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Local Lie algebra structure and momentum mapping

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§ 1. Introduction.

Momentum mappings for symplectic actions on a symplectic manifold are group theoretical analogues of the linear and angular momentum associated with the translational and rotational invariance. The existence of (coadjoint equivariant) momentum mappings is important in the mechanics because they give some conservative quantities. This result is known as Noether's theorem (cf. [**6**, **9**]). There are some works which discuss whether a given symplectic action admits a (coadjoint equivariant) momentum mapping or not (cf. [**6**, **9**]).

A Poisson manifold M is a differentiable manifold with a Lie algebra structure on $C^{\infty}(M)$ which is a derivation in each of its arguments. So, Poisson manifolds are a generalization of symplectic manifolds. A Poisson bracket $\{\ ,\ \}$ on M is one-to-one corresponding to an exterior contravariant 2-tensor field P on M satisfying the Schouten bracket $[P, P]=0$ (P is called a Poisson tensor on M) by the following relation $\{f, g\}=\langle P \mid df\wedge dg\rangle=-[[P, f], g]$ (cf. [**3**, **5**, **7**]). Weinstein [**10**] studied the local structure of general Poisson structures and says that every Poisson manifold is essentially a union of symplectic manifolds which fit together in a smooth way. But a general Poisson structure is quite different from the Poisson structure induced from a symplectic structure in some aspects. For example, the center of Poisson algebra $C^{\infty}(M)$ induced from the symplectic structure of M is the 0-dimensional de Rham cohomology group $H^{0}(M, \boldsymbol{R})$. The center of Poisson algebra $C^{\infty}(M)$ of the general Poisson manifold M (functions in the center of the Poisson algebra $C^{\infty}(M)$ are called Casimir functions on M) is not so obvious.

We can consider momentum mappings of natural actions on Poisson manifolds (we will call these Poisson actions) through analogy with symplectic actions on symplectic manifolds. Since Noether's theorem for momentum mappings of a Poisson action holds good (cf. [**3**, **5**]), the notion of momentum mappings is also important in Poisson manifolds with symmetry, so it is interesting to study the existence or coadjoint equivariancy of momentum mappings

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of Poisson actions. There are analogous theorems for momentum mappings of Poisson actions like the theorems of momentum mappings of symplectic actions (cf. [5]).

Corresponding to symplectic structures on even dimensional manifolds, we have contact structures on odd dimensional manifolds. The function space $C^\infty(M)$ of a contact manifold M has a natural Lie algebra structure with the Lagrange bracket and it is Lie algebra isomorphic to the Lie algebra of infinitesimal contact transformations. We can also consider momentum mappings of a group of strictly contact transformations. In this case, the coadjoint equivariancy of a momentum mapping is valid automatically.

There is a structure in $C^\infty(M)$ more general than contact structures, Poisson structures, and symplectic structures. This is called a local Lie algebra structure in $C^\infty(M)$ (cf. [2]). In this paper, we shall study local Lie algebra structures, Lie group actions preserving a local Lie algebra structure, and their momentum mappings.

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§ 2. A local Lie algebra structure in $C^\infty(M)$.

Though there is an idea of local Lie algebra structure in the space of sections of a vector bundle over a manifold M (cf. [2]), we only consider in this paper local Lie algebra structures in the space of sections of the trivial line bundle over a manifold M. The space of sections of this bundle is $C^\infty(M)$:= the space of real valued smooth functions on M.

DEFINITION 1 (cf. [2]). A local Lie algebra structure in $C^\infty(M)$ is a bracket operation in $C^\infty(M)$ satisfying the following conditions:

(1) it defines a Lie algebra structure in $C^\infty(M)$ over $\boldsymbol{R}$,

(2) $\{f, g\}$ is continuous both in f and g,

(3) $\mathrm{supp}\{f, g\} \subset \mathrm{supp} f \cap \mathrm{supp}\, g$ for each f and g, where $\mathrm{supp} f$ is the support of a function f.

Since $\{\,,\}$ is skew-symmetric, $\boldsymbol{R}$-bilinear, and its support is not increasing, there are a vector field ξ and a 2-vector field P on M from which the bracket $\{f, g\}$ of f and g is expressed as $\{f, g\} = f(\xi g) - g(\xi f) + \langle P | df \wedge dg \rangle$. ξ and P must satisfy the following relations: Lie differentiation $L_\xi P = 0$ and $\delta P \wedge P = \xi \wedge P + (1/2)\delta(P \wedge P)$, where the operator δ is defined by the formula

$$-\delta S = \sum_i \partial_i S^{ij\cdots k} \partial_j \wedge \cdots \wedge \partial_k .$$

As remarked in [2], this definition does not depend on the choice of local coordinate system. Since δ is related to the Schouten bracket, we recall the definition of the Schouten bracket. By $\wedge^k(M)$, we mean the space of exterior contravariant tensor fields on M of degree k. Let $\wedge^*(M)$ be the graded direct sum of $\wedge^k(M)$'s, that is, $\wedge^*(M)=\sum_{k=0}^{\dim M}\wedge^k(M)$. By $\wedge_k(M)$, we mean the space of exterior covariant tensor fields on M of degree k. $\wedge^0(M)=\wedge_0(M)=C^\infty(M)$. Let $\wedge_*(M)$ be the graded direct sum of $\wedge_k(M)$'s, that is, $\wedge_*(M)=\sum_{k=0}^{\dim M}\wedge_k(M)$.

DEFINITION 2 (cf. [5, 8]). The Schouten bracket is a homogeneous biderivation on $\wedge^*(M)$ of degree -1 uniquely determined by

(1) $[f, g]=0$ for all $f, g\in\wedge^0(M)$,

(2) $[X, f]=X\rfloor df=Xf$ for all $X\in\wedge^1(M)$, $f\in\wedge^0(M)$

and

(3) $[X, Y]=$ the Lie bracket of X and Y for $X, Y\in\wedge^1(M)$.

The Schouten bracket satisfies the following formulas: For each $S\in\wedge^s(M)$, $T\in\wedge^t(M)$, $U\in\wedge^u(M)$, $X\in\wedge^1(M)$, and $f_j\in\wedge^0(M)$, we have

(4) $[S, T\wedge U]=[S, T]\wedge U+(-1)^{(s-1)t}T\wedge[S, U]$,

(5) $[S\wedge T, U]=S\wedge[T, U]+(-1)^{t(u-1)}[S, U]\wedge T$,

(6) $[S, T]=(-1)^{(s-1)(t-1)+1}[T, S]$,

(7) $\underset{S,T,U}{\mathfrak{S}}(-1)^{(s-1)(u-1)}[[S, T], U]=\underset{S,T,U}{\mathfrak{S}}(-1)^{(s-1)(u-1)}[S, [T, U]]=0$,

where $\mathfrak{S}$ means the cyclic sum with respect to S, T, U. This formula is called the generalized Jacobi identity.

(8) $[X, S]=L_XS$, where L_X is the Lie differentiation with respect to X,

and

(9) $[[\cdots[[S, f_1], f_2], \cdots], f_s]=-\langle S|df_1\wedge df_2\wedge\cdots\wedge df_s\rangle$, $(s\geqq 2)$.

REMARKS. (i) We can get the Schouten bracket on $\wedge^*(M)$ inductively from (1), (2), (3), (4), and (6).

(ii) We can also define the Schouten bracket $[S, T]$ axiomatically; For $S\in\wedge^s(M)$, let i_S be the inner product of S, i.e., $\langle T|i_S\mu\rangle=\langle S\wedge T|\mu\rangle$, and let L_S be

$$L_S:=[d, i_S]_l:=d\circ i_S-(-1)^s i_S\circ d,$$

i.e., the left-twisted derivation of d and i_S, which is a generalization of the Lie differentiation. For $S\in\wedge^s(M)$ and $T\in\wedge^t(M)$, we have a derivation $j(S, T)$ on $\wedge_*(M)$ defined by

$$j(S, T):=[L_S, i_T]_r:=(-1)^{(s-1)t}L_S\circ i_T-i_T\circ L_S,$$

i.e., the right-twisted derivation of L_S and i_T. Then we can get the Schouten bracket $[S, T]$ by $i_{[S,T]}:=j(S, T)$ on $\wedge_*(M)$. The above definitions of L_S or

$j(S, T)$ are slightly different from those of [5]. The Schouten bracket $[S, T]$ in this paper is equal to $-[T, S]$ in [8].

(iii) If we define the degree of $S \in \wedge^s(M)$ as $s-1$, then the Schouten bracket makes $\wedge^*(M)$ a Lie superalgebra in the sense of [2].

The local Lie algebra structure in $C^\infty(M)$ can be written as $\{f, g\}=f[\xi, g]-g[\xi, f]-[[P, f], g]$. ξ and P must satisfy the following relations $[\xi, P]=0$ and $[P, P]+2\xi\wedge P=0$. These equations are corresponding to the Jacobi identity of $\{\,,\}$. Now we review some results of transitive local Lie algebra structures, which say that local Lie algebra structures are more general than symplectic or contact structures. "Transitive" means that the vector fields ξ and $[P, f]$ $(f \in C^\infty(M))$ span the tangent space of M everywhere.

PROPOSITION 2.1 ([2]). *Every transitive local Lie algebra structure* $\{\,,\}$ *in* $C^\infty(M^{2n})$ *is written locally as* $\{f, g\}=e^{-r}\{e^r f, e^r g\}_0$, *where* $\{\,,\}_0$ *is the Poisson bracket of a local symplectic structure and r is a function on some neighbourhood at each point.*

PROPOSITION 2.2 ([2]). *Let* M^{2n+1} *be an odd dimensional manifold. Then every transitive local Lie algebra structure in* $C^\infty(M^{2n+1})$ *is determined by the Lagrange bracket of some contact structure on M.*

By the analogy to Hamiltonian vector fields on symplectic manifolds or infinitesimal contact transformations on contact manifolds (cf. [1]), we consider a linear map β_0 of $\wedge^0(M)$ into $\wedge^1(M)$ defined by $\beta_0(f):=f\xi-[f, P]$. Also we define linear mappings β_s of $\wedge^s(M)$ into $\wedge^{s+1}(M)$ by $\beta_s(S):=S\wedge\xi-[S, P]$. Using the generalized Jacobi identity for the Schouten bracket and the Jacobi identity for the local Lie algebra structure, that is, $[\xi, P]=0$ and $[P, P]+2\xi\wedge P=0$, we have

PROPOSITION 2.3. (1) β_0 *is a Lie algebra homomorphism of* $\wedge^0(M)$ *into* $\wedge^1(M)$.

(2) $\beta_{s+1}\circ\beta_s(S)=-[[P, P], S]/2$ *holds for each* $S \in \wedge^s(M)$ *and* $s=0, \cdots, \dim M-1$.

(3) *If* $[P, P]=0$ *identically, then* $\{\wedge^s(M), \beta_s\}_{s=0}^{\dim M}$ *forms a cochain complex.*

REMARKS. (i) In the case of a Poisson manifold, (3) of the above Proposition is already known in [5].

(ii) Because $[P, P]=0$ is equivalent to $\xi\wedge P=0$, any local Lie algebra structure in $C^\infty(M^{2n+1})$ satisfying $[P, P]=0$ does not define a contact structure.

Casimir functions on a symplectic manifold are the constant functions on each connected component of M. We use the following notations: $\wedge^0_\xi(M):=\{f \in \wedge^0(M) \mid [f, \xi]=0\}$, and $Z:=$ the center of $\wedge^0(M):=$ the space of Casimir functions. For Casimir functions of a local Lie algebra structure in $C^\infty(M)$, we have

PROPOSITION 2.4 ([2]). (1) *$\wedge_\xi^0(M)$ is a subalgebra of $\wedge^0(M)$.*

(2) *f is a Casimir function if and only if f satisfies $[f, \xi]=0$ and $f\xi-[f, P]=0$, i.e., $Z=\wedge_\xi^0(M)\cap\mathrm{Ker}(\beta_0)$.*

PROOF. (1) Take $f, g\in\wedge_\xi^0(M)$. Since $\{f, g\}=-[[P, f], g]$ and $[\xi, P]=0$, we have $[\xi, \{f, g\}]=[\xi, -[[P, f], g]]=0$.

(2) Let f be a Casimir function. Since $0=\{f, 1\}=f[\xi, 1]-[\xi, f]-[[P, f], 1]=-[\xi, f]$ and so $\{f, g\}=f[\xi, g]-g[\xi, f]-[[P, f], g]=[f\xi-[P, f], g]$, we have $[\xi, f]=0$ and $f\xi-[P, f]=0$. If $[\xi, f]=0$ and $f\xi-[P, f]=0$, then $\{f, g\}=f[\xi, g]-[f\xi, g]=0$.

§ 3. Actions and momentum mappings.

Consider a finite dimensional connected Lie group G with its Lie algebra $\mathfrak{g}$. G acts on $\mathfrak{g}$ as the adjoint representation and acts on $\mathfrak{g}^*$ as the coadjoint representation: for each $a\in G$,

$$a\cdot\zeta := \mathrm{Ad}(a)(\zeta) \qquad (\zeta\in\mathfrak{g}), \quad \text{and}$$
$$a\cdot\mu := \mathrm{Ad}(a^{-1})^*(\mu) = \mu\circ\mathrm{Ad}(a^{-1}) \qquad (\mu\in\mathfrak{g}^*).$$

Now let G act on a manifold M by $\phi: G\times M\to M$. Then we have three more induced actions: for each $a\in G$,

(1) an action on a space of maps on M: $\underline{a}\cdot F:=F\circ\phi_{a^{-1}}$

(2) an action on a space of $\mathfrak{g}^*$-valued functions: $\underline{\underline{a}}\cdot F:=\mathrm{Ad}(a^{-1})^*\circ F$,

(3) an action on the Lie algebra $\wedge^1(M)$ of vector fields on M with the Lie bracket: $a\cdot X:=T(\phi_a)(X)$.

For each $\zeta\in\mathfrak{g}$, we have the infinitesimal generator (or the fundamental vector field) $\rho(\zeta)$ of $-\zeta$ on M defined by

$$\rho(\zeta)(f) = [\rho(\zeta), f] := \frac{d}{dt}((\underline{\exp t\zeta})\cdot f)\Big|_{t=0}$$

for each $f\in C^\infty(M)$. ρ is a Lie algebra homomorphism of $\mathfrak{g}$ into $\wedge^1(M)$ and satisfies $\rho(a\cdot\zeta)=a\cdot(\rho(\zeta))$ for each $a\in G$ and $\zeta\in\mathfrak{g}$.

Assume that G acts on the manifold M and preserves the local Lie algebra structure, that is, $\underline{a}\cdot\{f, g\}=\{\underline{a}\cdot f, \underline{a}\cdot g\}$ holds for each $a\in G$ and $f, g\in C^\infty(M)$.

Let X be a vector field on M and $\{\psi_t\}$ be the flow of X. $\{\psi_t\}$ preserves the local Lie algebra structure (i.e., $\psi_t^*\{f, g\}=\{\psi_t^*f, \psi_t^*g\}$ for each t, and $f, g\in\wedge^0(M)$) if and only if $[X, \xi]=0$ and $[X, P]=0$. We say that the vector field X preserves the local Lie algebra structure if $[X, \xi]=0$ and $[X, P]=0$

hold, that is, if the flow of X preserves the local Lie algebra structure. Thus we have

PROPOSITION 3.1. *The action of G preserves the local Lie algebra structure in $C^\infty(M)$ if and only if it leaves ξ and P invariant, that is, all the infinitesimal generators $\rho(\zeta)$ of the action of G preserve the local Lie algebra structure.*

PROPOSITION 3.2. *For $f\in\wedge^0(M)$, $\beta_0(f)$ preserves the local Lie algebra structure if and only if $[f, \xi]=0$, that is, $f\in\wedge^0_\xi(M)$. In particular, $\wedge^0_\xi(M)$ is G-invariant.*

PROOF. For each $f\in\wedge^0(M)$, we have

$$\begin{aligned}[\beta_0(f), \xi] &= [f, \xi]\xi-[[f, P], \xi]\\ &= [f, \xi]\xi-\{[[P, \xi], f]-[[\xi, f], P]\}\\ &= [f, \xi]\xi-[[f, \xi], P]\end{aligned}$$

and

$$\begin{aligned}[\beta_0(f), P] &= f[\xi, P]-[f, P]\wedge\xi-[[f, P], P]\\ &= -[f, P]\wedge\xi+[[P, P], f]/2\\ &= -[f, P]\wedge\xi-[\xi\wedge P, f]\\ &= -[f, P]\wedge\xi-\xi\wedge[P, f]-[\xi, f]P\\ &= [f, \xi]P.\end{aligned}$$

If $[f, \xi]=0$, then we have $[\beta_0(f), \xi]=0$ and $[\beta_0(f), P]=0$ from the above equations. Conversely, let $[\beta_0(f), \xi]=0$ and $[\beta_0(f), P]=0$, that is, $[f, \xi]\xi-[[f, \xi], P]=0$ and $[f, \xi]P=0$. Take an arbitrary point $x\in M$. If $P\neq 0$ at x, then we have $[f, \xi]=0$ at x. If $P=0$ at x, then we have $[f, \xi]\xi=0$ at x from the first equation. If $\xi\neq 0$ at x, then $[f, \xi]=0$. If $\xi=0$ at x, then $[f, \xi]=0$ in general. Therefore we have $[f, \xi]=0$. Using the generalized Jacobi's identity, we have

$$[[\rho(\zeta), f], \xi] = -[[f, \xi], \rho(\zeta)]-[[\xi, \rho(\zeta)], f] = 0$$

because G preserves the local Lie algebra structure. Thus we complete the proof of Proposition 3.2.

Let Γ be an abelian group of $\mathfrak{g}^*$-valued functions on M. Define the k-th cochain complex $C^k(G, \Gamma)$ ($k=0, 1, \cdots$) as the space of maps of $G\times G\times\cdots\times G$ (k-times) into Γ. Define additive maps $\partial_k : C^k(G, \Gamma)\to C^{k+1}(G, \Gamma)$ ($k=0, 1, \cdots$) by

$$\begin{aligned}(\partial_k F)(a_1, \cdots, a_k, a_{k+1}) &= a_1\cdot F(a_2, \cdots, a_{k+1})\\ &+\sum_{i=1}^{k}(-1)^i F(\cdots, a_i a_{i+1}, \cdots)+(-1)^{k+1}F(a_1, \cdots, a_k).\end{aligned}$$

By a direct calculation, we have

PROPOSITION 3.3. *$\partial_{k+1}\circ\partial_k=0$ holds for $k=0, 1, \cdots$ and so there are cohomology groups $H^k(G, \Gamma):=\mathrm{Ker}(\partial_k)/\mathrm{Im}(\partial_{k-1})$ of G with the coefficient Γ.*

By analogy to momentum mappings of symplectic actions, we have the notion of momentum mappings of the action of G which preserves the local Lie algebra structure. The definition of momentum mapping for the action is as follows:

DEFINITION 3. A map $J: M\to\mathfrak{g}^*$ is a momentum mapping of the action of G if $\beta_0\circ\hat{J}=\rho$ holds good on $\mathfrak{g}$, where $\hat{J}: \mathfrak{g}\to\wedge^0(M)$ is defined by $\hat{J}(\zeta)(x)=\langle\zeta, J(x)\rangle$ for each $\zeta\in\mathfrak{g}$ and $x\in M$ and is called the co-momentum mapping of J.

For momentum mappings of the action of G preserving a local Lie algebra structure, we have the same theorem as Noether's theorem by adding one more condition.

NOETHER'S THEOREM. *Let f be a G-invariant function on M. Then $\hat{J}(\zeta)$ is a first integral of $\beta_0(f)$ for each $\zeta\in\mathfrak{g}$ if f is invariant by ξ.*

For the existence of momentum mappings of the action of G, we have

PROPOSITION 3.4. *If the action of G admits a momentum mapping, then $\mathrm{Im}(\beta_1\circ\rho)$ is included in $[[P, P], \wedge^0(M)]$.*

PROOF. Since the action of G admits a momentum mapping, for each $\zeta\in\mathfrak{g}$ we have $g_\zeta\in\wedge^0_\xi(M)$ such that $\rho(\zeta)=\beta_0(g_\zeta)$. Therefore, we have $\beta_1\circ\rho(\zeta)=\beta_1\circ\beta_0(g_\zeta)=-[[P, P], g_\zeta]/2$ from Proposition 2.4 (2).

REMARK. Proposition 3.4 means that $\beta_1\circ\rho=0$ if the local Lie algebra structure is defined by the pair $(0, P)$, i.e., if it is a Poisson structure (cf. [**5**, **9**]).

J is called coadjoint equivariant if and only if $J(a\cdot x)=a\cdot(J(x))$ holds for each $a\in G$ and $x\in M$, or equivalently, $\underline{a}^{-1}\cdot(\hat{J}(\zeta))=\hat{J}(a^{-1}\cdot\zeta)$ holds for each $a\in G$ and $\zeta\in\mathfrak{g}$.

Now let $\theta(a, \zeta)$ be $\underline{a}^{-1}\cdot\hat{J}(\zeta)-\hat{J}(a^{-1}\cdot\zeta)$. It is an obstruction of coadjoint equivariancy of momentum mapping J.

PROPOSITION 3.5. *$\zeta\mapsto\theta(a, \zeta)$ is linear and $\theta(a, \zeta)$ is a Casimir function on M for each $a\in G$ and $\zeta\in\mathfrak{g}$.*

PROOF. Linearity of $\zeta\mapsto\theta(a, \zeta)$ is obvious from the definition of momentum mapping J. Since we have

$$\beta_0(\theta(a,\zeta)) = \beta_0(\underline{a}^{-1}\cdot\hat{J}(\zeta)) - \beta_0(\hat{J}(a^{-1}\cdot\zeta))$$
$$= a^{-1}\cdot(\beta_0\circ\hat{J}(\zeta)) - \rho(a^{-1}\cdot\zeta) = 0$$

for each $a\in G$ and $\zeta\in\mathfrak{g}$, $\theta(a,\zeta)$ is contained in $\mathrm{Ker}(\beta_0)$. From Proposition 3.2 $\hat{J}(\zeta)$, $\hat{J}(a^{-1}\cdot\zeta)$ and $\theta(a,\zeta)$ are contained in $\wedge_\xi^0(M)$. Therefore, $\theta(a,\zeta)$ is a Casimir function on M by Proposition 2.4.

LEMMA 1. *Each Casimir function on M is $\underline{G}$-invariant under the assumption that the action of G admits a momentum mapping.*

PROOF. Let f be an arbitrary Casimir function on M, that is, f satisfies $[f,\xi]=0$ and $f\xi-[f,P]=0$. It is sufficient only to prove that $[f,\rho(\zeta)]=0$ for each $\zeta\in\mathfrak{g}$ because G is connected. Since the action of G has a momentum mapping, there is a function g in $\wedge_\xi^0(M)$ satisfying

$$\rho(\zeta) = \beta_0(g) = g\xi - [g, P].$$

We have

$$[f,\rho(\zeta)] = [f, g\xi-[g,P]] = g[f,\xi]-[f,[g,P]]$$
$$= [g,[f,P]] = [g, f\xi] = f[g,\xi] = 0.$$

This lemma is the key lemma in our discussion of coadjoint equivariancy of momentum mappings. In the case of symplectic manifolds, each Casimir function is automatically $\underline{G}$-invariant because it is locally constant. From this Lemma 1, we can develop similar discussions as in the symplectic category and get a theorem which is a generalization of coadjoint equivariancy theorems of momentum mappings for symplectic or Poisson actions.

LEMMA 2. *$a\mapsto\Theta(a):=\underline{a}^{-1}\cdot J-\underline{\underline{a}}\cdot J$ is a cocycle in $C^1(G, Z\otimes\mathfrak{g}^*)$.*

LEMMA 3. *Let J_1 and J_2 be momentum mappings of the action of G on M, and Θ_1 and Θ_2 be the corresponding cocycles of J_1 and J_2 respectively. Then Θ_1 and Θ_2 are cohomologous in $C^1(G, Z\otimes\mathfrak{g}^*)$.*

The result which we obtained is the following:

THEOREM. *Let G be a connected Lie group acting on M, preserving the local Lie algebra structure in $C^\infty(M)$, and having a momentum mapping J. Then G admits a coadjoint equivariant momentum mapping if and only if $[\Theta]=0$ in $H^1(G, Z\otimes\mathfrak{g}^*)$, where $\Theta(a)=\underline{a}^{-1}\cdot J-\underline{\underline{a}}\cdot J$ and Z is the space of all Casimir functions on M.*

PROOF. Assume that $[\Theta]=0$ in $H^1(G, Z\otimes\mathfrak{g}^*)$. Then there is a 0-dimensional cochain $F\in C^0(G, Z\otimes\mathfrak{g}^*)=Z\otimes\mathfrak{g}^*$ such that $\Theta=\partial_0 F$. Then $\Theta(a)=\underline{\underline{a}}\cdot F-F$ holds for each $a\in G$. Define $\hat{J}_1$ by $\hat{J}_1(\zeta):=\hat{J}(\zeta)+\langle F,\zeta\rangle$. Since $\zeta\mapsto\hat{J}_1(\zeta)$ is linear from $\mathfrak{g}$ into $C^\infty(M)$ and

$$\beta_0(\hat{J}_1(\zeta)) = \beta_0(\hat{J}(\zeta)) + \beta_0(\langle F, \zeta\rangle)$$
$$= \rho(\zeta)$$

holds good, J_1 is a momentum mapping of the action of G. Coadjoint equivariancy of J_1 comes from

$$\begin{aligned}\hat{J}_1(a\cdot\zeta) &= \hat{J}(a\cdot\zeta) + \langle F, a\cdot\zeta\rangle \\ &= \underline{a}\cdot\hat{J}(\zeta) - \langle\Theta(a^{-1}), \zeta\rangle + \langle F, a\cdot\zeta\rangle \\ &= \underline{a}\cdot(\hat{J}(\zeta) - \langle F, \zeta\rangle) - \langle\Theta(a^{-1}) - \underline{\underline{a}}\cdot F, \zeta\rangle \\ &= \underline{a}\cdot\hat{J}_1(\zeta) - \langle F + \Theta(a^{-1}) - \underline{\underline{a}}\cdot F, \zeta\rangle \\ &= \underline{a}\cdot\hat{J}_1(\zeta)\end{aligned}$$

using that Casimir function F is $\underline{G}$-invariant. If the action of G has a coadjoint equivariant momentum mapping, then $\Theta=0$ identically, and of course $[\Theta]=0$. This completes the proof of our theorem.

COROLLARY. *If the Lie group G is compact, then the action of G admits a coadjoint equivariant momentum mapping under the assumptions of the theorem.*

PROOF. It is sufficient only to show that $H^1(G, Z\otimes\mathfrak{g}^*)=0$ if G is compact. Let F be a k-cocycle. Then we have

$$\begin{aligned}0 = (\partial_k F)(a_1, \cdots, a_k, a_{k+1}) &= \underline{\underline{a}}_1\cdot F(a_2, \cdots, a_{k+1}) \\ &+ \sum_{i=1}^{k}(-1)^i F(\cdots, a_i a_{i+1}, \cdots) + (-1)^{k+1}F(a_1, \cdots, a_k).\end{aligned}$$

By integrating the above in a_{k+1} with the normalized Haar measure of G, we have

$$\begin{aligned}0 = \underline{\underline{a}}_1\cdot f(a_2, \cdots, a_k) &+ \sum_{i=1}^{k-1}(-1)^i f(\cdots, a_i a_{i+1}, \cdots) \\ &+ (-1)^k f(a_1, \cdots, a_{k-1}) + (-1)^{k+1}F(a_1, \cdots, a_k),\end{aligned}$$

where $f(a_2, \cdots, a_k) = \int_G F(a_2, \cdots, a_k, a_{k+1})dG(a_{k+1})$. Therefore, we have $F=(-1)^k\partial_{k-1}(f)$, and $H^k(G, Z\otimes\mathfrak{g}^*)=\{0\}$ $(k=1, 2, \cdots)$.

REMARKS. (i) In the case of symplectic actions, if M is connected, then the center $Z=\boldsymbol{R}$, $H^1(G, Z\otimes\mathfrak{g}^*)$ is the usual cohomology group of G relative to the G-module $\mathfrak{g}^*$, and $[\Theta]$ is just the symplectic cohomology of a symplectic action of G in the sense of Souriau [**6**].

(ii) Let J be a momentum mapping of a symplectic action of G on a connected symplectic manifold M, and Θ be the corresponding 1-cocycle of J. Then $(a, \mu)\mapsto a\cdot\mu+\Theta(a^{-1})$ is an affine action on $\mathfrak{g}^*$ and J is equivariant with respect to the action of G on M and this affine action of G on $\mathfrak{g}^*$ (cf. [**6**]). $[\Theta]=0$

holds in $H^1(G, \mathfrak{g}^*)$ if and only if the above affine action of G on $\mathfrak{g}^*$ has a fixed point. Even in the case of Poisson actions, we do not have similar properties as above in general (cf. [**3**]).

When the local Lie algebra structure is defined from a symplectic structure, $(\zeta, \eta)\mapsto\{\hat{J}(\zeta), \hat{J}(\eta)\}-\hat{J}([\zeta, \eta])$ is a 2-cocycle of $\mathfrak{g}$ with the coefficient of the trivial $\mathfrak{g}$-module $\boldsymbol{R}$ and we have some results of coadjoint equivariancy of momentum mappings of symplectic actions stated by the cohomology groups of the Lie algebra $\mathfrak{g}$ of G with the coefficient of the trivial $\mathfrak{g}$-module $\boldsymbol{R}$. Lemma 1 admits us to generalize some of these results of symplectic actions in our case.

Let Γ' be a subspace of $C^\infty(M)=\wedge^0(M)$. $\mathfrak{g}$ acts on Γ' naturally as follows: $\zeta\cdot f:=\rho(\zeta)f=[\rho(\zeta), f]$. Then Γ' is a representation of $\mathfrak{g}$. Define the k-th cochain complex $C^k(\mathfrak{g}, \Gamma')$ ($k=0, 1, \cdots$) as the space of skew-symmetric multilinear maps of $\mathfrak{g}\times\mathfrak{g}\times\cdots\times\mathfrak{g}$ (k-times) into Γ'. Define additive maps $\partial_k: C^k(\mathfrak{g}, \Gamma')\to C^{k+1}(\mathfrak{g}, \Gamma')$ ($k=0, 1, \cdots$) by

$$(\partial_k F)(\zeta_1, \cdots, \zeta_k, \zeta_{k+1})=\sum_{i=1}^{k+1}(-1)^{i+1}\zeta_i\cdot F(\zeta_1, \cdots, \hat{\zeta}_i, \cdots, \zeta_{k+1})$$
$$+\sum_{i<j}(-1)^{i+j}F([\zeta_i, \zeta_j], \zeta_1, \cdots, \hat{\zeta}_i, \cdots, \hat{\zeta}_j, \cdots, \zeta_{k+1}),$$

where "^" means omitting the element at that position. By a direct calculation $\partial_{k+1}\circ\partial_k=0$ holds for $k=0, 1, \cdots$ and so we have the cohomology groups $H^k(\mathfrak{g}, \Gamma')$ of $\mathfrak{g}$ with the coefficient $\mathfrak{g}$-module Γ' ($k=0, 1, 2, \cdots$).

Let J be a momentum mapping of the action of G preserving the local Lie algebra structure. J is coadjoint equivariant if and only if $\underline{a}\cdot\hat{J}(\zeta)=\hat{J}(a\cdot\zeta)$ for each $a\in G$ and $\zeta\in\mathfrak{g}$. This is equivalent to $[\rho(\eta), \hat{J}(\zeta)]=\hat{J}([\eta, \zeta])$ for each $\zeta, \eta\in\mathfrak{g}$ if G is connected. From Proposition 3.2, $\hat{J}(\eta)\in\wedge^0_\xi(M)$ holds, and $\rho(\eta)=\beta_0\circ\hat{J}(\eta)$ from the definition of momentum mapping. Therefore, we have $\rho(\eta)=-[\hat{J}(\eta), P]$ and so $[\rho(\eta), \hat{J}(\zeta)]=\{\hat{J}(\eta), \hat{J}(\zeta)\}$. This means that J is a coadjoint equivariant momentum mapping if and only if the co-momentum mapping $\hat{J}: \mathfrak{g}\to\wedge^0_\xi(M)$ of J is Lie algebra homomorphic. So, for each momentum mapping J, we shall study the mapping

$$(\eta, \zeta)\longmapsto\{\hat{J}(\eta), \hat{J}(\zeta)\}-\hat{J}([\eta, \zeta]),$$

which measures the difference from Lie algebra homomorphism. We see that this mapping is a 2-cocycle of $\mathfrak{g}$ with the coefficient $Z:=$ the center of $\wedge^0(M)$, i.e., the space of Casimir functions on M, and it defines a cohomology class in $H^2(\mathfrak{g}, Z)$, which is a generalization of the infinitesimal symplectic cohomology in [**6**]. We have

PROPOSITION 3.6. *Assume that the action of G on M preserves the local Lie algebra structure and has a momentum mapping. If $H^2(\mathfrak{g}, Z)=0$, then the action of G admits a coadjoint equivariant momentum mapping, where Z is the center* *of* $C^{\infty}(M)$.

PROOF. Let J be a momentum mapping. Then

$$c(\eta, \zeta) := \{\hat{J}(\eta), \hat{J}(\zeta)\} - \hat{J}([\eta, \zeta])$$

is a 2-cocycle of $\mathfrak{g}$ and we have a 1-cocycle $c_1 \in C^1(\mathfrak{g}, Z)$ satisfying $c = \partial_1 c_1$ because of $H^2(\mathfrak{g}, Z) = 0$. From Lemma 1, we have

$$\begin{aligned} c(\eta, \zeta) &= (\partial_1 c_1)(\eta, \zeta) \\ &= [\rho(\eta), c_1(\zeta)] - [\rho(\zeta), c_1(\eta)] - c_1([\eta, \zeta]) \\ &= -c_1([\eta, \zeta]). \end{aligned}$$

Since

$$\beta_0(\hat{J}(\zeta) - c_1(\zeta)) = \beta_0 \circ \rho(\zeta)$$

and

$$\begin{aligned} \{\hat{J}(\eta) - c_1(\eta), \hat{J}(\zeta) - c_1(\zeta)\} &= \{\hat{J}(\eta), \hat{J}(\zeta)\} \\ &= \hat{J}([\eta, \zeta]) - c_1([\eta, \zeta]) \end{aligned}$$

hold good, $\zeta \mapsto \hat{J}(\zeta) - c_1(\zeta)$ is a Lie algebra homomorphic co-momentum mapping of the action of G.

REMARK. The second cohomology group $H^2(\mathfrak{g}, Z)$ of $\mathfrak{g}$ is not equal to the first cohomology group $H^1(G, Z \otimes \mathfrak{g}^*)$ of G (cf. [**6**]).

By using the Jacobi identity of the local Lie algebra structure, the co-momentum mappings being Lie algebra homomorphic and Lemma 1, we also have a generalization of the theorem in [**4**].

PROPOSITION 3.7. *Let* $G_1 \rtimes G_2$ *be a connected semidirect product Lie group acting on* M *and preserving the local Lie algebra structure. Assume that the actions of* G_1 *and* G_2 *have coadjoint equivariant momentum mappings respectively. If* $H^1(\mathfrak{g}_1, \boldsymbol{R}) = 0$, *then* $G_1 \rtimes G_2$ *has a coadjoint equivariant momentum mapping.*

§ 4. Examples.

First we recall

PROPOSITION 4.1 (cf. [**2**]). *Let* $\{\ ,\ \}$ *be a local Lie algebra structure in* $C^{\infty}(M)$ *defined by the pair* (ξ, P). *Then the distribution spanned by* ξ *and* $[f, P]$ $(f \in \wedge^0(M))$ *is involutive.*

REMARK. If there exists an integral submanifold of the distribution in Proposition 4.1, then the restricted local Lie algebra structure is transitive, and so the manifold must be a contact manifold or a symplectic manifold locally by Proposition 2.1 or 2.2. In the case of Poisson manifolds, we refer to [**10**] for more details.

If we have a momentum mapping for the action of G preserving the local Lie algebra structure defined by the pair (ξ, P), then $\mathrm{Im}(\rho)\subset\mathrm{Im}(\beta_0)$ holds. Since $\dim\mathrm{Im}(\beta_0)$ is equal to the rank of the local Lie algebra structure, we get an elementary necessary condition of the existence of a momentum mapping:

PROPOSITION 4.2. *Let G be a group acting on M, preserving the local Lie algebra structure in $C^\infty(M)$ defined by (ξ, P). If the action of G has a momentum mapping, then the dimension of each G-orbit $G\cdot x$ is no more than the rank of the local Lie algebra structure at x.*

REMARK. If the local Lie algebra structure is transitive, for example, if it is induced from a symplectic structure, then the above Proposition 4.2 is trivial and does not work effectively.

Taking the above propositions into consideration, we will construct some examples of local Lie algebra structures of $\boldsymbol{R}^4$ and group actions with or without momentum mappings.

EXAMPLE 1. Let (x^1, x^2, y^1, y^2) be the canonical coordinate system of $\boldsymbol{R}^4$, ξ be $\sum_{j=1}^{2}(x^j\partial/\partial y^j - y^j\partial/\partial x^j)$, and

$$\begin{aligned} P=&\frac{1}{4}\sum_{i,j}(x^iy^j-x^jy^i)\left(\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}+\frac{\partial}{\partial y^i}\wedge\frac{\partial}{\partial y^j}\right)\\ &+\frac{1}{2}\sum_{i,j}(-x^ix^j-y^iy^j)\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial y^j}\\ &+\frac{c}{2}\sum_{i}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial y^i}, \end{aligned}$$

where c is a constant. Then the pair of (ξ, P) defines a local Lie algebra structure in $\boldsymbol{R}^4$. (If $c=1$ and restrict ξ and P to $S^3(1)$, then the local Lie algebra structure (ξ, P) is the one of the standard contact structure of $S^3(1)$.)

Let G be the connected Lie subgroup of $GL(4, \boldsymbol{R})$ whose Lie algebra is generated by

$$\zeta_1=\begin{bmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0 \end{bmatrix},\quad \zeta_2=\begin{bmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0 \end{bmatrix},$$

$$\zeta_3=\begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix},\quad \zeta_4=\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

ζ_j's satisfy $[\zeta_1, \zeta_2]=2(\zeta_3-\zeta_4)$, $[\zeta_1, \zeta_3]=-\zeta_2$, $[\zeta_1, \zeta_4]=\zeta_2$, $[\zeta_2, \zeta_3]=\zeta_1$, $[\zeta_2, \zeta_4]=-\zeta_1$, $[\zeta_3, \zeta_4]=0$, and $\xi=-\rho(\zeta_3+\zeta_4)$. Then G preserves the local Lie algebra structure. If $c=0$, then the action of G does not have any momentum mappings. If $c\neq 0$, then the action of G has a coadjoint equivariant momentum mapping J defined by $\hat{J}(\zeta_k)=f_k$, where

$$f_1=\frac{2}{c}(-x^1y^2+x^2y^1), \qquad f_2=\frac{2}{c}(x^1x^2+y^1y^2),$$

$$f_3=\frac{1}{c}(x^1x^2+y^1y^2), \quad \text{and} \quad f_4=\frac{1}{c}(x^2x^2+y^2y^2).$$

EXAMPLE 2. Now we consider Poisson structures in $\boldsymbol{R}^4$. Let (x^1, x^2, x^3, x^4) be the natural coordinates of $\boldsymbol{R}^4$ and $D_i=\partial/\partial x^i$ $(i=1, 2, 3, 4)$.

(2.1) The natural linear action of $GL(2, \boldsymbol{R})$ on $\boldsymbol{R}^4$ is a Poisson action for a Poisson tensor $P=(x^1D_1+x^2D_2)\wedge(x^3D_3-x^4D_4)+u(x^3x^4)D_3\wedge D_4$, where u is a function of one variable, but it has no momentum mapping.

(2.2) The natural linear action of $SL(2, \boldsymbol{R})$ on $\boldsymbol{R}^4$ is a Poisson action for a Poisson tensor

$$P=x^3x^4D_1\wedge D_2+(x^1D_1+x^2D_2)\wedge(-x^3D_3+x^4D_4)+2x^3x^4D_3\wedge D_4$$

without momentum mappings. The action of $SL(2, \boldsymbol{R})$ is also a Poisson action for $P=D_1\wedge D_2+v(x^3, x^4)D_3\wedge D_4$ for any function v, and it has a coadjoint equivariant momentum mapping

$$\langle J(x), \zeta\rangle=\det\begin{bmatrix} x^1 & \zeta\begin{bmatrix} x^1 \\ x^2\end{bmatrix} \\ x^2 & \end{bmatrix}\Big/2.$$

(2.3) The natural $S^1\times S^1$-action on $\boldsymbol{R}^4$ is a Poisson action for

$$\begin{aligned} P=&(x^3x^3+x^4x^4)D_1\wedge D_2-x^2x^3D_1\wedge D_3-x^2x^4D_1\wedge D_4 \\ &+x^1x^3D_2\wedge D_3+x^1x^4D_2\wedge D_4, \end{aligned}$$

but has no momentum mapping.

(2.4) The natural linear action of $SO(3)$ on $\boldsymbol{R}^4$ is a Poisson action for

$$\begin{aligned} P=&u(x^1x^1+x^2x^2+x^3x^3)(x^1D_2\wedge D_3+x^2D_3\wedge D_1+x^3D_1\wedge D_2) \\ &+v(x^1x^1+x^2x^2+x^3x^3, x^4)(x^1D_1+x^2D_2+x^3D_3)\wedge D_4, \end{aligned}$$

where u and v are functions satisfying the following: For some positive ε, $v(r, s)=0$ for $|r|\leqq\varepsilon$ and $u(r)$ is a non-zero function which is equal to $|r|^{1/2}$ for $|r|\geqq\varepsilon/2$. Then the Poisson action of $SO(3)$ has a coadjoint equivariant momentum mapping

$$\langle J(x), (\zeta_1, \zeta_2, \zeta_3)\rangle=-\Big(\sum_{i=1}^{3}x^i\zeta_i\Big)\Big/u\Big(\sum_{i=1}^{3}x^ix^i\Big),$$

where the dual space of the Lie algebra of $SO(3)$ is identified with $\boldsymbol{R}^3$ in the standard way.

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