\tau(e_\omega)+\sum_\omega \tau(e_\omega)H(M_{e_\omega}|(M^G)_{e_\omega}) \\ &= \sum_\omega \tau(e_\omega)\{\log|G/K|+\log(|\omega|d_\omega^2/\tau(e_\omega))\} \\ &= \log|G/K|+\sum_\omega \tau(e_\omega)\log(|\omega|d_\omega^2/\tau(e_\omega)). \end{aligned}$$

The second equality is clear from $\tau(e_\omega)=|\omega|\tau(f_\chi)$ ($\chi\in\omega$). When M is a finite type I factor, these formulas follow from similar arguments to the above as a special case that $G=K(\alpha)$. [Q. E. D.]

Next, we shall concentrate our interest on the values of the relative entropy $H(M|M^\alpha)$ when α varies over all actions of G on M, and on such actions α that $H(M|M^\alpha)$ attains the maximum value. For an action α of G on a finite factor M, we name α a *Jones action* if $\tau(e_\omega)=d_\omega^2|\omega|/|K(\alpha)|$, in other words, $\tau(f_\chi)=d_\chi^2/|K(\alpha)|$. Here, our situation goes back to a general case as described in the beginning part of this section. By the reduction theory ([**5**]), we get the factor decomposition of a given finite von Neumann algebra M into a direct integral as

$$M\cong\int_Y^\oplus M(\zeta)d\nu(\zeta) \text{ and } Z(M)\cong L^\infty(Y,\nu).$$

We denote by $H(\zeta)$ the stabilizer of the action α on $L^\infty(Y,\nu)$ at $\zeta\in Y$. Then, the action of G on M induces the action α^ζ of $H(\zeta)$ on $M(\zeta)$ for ν-almost all $\zeta\in Y$. The following is easily obtained from three formulas in each case, given in Theorem 2.6, Proposition 2.2 and Proposition 2.1.

COROLLARY 2.7. *Let α be an action of a finite group G on a finite von Neumann algebra M on a separable Hilbert space with a faithful normal normalized trace τ. Then, $0\leqq H(M|M^\alpha)\leqq\log|G|$. Moreover, $H(M|M^\alpha)=\log|G|$ if and only if the action α keeps the trace τ invariant and for ν-almost all $\zeta\in Y$, the reduced actions α^ζ of $H(\zeta)$ on $M(\zeta)$ are Jones actions.*

REMARK 2.8. In [**6**], V. Jones gave complete classifications of all actions of a finite group G on the hyperfinite type II_1 factor R. Here we note that, for an action α of G on R, α is conjugate to a Jones action if and only if α is conjugate to a model action constructed by V. Jones [**6**]. Thus, by Corollary 2.7, we see that there is one and only one action α up to conjugacy in each cocycle conjugacy class such that $H(R|R^{\alpha})$ attaines $\log|G|$, which is nothing but Jones' model action. Moreover, in each conjugacy class characterized by a normal subgroup K of G and $[\lambda, \mu]\in\Lambda(G, K)$, for an arbitrary value c: $\log|G/K|\leqq c\leqq\log|G|$, one knows that there exists an action α (not necessarily unique) with $H(R|R^{\alpha})=c$. Since $H(R|R^{\alpha})$ is computed in association with the conjugacy invariants, we note that all values of $H(R|R^{\alpha})$ for an arbitrary action α of G on R are computable due to Jones' work [**6**].

Finally, we remark on the maximum value of the relative entropy $H(M\rtimes_{\alpha}G|M)$ for an action of a finite group G on a finite von Neumann algebra M. One always have $H(M\rtimes_{\alpha}G|M)\leqq\log|G|$ (see [**12**] and [**15**]). A sufficient condition that $H(M\rtimes_{\alpha}G|M)$ attains $\log|G|$ is studied by the second named author [**15**] with a direct elementary proof but another proof may be given in a slightly more general situation and more in the spirit of the paper, as follows. If $M(n, \boldsymbol{C})\subset M^{\alpha}$ with n larger than the dimensions of irreducible representations of G, then

$$H(M\rtimes_{\alpha}G|M)\geqq H(M(n, \boldsymbol{C})\otimes R(G)|M(n, \boldsymbol{C})),$$

where $R(G)$ is the group ring of G, and by 6.2 in [**12**] the last term equals $\log|G|$. Similar results hold for a twisted crossed product $W^{*}(M, G, \mu)$ by a multiplier μ of G.

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