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A stochastic solution of a high order parabolic equation

By Kunio NISHIOKA

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§ 0. Introduction.

Our purpose in this paper is to solve the following initial value problem by a stochastic method, using an extension of a Girsanov type formula as in [4].

$$\frac{\partial W}{\partial t}(t, x)=(A+B)W(t, x), \qquad t>0, \ x\in \boldsymbol{R}^{d}, \tag{0.1, i}$$

$$W(0, x)=f(x), \tag{0.1, ii}$$

where

$$A=(-1)^{q-1}\rho\sum_{k=1}^{d}\left(\frac{\partial}{\partial x_{k}}\right)^{2q},$$

q is a natural number, and ρ is a complex number such that $\mathrm{Re}\,\rho>0$, and

$$B=\sum_{|\alpha|\leqq 2q}b_{\alpha}(x)\left(\frac{\partial}{\partial x}\right)^{\alpha},$$

$f(x)$ and $b_{\alpha}(x)$ are complex valued functions in a certain class $\mathcal{F}^{0}(\boldsymbol{R}^{d})$ (see § 1), and $|\alpha|=\sum_{k=1}^{d}\alpha_{k}$ and $(\partial/\partial x)^{\alpha}=\prod_{k=1}^{d}(\partial/\partial x_{k})^{\alpha_{k}}$ for multi index $\alpha=(\alpha_{1}, \cdots, \alpha_{d})$. For $b_{\alpha}(x)$, $|\alpha|=2q$, we assume a sufficient condition, under which (0.1) is strongly parabolic.

As in [4], we consider *A-process*, which is a "Markov process" *related to*

$$\frac{\partial u}{\partial t}(t, x)=Au(t, x), \qquad t>0, \ x\in \boldsymbol{R}^{d}, \tag{0.2}$$

i.e., the density of the "transition probability" of the process is the fundamental solution of (0.2). In general, this transition probability is not positive even for real ρ. Therefore, if a completely additive measure related to A-process should be realized on a path space, then the measure would not be of bounded variation, shown as in [1, 2, 4]. Thus, A-process is not a Markov process in the usual sense.

In [4], we defined "stochastic integrals" of A-process, and each stochastic integral corresponds to a differential operator of order up to $2q-1$. Here we will define "singular stochastic integrals", to which differential operators of orders up to $2q$ correspond. If the singular stochastic integrals are once established, then a Girsanov type formula, obtained in [4], will enable us to solve (0.1). Justification of this procedure is the theme of this article.

At first, we consider *ε-process*, that is, a "Markov process" related to the following parabolic equation of order higher than (0.2):

$$\frac{\partial u}{\partial t}(t, x)=\left[(-1)^{p-1}\varepsilon\sum_{k=1}^{d}\left(\frac{\partial}{\partial x_k}\right)^{2p}+A\right]u(t, x), \qquad t>0,\ x\in \boldsymbol{R}^d, \tag{0.3}$$

where ε is a positive number and p is a natural number such that $p>q$. In § 2, we define stochastic integrals of ε-process by the same manner as in [4].

Next, in § 3, we let ε tend to zero for (0.3), then, with a suitable choice of integrands, the stochastic integrals of ε-process converge to the singular stochastic integrals of A-process. Here the differential operators of orders up to $2q$ correspond to these singular stochastic integrals, and the sense of the convergence is a little wider than "the weak sense", in [**1**, **3**, **4**].

The content of § 4 is a construction of "Girsanov density" for the singular stochastic integrals and some of its properties. For instance, our Girsanov type formula solves "martingale problem" for $(A+B)$. Even in the case $A=\Delta$, i. e. the Brownian motion, this is new in comparison with the usual Girsanov formula.

In § 5, we specify a stochastic solution of (0.1) by the Girsanov type formula and prove the uniqueness and regularity of the stochastic solution.

§ 1. Preliminaries.

Let $\mathcal{M}^{\kappa}(\boldsymbol{R}^d)$, $\kappa\geqq 0$, be the space of complex valued measures μ on $\boldsymbol{R}^d$ with $\|\mu\|_{\kappa}\equiv\int(1+|\xi|)^{\kappa}|\mu|(d\xi)<\infty$. $\mathcal{F}^{\kappa}(\boldsymbol{R}^d)$ is the space of all Fourier transforms $f(x)=\int\exp\{i\langle\xi, x\rangle\}\mu_f(d\xi)$ of μ_f in $\mathcal{M}^{\kappa}(\boldsymbol{R}^d)$, where $\langle\xi, x\rangle$ is the inner product in $\boldsymbol{R}^d$, and we define $\|f\|_{\kappa}\equiv\|\mu_f\|_{\kappa}$. $\mathcal{M}^{\kappa}(\boldsymbol{R}^d)$ is a commutative Banach algebra with norm $\|\ \|_{\kappa}$ under convolution. We define $\mathcal{M}^{\infty}(\boldsymbol{R}^d)\equiv\bigcap_{\kappa\geqq 0}\mathcal{M}^{\kappa}(\boldsymbol{R}^d)$ and $\mathcal{F}^{\infty}(\boldsymbol{R}^d)\equiv\bigcap_{\kappa\geqq 0}\mathcal{F}^{\kappa}(\boldsymbol{R}^d)$. $\mathcal{F}^{\infty}(\boldsymbol{R}^d)$ contains the Schwartz class $\mathcal{S}$, constants, $\sin x_k$, $\cos x_k$, etc.

We define some "stochastic terms" about A-process and ε-process as in [4]. *The path space* $\boldsymbol{C}$ is the set of all continuous functions $w(\cdot)=(w_1(\cdot), \cdots, w_d(\cdot))$: $[0, \infty)\rightarrow\boldsymbol{R}^d$. We say that a function $f(w)$ on $\boldsymbol{C}$ is a *tame function*, if $f(w)$ is a Borel function of a finite number of observations, that is

$$f(w)=g(w(t_1), \cdots, w(t_N))$$

for a Borel function g on $\boldsymbol{R}^{d\times N}$. Moreover, if g is in $\mathcal{F}^{\kappa}(\boldsymbol{R}^{d\times N})$ (resp., polynomial), then we say that $f(w)$ is an $\mathcal{F}^{\kappa}$ (resp., a *polynomial*) *tame function.*

The Fourier transform of the fundamental solution $p^{\varepsilon}(t, x)$ of (0.3) is

$$\exp\left\{-\sum_{k}(\varepsilon\xi_k^{2p}+\rho\xi_k^{2q})t\right\},$$

and $p^{\varepsilon}(t, x)$ is in the Schwartz class $\mathcal{S}$ in x for each positive t. The *expectation* $E_x^{\varepsilon}[f(w)]$ of a tame function $f(w)=g(w(t_1), \cdots, w(t_N))$, $0\leqq t_1\leqq\cdots\leqq t_N$, is defined by the following, if the integral on the right hand side exists:

$$E_x^{\varepsilon}[f(w)]=\int\cdots\int dy^{(1)}\cdots dy^{(N)}\left(\prod_n p^{\varepsilon}(t_n-t_{n-1}, y^{(n)}-y^{(n-1)})\right) \times g(y^{(1)}, \cdots, y^{(N)}), \tag{1.1}$$

where $t_0=0$ and $y^{(0)}=x$. ε-process has *Markov property,* that is: for f in $\mathcal{F}^0(\boldsymbol{R}^{d\times N})$, g in $\mathcal{F}^0(\boldsymbol{R}^{d\times N'})$, and $0\leqq s_1\leqq\cdots\leqq s_N\leqq t_1\leqq\cdots\leqq t_{N'}$,

$$\begin{aligned}&E_x^{\varepsilon}[f(w(s_1), \cdots, w(s_N))g(w(t_1), \cdots, w(t_{N'}))]\\&=E_x^{\varepsilon}[f(w(s_1), \cdots, w(s_N))E_{w(s_N)}^{\varepsilon}[g(w(t_1-s_N), \cdots, w(t_{N'}-s_N))]].\end{aligned} \tag{1.2}$$

We say that a sequence of tame functions $\{f_n\}$ *converges in the ε-weak sense,* if $\lim_{n\to\infty}E_x^{\varepsilon}[f_n g]$ exists for each $\mathcal{F}^{\infty}$ tame function g and each x.

$J=(J_1, \cdots, J_d)$ is a *multi index of a stochastic integral* (in abbreviation, *S. I. multi index*) if J_k, $k=1, \cdots, d$, are natural numbers such that

$$2p\geqq J_k\geqq 1, \qquad k=1, \cdots, d, \tag{1.3, i}$$

$$|J|\equiv\sum_{k=1}^{d}J_k\geqq 2p(d-1)+1. \tag{1.3, ii}$$

For A-process, we use the similar terms as above; *the expectation* $E_x[f(w)]$, *Markov property, the weak sense convergence,* etc. (cf. [4]).

As a relation of the both processes: for an $\mathcal{F}^0$ or a polynomial tame function $f(w)$, $\lim_{\varepsilon\to 0}E_x^{\varepsilon}[f(w)]=E_x[f(w)]$.

§ 2. Stochastic integrals of ε-process.

We fix a positive number T throughout this article. For a large natural number M, let $\delta=T/M$, $s_m=mT/M$ $(m=0, 1, \cdots, M)$, and let

$$\delta w_k(s_m)=w_k(s_{m+1})-w_k(s_m), \qquad k=1, \cdots, d.$$

For S. I. multi index J, we define

$$(\delta w(s_m))^J\equiv\left(\frac{1}{\delta}\right)^{d-1}\prod_{k=1}^{d}(\delta w_k(s_m))^{J_k}, \tag{2.1}$$

where we use the convention for $J_k=2p$

$$(\delta w_k(s_m))^{J_k}=(-1)^{p-1}\varepsilon(2p)!\,\delta. \tag{2.2}$$

We denote the characteristic function of an interval $[0, t]$ by $\chi_{[0,t]}$.

2.1. THEOREM. *Let $a(x)$ and $a_n(x)$ $(n=1, \cdots, N)$ be functions in $\mathcal{F}^\infty(\boldsymbol{R}^d)$, and let J and $J(n)$ $(n=1, \cdots, N)$ be S.I. multi indices. Then, the following sequences of tame functions converge ε-weakly for each positive ε, as $M\to\infty$:*

$$\sum_{m=0}^{M}\chi_{[0,t]}(s_m)a(w(s_m))(\delta w(s_m))^J,$$

$$\sum_{m_1=N-1}^{M}\sum_{m_2=N-2}^{m_1-1}\cdots\sum_{m_N=0}^{m_{N-1}-1}\chi_{[0,t]}(s_{m_1})\Big(\sum_{n=1}^{N}a_n(w(s_{m_n}))(\delta w(s_{m_n}))^{J(n)}\Big). \tag{2.3}$$

2.2. DEFINITION. We call the ε-weak limits above *stochastic integrals of ε-process,* and we use the symbolical notations:

$$\varepsilon\text{-}\int_0^t a(w(s))(dw(s))^J,$$

$$\varepsilon\text{-}\int_0^t (dw(s_1))^{J(1)}\int_0^{s_1}(dw(s_2))^{J(2)}\cdots\int_0^{s_{N-1}}(dw(s_N))^{J(N)} \times a_1(w(s_1))a_2(w(s_2))\cdots a_N(w(s_N)).$$

PROOF OF THEOREM 2.1. We shall prove the weak convergence of (2.3) for $a_n(x)=\int\exp\{i\langle\xi^{(n)}, x\rangle\}\mu_n(d\xi^{(n)})$, $n=1, \cdots, N$.

Step 1. Let $\boldsymbol{\Phi}=(\Phi_1, \Phi_2)$ be an ordered partition of the set $\{1, \cdots, d\}$ into two parts, where Φ_1 or Φ_2 may be empty.

For S.I. multi index J and an ordered partition $\boldsymbol{\Phi}$, a constant $C(J, \boldsymbol{\Phi})$ is defined by

$$C(J, \boldsymbol{\Phi})=\begin{cases}\Big(\prod_{k\in\Phi_1}\dfrac{(2p)!}{(2p-J_k)!}\Big)\Big(\prod_{k\in\Phi_2}\dfrac{(2q)!}{(2q-J_k)!}\Big), & \text{if } J_k\leqq 2q \text{ for } k\in\Phi_2,\\ 0, & \text{otherwise.}\end{cases} \tag{2.4}$$

Step 2. Assume that $g(w)=g(w(u))=\int\exp\{i\langle\zeta, w(u)\rangle\}\mu_g(d\zeta)$, $u\geqq t$, and set $H(n)=(H_1(n), \cdots, H_d(n))$ with

$$H_k(n)\equiv\zeta_k+\sum_{r=0}^{n-1}\xi_k^{(r)}, \qquad k=1, \cdots, d,\ n=1, \cdots, N+1, \tag{2.5}$$

where $\xi_k^{(0)}\equiv 0$. By a similar argument as in [4],

(2.6)
$$\lim_{M\to\infty} E_x^\varepsilon[(2.3)g(w(u))]$$
$$= \int \mu_g(d\zeta)\int \mu_1(d\xi^{(1)}) \cdots \int \mu_N(d\xi^{(N)})\int_{t>s_1>\cdots>s_N>0} ds_1 \cdots ds_N$$
$$\times \exp\{i\langle \zeta+\xi^{(1)}+\cdots+\xi^{(N)}, x\rangle\}\Big[\prod_n\Big\{\sum_{\Phi} C(J(n), \Phi)\varepsilon^{|\Phi_1|}\rho^{|\Phi_2|}$$
$$\times(-1)^{p|\Phi_1|+q|\Phi_2|+d}\Big(\prod_{k\in\Phi_1}(iH_k(n))^{2p-J_k(n)}\Big)\Big(\prod_{k\in\Phi_2}(iH_k(n))^{2q-J_k(n)}\Big)\Big\}\Big]$$
$$\times\Big[\prod_n\prod_k \exp\{(-\varepsilon(H_k(n))^{2p}-\rho(H_k(n))^{2q})(s_{n-1}-s_n)\}\Big]$$
$$\times\Big[\prod_k \exp\{(-\varepsilon(H_k(N+1))^{2p}-\rho(H_k(N+1))^{2q})s_N\}\Big], \qquad s_0\equiv u,$$

where $|\Lambda|$ denotes the number of elements in Λ for a finite set Λ, and $\sum_\Phi$ denotes the sum over all Φ's.

Step 3. By (2.6) and a similar argument as in [4], we have:

(i) $\|\lim_M E^\varepsilon_\cdot[(2.3)g(w(u))]\|_\kappa<\infty$ for each κ,

(ii) if a sequence $\{g^{(r)}(x)\}$ in $\mathcal{F}^\infty(\boldsymbol{R}^d)$ converges to a function $g(x)$ in $\mathcal{F}^\infty(\boldsymbol{R}^d)$ with respect to $\|\ \|_\kappa$ sense for each κ, then, for each κ,

$$\lim_{M,r\to\infty}\|E^\varepsilon_\cdot[(2.3)g^{(r)}(w(u))]-\lim_{M\to\infty}E^\varepsilon_\cdot[(2.3)g(w(u))]\|_\kappa=0.$$

Step 4. To prove the ε-weak convergence of (2.3) for a general $\mathcal{F}^\infty$ tame function $g(w)$, it is sufficient to look at the case $g(w)=g(w(u_1), w(u_2))$, $u_1\geqq u_2$.

If $u_1\geqq u_2\geqq t$, then the proof is essentially a repetition of Step 2, by (1.2).

If $u_1\geqq t\geqq u_2$, or if $t\geqq u_1\geqq u_2$, then Markov property (1.2) and (i), (ii) in Step 3 complete the proof as in [4]. □

As in [4], stochastic integrals of ε-process correspond to differential operators in $\boldsymbol{R}^d$.

2.3. Corollary. *Let f be an $\mathcal{F}^\infty$ tame function, that is $f=g(x^{(1)}, \cdots, x^{(R)})$ with $g\in\mathcal{F}^\infty(\boldsymbol{R}^{d\times R})$, $x^{(1)}=w(u_1), \cdots, x^{(R)}=w(u_R)$ $(0\equiv u_0\leqq u_1\leqq\cdots\leqq u_R\leqq T)$. Then,*

$$\lim_{\tau\downarrow t}\frac{1}{\tau-t}E_x^\varepsilon\Big[\Big\{\varepsilon\text{-}\int_t^\tau a(w(s))(dw(s))^J\Big\}f\Big]$$
$$=\begin{cases}\displaystyle\sum_\Phi C(J,\Phi)(-1)^{p|\Phi_1|+q|\Phi_2|+d}\varepsilon^{|\Phi_1|}\rho^{|\Phi_2|} \\ \displaystyle\quad\times E_x^\varepsilon\Big[a(w(t))\Big(\Big\{\prod_{k\in\Phi_1}\Big(\sum_{r=R'}^R\frac{\partial}{\partial x_k^{(r)}}\Big)^{2p-J_k}\Big\}\Big\{\prod_{k\in\Phi_2}\Big(\sum_{r=R'}^R\frac{\partial}{\partial x_k^{(r)}}\Big)^{2q-J_k}\Big\}\cdot g\Big)\Big] \\ \qquad\qquad\qquad\qquad\qquad\qquad\text{for } u_{R'-1}\leqq t<u_{R'}, \\ \displaystyle\sum_\Phi C(J,\Phi)(-1)^{p|\Phi_1|+q|\Phi_2|+d}\varepsilon^{|\Phi_1|}\rho^{|\Phi_2|}E_x^\varepsilon[a(w(t))g] \qquad \text{for } u_R\leqq t,\end{cases}$$

where $\chi(J,\Phi)=1$ if $J_k=2p$ for $k\in\Phi_1$ and if $J_k=2q$ for $k\in\Phi_2$, and $\chi(J,\Phi)=0$ otherwise.

PROOF. Let $\mu_g(d\zeta^{(1)}, \cdots, d\zeta^{(R)})$ be the measure in $\mathcal{M}^\infty(\boldsymbol{R}^{d\times R})$ corresponding to g, and apply (1.2) to the tame function

$$\frac{1}{\tau-t}E_x\Big[\sum_{m=0}^{M}\chi_{[t,\tau]}(s_m)a(w(s_m))(\delta w(s_m))^J f\Big], \tag{2.7}$$

under the assumption $u_{R'-1}\leqq t<u_{R'}$. As $M\to\infty$, we obtain a little modification of (2.6):

$$\begin{aligned}
&\lim_{M\to\infty}(2.7)\\
&=\frac{1}{\tau-t}\int\mu_g(d\zeta^{(1)}, \cdots, d\zeta^{(R)})\int\mu_a(d\xi)\int_t^\tau ds\exp\{i\langle\zeta^{(1)}+\cdots+\zeta^{(R)}, x\rangle\}\\
&\quad\times\Big[\sum_{\Phi}C(J, \Phi)\varepsilon^{|\Phi_1|}\rho^{|\Phi_2|}(-1)^{p|\Phi_1|+q|\Phi_2|+d}\\
&\qquad\times\Big(\prod_{k\in\Phi_1}(i\zeta_k^{(R')}+\cdots+i\zeta_k^{(R)})^{2p-J_k}\Big)\Big(\prod_{k\in\Phi_2}(i\zeta_k^{(R')}+\cdots+i\zeta_k^{(R)})^{2q-J_k}\Big)\Big]\\
&\quad\times\Big[\prod_k\exp\Big\{\varepsilon\int_0^\infty d\theta\Big(\sum_{r=1}^R\zeta_k^{(r)}\chi_{[0,u_r]}(\theta)+\xi_k\chi_{[0,s]}(\theta)\Big)^{2p}\\
&\qquad\qquad-\rho\int_0^\infty d\theta\Big(\sum_{r=1}^R\zeta_k^{(r)}\chi_{[0,u_r]}(\theta)+\xi_k\chi_{[0,s]}(\theta)\Big)^{2q}\Big\}\Big].
\end{aligned}$$

In the case $t\geqq u_R$, the similar equality as the above is obtained. Thus, as τ tends to t, Corollary 2.3 follows. □

2.4. COROLLARY. *For functions f and g in $\mathcal{F}^\infty(\boldsymbol{R}^d)$,*

$$\begin{aligned}
&E_x^\varepsilon\Big[\Big\{\varepsilon\text{-}\int_0^{t+u}(dw(s_1))^{J(1)}\cdots\int_0^{s_{N-1}}(dw(s_N))^{J(N)}\Big(\prod_{n=1}^N a_n(w(s_n))\Big)\Big\}f(w(t))g(w(t+u))\Big] \qquad (2.8)\\
&=E_x^\varepsilon\Big[\Big\{\varepsilon\text{-}\int_0^{t}(dw(s_1))^{J(1)}\cdots\int_0^{s_{N-1}}(dw(s_N))^{J(N)}\\
&\qquad\times\Big(\prod_{n=1}^N a_n(w(s_n))\Big)\Big\}f(w(t))E_{w(t)}^\varepsilon[g(w(u))]\Big]\\
&\quad+E_x^\varepsilon\Big[f(w(t))E_{w(t)}^\varepsilon\Big[\Big\{\varepsilon\text{-}\int_0^{u}(dw(s_1))^{J(1)}\cdots\int_0^{s_{N-1}}(dw(s_N))^{J(N)}\\
&\qquad\qquad\times\Big(\prod_{n=1}^N a_n(w(s_n))\Big)\Big\}g(w(u))\Big]\Big]\\
&\quad+\sum_{l=1}^{N-1}E_x^\varepsilon\Big[f(w(t))\Big\{\varepsilon\text{-}\int_0^{t}(dw(s_{l+1}))^{J(l+1)}\cdots\int_0^{s_{N-1}}(dw(s_N))^{J(N)}\Big(\prod_{n=l+1}^{N-1}a_n(w(s_n))\Big)\Big\}\\
&\qquad\times E_{w(t)}^\varepsilon\Big[\Big\{\varepsilon\text{-}\int_0^{u}(dw(s_1))^{J(1)}\cdots\int_0^{s_{l-1}}(dw(s_l))^{J(l)}\\
&\qquad\qquad\times\Big(\prod_{n=1}^{l}a_n(w(s_n))\Big)\Big\}g(w(u))\Big]\Big],
\end{aligned}$$

where $\sum_{l=1}^{N-1}\{\ \}=0$ for $N=1$.

PROOF. (1.2) combined with Step 3 in the proof of Theorem 2.1 imply Corollary. The proof is essentially the same as that in [4]. □

§ 3. Singular stochastic integrals.

From now on, we consider only the multi indices $\alpha=(\alpha_1, \cdots, \alpha_d)$ such that $|\alpha|\leqq 2q$, and define S. I. multi index $J(\alpha)=(J_1(\alpha), \cdots, J_d(\alpha))$ by

$$J_k(\alpha)=2p-\alpha_k, \qquad k=1, \cdots, d. \tag{3.1}$$

Here we consider a new order partition $\boldsymbol{\Psi}=(\Psi_1, \Psi_2)$ of $\{1, \cdots, d\}$ such that Ψ_1 may be empty, but Ψ_2 is not empty.

For each α and $\boldsymbol{\Psi}$, $\Gamma(\alpha, \boldsymbol{\Psi})$ is the set of multi indices α' satisfying

$$\alpha'_k=\begin{cases} \alpha_k, & \text{for } k\in\Psi_1 \\ \alpha_k+2(p-q), & \text{for } k\in\Psi_2. \end{cases} \tag{3.2}$$

Hence $\Gamma(\alpha, \boldsymbol{\Psi})$ consists of at most one element. For a measure ν_α in $\mathcal{M}^0(\boldsymbol{R}^d)$ ($|\alpha|\leqq 2q$), we define a new measure $\mu^{(\varepsilon)}_{\alpha,\beta}$ by

$$\mu^{(\varepsilon)}_{\alpha,\beta}=C(\alpha, \beta, \varepsilon)\nu_\alpha, \qquad |\beta|\leqq|\alpha|, \tag{3.3}$$

where

$$C(\alpha, \beta, \varepsilon)=\begin{cases} 0, & \text{if } |\beta|\geqq|\alpha| \text{ and if } \beta\neq\alpha, \\ (\varepsilon^d C(J(\alpha), \boldsymbol{d}))^{-1}(-1)^{(p+1)d}, & \text{if } \beta=\alpha, \\ -\sum\limits_{\boldsymbol{\Psi}}\sum\limits_{\beta'\in\Gamma(\beta,\boldsymbol{\Psi})}(\varepsilon^d C(J(\beta), \boldsymbol{d}))^{-1}C(J(\beta'), \boldsymbol{\Psi}) & \\ \quad\times(-1)^{p(d+|\Psi_1|)+q|\Psi_2|}\varepsilon^{|\Psi_1|}\rho^{|\Psi_2|}C(\alpha, \beta', \varepsilon), & \text{if } |\beta|<|\alpha|, \end{cases} \tag{3.4}$$

and $C(J(\alpha), \boldsymbol{d})$ is given by (2.4) with $\Phi_1=\{1, \cdots, d\}$ and $\Phi_2=\emptyset$. Note that; (3.4) is well defined, since

$$|\beta'|=|\beta|+2(p-q)|\Psi_2|\geqq|\beta|+1.$$

Let $b_\alpha(x)$ and $a^{(\varepsilon)}_{\alpha,\beta}(x)$, $|\beta|\leqq|\alpha|$, be the Fourier transforms of the measures ν_α and $\mu^{(\varepsilon)}_{\alpha,\beta}$ respectively:

$$b_\alpha(x)=\int\exp\{i\langle\xi, x\rangle\}\nu_\alpha(d\xi), \tag{3.5, i}$$

$$a^{(\varepsilon)}_{\alpha,\beta}(x)=\int\exp\{i\langle\xi, x\rangle\}\mu^{(\varepsilon)}_{\alpha,\beta}(d\xi). \tag{3.5, ii}$$

3.1. LEMMA. (i) *For positive numbers γ and t,*

$$\sup_{y\in\boldsymbol{R}^1}|y|^\gamma\exp\{-(\operatorname{Re}\rho)y^{2q}t\}\leqq\left(\frac{\gamma}{2q\,e\,(\operatorname{Re}\rho)t}\right)^{\gamma/2q}.$$

(ii) *For a multi index $\alpha=(\alpha_1, \cdots, \alpha_d)$,*

$$\left|\prod_{k=1}^d y_k^{\alpha_k}\right|\leqq\sum_{k=1}^d|y_k|^{|\alpha|}.$$

PROOF. Direct computations prove Lemma 3.1. □

Now we let ε tend to zero for the stochastic integrals of ε-process.

3.2. THEOREM. *For measures ν_α's in $\mathcal{M}^0(\boldsymbol{R}^d)$ and an $\mathcal{F}^0$ tame function $g(w)$, the following converge in $\|\ \|_0$ sense as $\varepsilon \to 0$:*

$$\sum_{|\beta| \leq |\alpha|} E_x^\varepsilon\left[\left\{\varepsilon\text{-}\int_0^t a_{\alpha,\beta}^{(\varepsilon)}(w(s))(dw(s))^{J(\beta)}\right\}g(w)\right],$$

$$\sum_{|\beta^{(1)}| \leq |\alpha^{(1)}|} \cdots \sum_{|\beta^{(N)}| \leq |\alpha^{(N)}|} E_x^\varepsilon\left[\left\{\varepsilon\text{-}\int_0^t (dw(s_1))^{J(\beta^{(1)})} \cdots \int_0^{s_{N-1}} (dw(s_N))^{J(\beta^{(N)})}\right.\right.$$
$$\tag{3.6} \left.\left. \times a_{\alpha^{(1)},\beta^{(1)}}^{(\varepsilon)}(w(s_1)) \cdots a_{\alpha^{(N)},\beta^{(N)}}^{(\varepsilon)}(w(s_N))\right\}g(w)\right],$$

where $J(\beta)$'s and $a_{\alpha,\beta}^{(\varepsilon)}$'s are given in (3.1), (3.3), (3.4), *and* (3.5), *respectively.*

3.3. DEFINITION. (i) The limits in Theorem 3.2 are denoted by

$$E_x\left[\left\{\text{s-}\int_0^t b_\alpha(w(s))(dw(s))^{I(\alpha)}\right\}g(w)\right],$$

$$E_x\left[\left\{\text{s-}\int_0^t (dw(s_1))^{I(\alpha^{(1)})} \cdots \int_0^{s_{N-1}} (dw(s_N))^{I(\alpha^{(N)})}\right.\right.$$
$$\left.\left. \times b_{\alpha^{(1)}}(w(s_1)) \cdots b_{\alpha^{(N)}}(w(s_N))\right\}g(w)\right]$$

respectively, where $I(\alpha) = (I_1(\alpha), \cdots, I_d(\alpha))$ is defined by

$$I_k(\alpha) = 2q - \alpha_k, \qquad k = 1, \cdots, d.$$

(ii) Symbolically we call $\text{s-}\int_0^t b_\alpha(w(s))(dw(s))^{I(\alpha)}$ and the other integral in the bracket $\{\ \}$ *singular stochastic integrals of A-process*, which are linear functionals over the space of $\mathcal{F}^0$ tame functions combined with $E_x[\cdot]$.

PROOF OF THEOREM 3.2. We may assume that $g(w) = g(w(u)) = \int \exp\{i\langle\zeta, w(u)\rangle\}\mu_g(d\zeta)$, $u \geqq t$, because Markov property for the stochastic integrals of ε-process (Corollary 2.4) and (3.10) and (3.11) later reduce the problem to this case.

Step 1. First we prove the convergence of (3.6), when $a_{\alpha,\beta}^{(\varepsilon)}$ and b_α are in $\mathcal{F}^\infty(\boldsymbol{R}^d)$ and $g(w)$ is an $\mathcal{F}^\infty$ tame function. Let $H_k(n)$ be given by (2.5). From (2.6), we observe

$$\{(3.6)\} = \sum_{\beta^{(1)}} \cdots \sum_{\beta^{(N)}} \int_{t>s_1>\cdots>s_N>0} ds_1 \cdots ds_N \int \mu_g(d\zeta)$$
$$\times \int \mu_{\alpha^{(1)},\beta^{(1)}}^{(\varepsilon)}(d\xi^{(1)}) \cdots \int \mu_{\alpha^{(N)},\beta^{(N)}}^{(\varepsilon)}(d\xi^{(N)}) \exp\{i\langle\zeta + \xi^{(1)} + \cdots + \xi^{(N)}, x\rangle\}$$

$$\times\Big[\prod_n\Big\{\sum_{\Phi} C(J(\beta^{(n)}),\ \Phi)\varepsilon^{|\Phi_1|}\rho^{|\Phi_2|}(-1)^{p|\Phi_1|+q|\Phi_2|+d}$$

$$\times\Big(\prod_{k\in\Phi_1}(iH_k(n))^{2p-J_k(\beta^{(n)})}\Big)\Big(\prod_{k\in\Phi_2}(iH_k(n))^{2q-J_k(\beta^{(n)})}\Big)\Big\}$$

$$\times\exp\Big\{\sum_k(-\varepsilon(H_k(n))^{2p}-\rho(H_k(n))^{2q})(s_{n-1}-s_n)\Big\}\Big]$$

$$\times\Big[\prod_k\exp\{(-\varepsilon(H_k(N+1))^{2p}-\rho(H_k(N+1))^{2q})s_N\}\Big]$$

$$=\int_{t>s_1>\cdots>s_N>0}ds_1\cdots ds_N\int\mu_g(d\zeta)\int\nu_{\alpha^{(1)}}(d\xi^{(1)})\cdots\int\nu_{\alpha^{(N)}}(d\xi^{(N)})$$

$$\times\exp\{i\langle\zeta+\xi^{(1)}+\cdots+\xi^{(N)},\ x\rangle\}$$

$$\times\Big[\prod_n\prod_k(iH_k(n))^{\alpha_k^{(n)}}\exp\{(-\varepsilon(H_k(n))^{2p}-\rho(H_k(n))^{2q})(s_{n-1}-s_n)\}\Big]$$

$$\times\Big[\prod_k\exp\{(-\varepsilon(H_k(N+1))^{2p}-\rho(H_k(N+1))^{2q})s_N\}\Big],\qquad s_0\equiv u,$$

where the second equality follows from (3.3) and (3.4). Hence

$$\lim_{\varepsilon\to0}\{(3.6)\}=\int_{t>s_1>\cdots>s_N>0}ds_1\cdots ds_N\int\mu_g(d\zeta)\tag{3.7}$$

$$\times\int\nu_{\alpha^{(1)}}(d\xi^{(1)})\cdots\int\nu_{\alpha^{(N)}}(d\xi^{(N)})\exp\{i\langle\zeta+\xi^{(1)}+\cdots+\xi^{(N)},\ x\rangle\}$$

$$\times\Big[\prod_n\prod_k(iH_k(n))^{\alpha_k^{(n)}}\exp\{-\rho(H_k(n))^{2q}(s_{n-1}-s_n)\}\Big]$$

$$\times\Big[\prod_k\exp\{-\rho(H_k(N+1))^{2q}s_N\}\Big]$$

$$\equiv K(\alpha^{(1)},\cdots,\alpha^{(N)}).$$

Step 2. We shall prove that (3.7) holds for ν_α and μ_g in $\mathcal{M}^0(\boldsymbol{R}^d)$. First, we seek a bound of $\|K(\alpha^{(1)},\cdots,\alpha^{(N)})\|_0$ under the assumption

$$|\alpha^{(n)}|=2q,\qquad\text{if } n=1,\cdots,L,\tag{3.8}$$
$$<2q,\qquad\text{if } n=L+1,\cdots,N.$$

Set $\theta(t)\equiv\|\mu_g\|_0^{-1}\int|\mu_g|(d\xi)(1-\exp\{-(\mathrm{Re}\,\rho)\sum_k\xi_k^{2q}t\})$, then

$$|\theta(t)|\leqq1,\qquad\lim_{t\to0}\theta(t)=0.\tag{3.9}$$

If $N>L$ in (3.8), then Lemma 3.1, combined with (3.7), imply

$$\|K(\alpha^{(1)},\cdots,\alpha^{(N)})\|_0$$

$$\leqq\int_{t>s_1>\cdots>s_N>0}ds_1\cdots ds_N\int d|\mu_g|\int d|\nu_{\alpha^{(1)}}|\cdots\int d|\nu_{\alpha^{(N)}}|$$

$$\times\Big[\prod_{n=1}^{L}\Big(\sum_k(H_k(n))^{2q}\Big)\exp\Big\{-(\mathrm{Re}\,\rho)\sum_k(H_k(n))^{2q}(s_{n-1}-s_n)\Big\}\Big]$$

$$\times\Big[\prod_{n=L+1}^{N}\prod_k\{2q(\mathrm{Re}\,\rho)\,e\,(s_{n-1}-s_n)\}^{-\alpha_k^{(n)}/2q}\Big]$$

$$\leqq \|\mu_g\|_0 \|\nu_{\alpha^{(1)}}\|_0 \cdots \|\nu_{\alpha^{(N)}}\|_0 (\mathrm{Re}\,\rho)^{-L}$$
$$\times\Big[\prod_{n=L+1}^{N} t^{1-|\alpha^{(n)}|/2q}((\mathrm{Re}\,\rho)\,e)^{-|\alpha^{(n)}|/2q} B\Big(1+\sum_{r=n+1}^{N}\Big(1-\frac{|\alpha^{(r)}|}{2q}\Big),\ 1-\frac{|\alpha^{(n)}|}{2q}\Big)\Big],$$

where $B(\cdot,\cdot)$ is the Beta function and $\prod_{n=N+1}^{N}\{\ \}=1$. Thus, for constants $c_1=c_1(q)$ and $c_2=c_2(q,\rho)$,

$$\|K(\alpha^{(1)},\cdots,\alpha^{(N)})\|_0 \leqq c_1\|\mu_g\|_0\|\nu_{\alpha^{(1)}}\|_0\cdots\|\nu_{\alpha^{(N)}}\|_0 \times(\mathrm{Re}\,\rho)^{-L}(c_2\max(t,t^{1/2q}))^{N-L}\Big(\Big[\frac{N-L}{2q}\Big]!\Big)^{-1}. \tag{3.10}$$

If $L=N$ in (3.8), then Lemma 3.1 and an analogous computation as before derive

$$\|K(\alpha^{(1)},\cdots,\alpha^{(N)})\|_0 \leqq \|\mu_g\|_0\|\nu_{\alpha^{(1)}}\|_0\cdots\|\nu_{\alpha^{(N)}}\|_0(\mathrm{Re}\,\rho)^{-N}\theta(t). \tag{3.11}$$

Since the orders of $\alpha^{(n)}$, such that $|\alpha^{(n)}|=2q$, do not effect the bound in (3.10), (3.7) is valid for ν_α and μ_g in $\mathcal{M}^0(\boldsymbol{R}^d)$ by (3.10) and (3.11). □

The singular stochastic integrals correspond to differential operators in $\boldsymbol{R}^d$:

3.4. COROLLARY. *Let f be an $\mathcal{F}^\infty$ tame function, that is $f=g(x^{(1)},\cdots,x^{(R)})$ with $g\in\mathcal{F}^\infty(\boldsymbol{R}^{d\times R})$, $x^{(1)}=w(u_1),\cdots,x^{(R)}=w(u_R)$ $(0\equiv u_0\leqq u_1\leqq\cdots\leqq u_R\leqq T)$. Then,*

$$\lim_{\tau\downarrow t}\frac{1}{\tau-t}E_x\Big[\Big\{\text{s-}\int_t^\tau b_\alpha(w(s))(dw(s))^{I(\alpha)}\Big\}f\Big]$$
$$=\begin{cases} E_x\Big[b_\alpha(w(t))\Big(\Big(\prod_k\Big(\sum_{r=R'}^{R}\frac{\partial}{\partial x_k^{(r)}}\Big)^{\alpha_k}\Big)g\Big)\Big] & \text{for } u_{R'-1}\leqq t<u_{R'},\\ E_x[\chi(\alpha)b_\alpha(w(t))g] & \text{for } u_R\leqq t,\end{cases}$$

where $\chi(\alpha)=1$ if $|\alpha|=0$ and $\chi(\alpha)=0$ otherwise.

PROOF. A little modification of (3.7) implies Corollary 3.4 as in the case of Corollary 2.3. □

We note some properties of the singular stochastic integrals.

3.5. COROLLARY. (i) *Let $\{b_{\alpha^{(n)}}^{(r)}\}_{r=1,2,\dots}$ $(n=1,\cdots,N)$ and $\{g^{(r)}\}_{r=1,2,\dots}$ be sequences in $\mathcal{F}^0(\boldsymbol{R}^d)$, which converge to $b_{\alpha^{(n)}}$ $(n=1,\cdots,N)$ and g in $\|\ \|_0$ sense, respectively. Then*

$$\lim_{r\to\infty}\Big\|E_\cdot\Big[\Big\{\text{s-}\int_0^t(dw(s_1))^{I(\alpha^{(1)})}\cdots\int_0^{s_{N-1}}(dw(s_N))^{I(\alpha^{(N)})}\Big(\prod_n b_{\alpha^{(n)}}(w(s_n))\Big)\Big\}g(w(u))\Big]$$
$$-E_\cdot\Big[\Big\{\text{s-}\int_0^t(dw(s_1))^{I(\alpha^{(1)})}\cdots\int_0^{s_{N-1}}(dw(s_N))^{I(\alpha^{(N)})}$$
$$\times\Big(\prod_n b_{\alpha^{(n)}}^{(r)}(w(s_n))\Big)\Big\}g^{(r)}(w(u))\Big]\Big\|_0=0.$$

(ii) *If* $|\alpha|\neq 0$, *then the singular stochastic integrals are martingales in the following sense: for* $g(x^{(1)}, \cdots, x^{(N)})$ *in* $\mathcal{F}^0(\boldsymbol{R}^{d\times N})$ *and* $0\leqq t_1\leqq\cdots\leqq t_N\leqq u$,

$$E_x\left[\left\{\text{s-}\int_0^u b(w(s))(dw(s))^{I(\alpha)}\right\}g(w(t_1), \cdots, w(t_N))\right]$$
$$= E_x\left[\left\{\text{s-}\int_0^{t_N} b(w(s))(dw(s))^{I(\alpha)}\right\}g(w(t_1), \cdots, w(t_N))\right].$$

(iii) *The singular stochastic integrals are Markovian, that is: for functions* f *and* g *of* $\mathcal{F}^0(\boldsymbol{R}^d)$,

$$\begin{aligned}(3.12)\qquad & E_x\left[\left\{\text{s-}\int_0^{t+u}(dw(s_1))^{I(\alpha^{(1)})}\cdots\int_0^{s_{N-1}}(dw(s_N))^{I(\alpha^{(N)})}\right.\right.\\ &\qquad\left.\left.\times\left(\prod_n b_{\alpha^{(n)}}(w(s_n))\right)\right\}f(w(t))g(w(t+u))\right]\\ &= E_x\left[\left\{\text{s-}\int_0^{t}(dw(s_1))^{I(\alpha^{(1)})}\cdots\int_0^{s_{N-1}}(dw(s_N))^{I(\alpha^{(N)})}\right.\right.\\ &\qquad\left.\left.\times\left(\prod_n b_{\alpha^{(n)}}(w(s_n))\right)\right\}f(w(t))E_{w(t)}[g(w(u))]\right]\\ &\quad+E_x\left[f(w(t))E_{w(t)}\left[\left\{\text{s-}\int_0^{u}(dw(s_1))^{I(\alpha^{(1)})}\cdots\int_0^{s_{N-1}}(dw(s_N))^{I(\alpha^{(N)})}\right.\right.\right.\\ &\qquad\left.\left.\left.\times\left(\prod_n b_{\alpha^{(n)}}(w(s_n))\right)\right\}g(w(u))\right]\right]\\ &\quad+\sum_{l=1}^{N-1}E_x\left[f(w(t))\left\{\text{s-}\int_0^{t}(dw(s_{l+1}))^{I(\alpha^{(l+1)})}\right.\right.\\ &\qquad\left.\times\cdots\times\int_0^{s_{N-1}}(dw(s_N))^{I(\alpha^{(N)})}\left(\prod_{n=l+1}^{N-1} b_{\alpha^{(n)}}(w(s_n))\right)\right\}\\ &\qquad\times E_{w(t)}\left[\left\{\text{s-}\int_0^{u}(dw(s_1))^{I(\alpha^{(1)})}\cdots\int_0^{s_{l-1}}(dw(s_l))^{I(\alpha^{(l)})}\right.\right.\\ &\qquad\left.\left.\left.\times\left(\prod_{n=1}^{l} b_{\alpha^{(n)}}(w(s_n))\right)\right\}g(w(u))\right]\right],\end{aligned}$$

where $\sum_{l=1}^{N-1}\{\ \}=0$ *for* $N=1$.

PROOF. We easily obtain (i) from (3.10) and (3.11), and (ii) from (3.7). We prove (iii). By (3.10) and (3.11), the expectation of the singular stochastic integral $E_x\left[\left\{\text{s-}\int_0^u\cdots\right\}g(w(u))\right]$ is in $\mathcal{F}^0(\boldsymbol{R}^d)$ for each u, and the right hand side of (3.12) is well defined. Note that: (a) (3.6), the stochastic integral of ε-process, converges to the singular integrals of A-process in $\|\ \|_0$ sense for each t as $\varepsilon\to 0$, (b) $\|(3.6)\|_0\leqq c\|g\|_0$, and (c) $\|K(\alpha^{(1)}, \cdots, \alpha^{(N)})\|_0\leqq c\|g\|_0$, where c is a constant independent of ε. By (a), (b), and (c),

$$\sum_{\beta^{(1)}}\cdots\sum_{\beta^{(N)}}\{(2.8) \text{ with } J(n)=J(\beta^{(n)}) \text{ and } a_n=a^{(\varepsilon)}_{\alpha^{(n)},\beta^{(n)}}\}$$

converges to (3.12) in $\|\ \|_0$ sense for each t, as $\varepsilon\to 0$. Hence the proof is complete. □

3.6. REMARK. (i) Let $b(x)$ be a function in $\mathcal{T}^{\infty}(\boldsymbol{R}^d)$. When $|\alpha|\leqq 2q-1$, $I(\alpha)$ satisfies (1.3) with $p=q$, and the stochastic integral $\int_0^t b(w(s))(dw(s))^{I(\alpha)}$ for A-process exists as in [4] and coincides with the singular integral s-$\int_0^t b(w(s))(dw(s))^{I(\alpha)}$ except for a constant factor. However, when $|\alpha|=2q$, $I(\alpha)$ does not satisfy (1.3) with $p=q$, and there is no stochastic integral which corresponds to s-$\int_0^t b(w(s))(dw(s))^{I(\alpha)}$, while the latter exists.

For instance, in the Brownian motion case, Itô's stochastic integrals correspond to s-$\int_0^t b(w(s))(dw(s))^{I(\alpha)}$ when $|\alpha|=1$. But, for $|\alpha|=2$, there is neither Itô's integral nor such quantity which corresponds to s-$\int_0^t b(w(s))(dw(s))^{I(\alpha)}$.

(ii) When $|\alpha|=0$, the singular stochastic integral s-$\int_0^t b(w(s))(dw(s))^{I(\alpha)}$ should be written as $\int_0^t b(w(s))ds$. This is implied by (2.1), (2.2), and Definitions 2.2, 3.3 (cf. Remark 3.3, (i) in [4]).

3.7. REMARK. For the weak existence of singular stochastic integral, we need not assume that $|\alpha|\leqq 2q$. For instance, if we replace the assumption by $|\alpha|\leqq 2p-1$, then the singular stochastic integrals, which correspond to differential operators up to order $2p-1$, are defined similarly. But, in this case, the integrands b_α and tame function g should be taken in $\mathcal{T}^{\infty}(\boldsymbol{R}^d)$ and $\mathcal{T}^{\infty}$ tame functions.

§ 4. A Girsanov type formula for singular stochastic integrals.

As is well known, in the Brownian motion case, the Girsanov density

$$Z(t) \equiv \exp\left\{\sum_{k=1}^{d}\left(\int_0^t a_k(B(s))dB_k(s)-\frac{1}{2}\int_0^t (a_k(s))^2 ds\right)\right\} \tag{4.1}$$

satisfies the stochastic differential equation

$$Z(t)=1+\sum_k \int_0^t a_k(B(s))Z(s)dB_k(s).$$

Therefore $Z(t)$ is given by

$$\begin{aligned} Z(t)=1+\sum_{N=1}^{\infty}\sum_{k_1=1}^{d}\cdots\sum_{k_N=1}^{d}\int_0^t dB_{k_1}(s_1)\cdots\int_0^{s_{N-1}} dB_{k_N}(s_N) \\ \times a_{k_1}(B(s_1))\cdots a_{k_N}(B(s_N)). \end{aligned} \tag{4.2}$$

In this section, we shall define the " Girsanov density " of the singular stochastic integrals as an analogy of (4.2). Set

$$b^* = \sum_{|\alpha|=2q} \|\nu_\alpha\|_0 \quad (\equiv \sum_{|\alpha|=2q} \|b_\alpha\|_0),$$
$$b^{**} = \sum_{|\alpha|\leqq 2q-1} \|\nu_\alpha\|_0 \quad (\equiv \sum_{|\alpha|\leqq 2q-1} \|b_\alpha\|_0).$$

4.1. LEMMA. *For a large number C and a constant $c_3 = c_3(q, \rho)$,*

$$\Big\| E.\Big[\Big\{\sum_{|\alpha^{(1)}|\leqq 2q} \cdots \sum_{|\alpha^{(N)}|\leqq 2q} \text{s-}\int_0^t (dw(s_1))^{I(\alpha^{(1)})} \times \cdots \times \int_0^{s_{N-1}} (dw(s_N))^{I(\alpha^{(N)})} b_{\alpha^{(1)}}(w(s_1)) \cdots b_{\alpha^{(N)}}(w(s_N))\Big\} g(w(u))\Big]\Big\|_0$$
$$\leqq \|\mu_g\|_0 \Big(\frac{b^*}{\operatorname{Re}\rho} + \frac{1}{C}\Big)^N C^{2q} \exp\{c_3(Cb^{**})^{2q} T^{2q}\}.$$

PROOF. From (3.10) and (3.11),

$$\Big\|\sum_{\alpha^{(1)}} \cdots \sum_{\alpha^{(N)}} K(\alpha^{(1)}, \cdots, \alpha^{(N)})\Big\|_0 \tag{4.3}$$
$$\leqq \sum_{r=0}^{N-1} c_1 \binom{N}{r} \|\mu_g\|_0 \Big(\frac{b^*}{\operatorname{Re}\rho}\Big)^r (b^{**})^{N-r} \Big(\Big[\frac{N-r}{2q}\Big]!\Big)^{-1} (c_2 \max(t, t^{1/2q}))^{N-r}$$
$$+ \|\mu_g\|_0 \Big(\frac{b^*}{\operatorname{Re}\rho}\Big)^N \theta(t).$$

Note (3.9) and that

$$|y|^{N-r} \Big(\Big[\frac{N-r}{2q}\Big]!\Big)^{-1} \leqq \Big(\frac{1}{C}\Big)^{N-r} C^{2q} \exp\{(C|y|^{2q}\}$$

for a large number C. Now, Lemma 4.1 clearly follows from (3.7) and (4.3). □

4.2. THEOREM. *For functions $b_\alpha(x)$ in $\mathcal{F}^0(\boldsymbol{R}^d)$, suppose*

$$b^* \equiv \sum_{|\alpha|=2q} \|b_\alpha\|_0 < \operatorname{Re}\rho \tag{4.4}$$

(see Remark 5.4). *Then, for any $\mathcal{F}^0$ tame function g,*

$$E_x[g(w)] + \sum_{N=1}^{\infty} \sum_{|\alpha^{(1)}|\leqq 2q} \cdots \sum_{|\alpha^{(N)}|\leqq 2q} E_x\Big[\Big\{\text{s-}\int_0^t (dw(s_1))^{I(\alpha^{(1)})} \times \cdots \times \int_0^{s_{N-1}} (dw(s_N))^{I(\alpha^{(N)})} b_{\alpha^{(1)}}(w(s_1)) \cdots b_{\alpha^{(N)}}(w(s_N))\Big\} g(w)\Big] \tag{4.5}$$

converges in $\| \ \|_0$ sense.

4.3. DEFINITION. We denote

$$\{(4.5)\} = E_x[Z(t, w) g(w)],$$

and call $Z(t, w)$ *Girsanov density*, symbolically.

4.4. REMARK. In [**4**], the Girsanov density was defined by an analogy of the expansion of (4.1). But, now we cannot take that way, because the quantities as $\left(s\text{-}\int_0^t b(w(s))(dw(s))^{I(\alpha)}\right)^n$, $n \geqq 2$, do not exist even in the weak sense, when $|\alpha|=2q$. Therefore, we define $Z(t, w)$ by an analogy of (4.2) (see Corollary 4.7).

PROOF OF THEOREM 4.2. Let $g(w)=g(w(u))$, $u \geqq t$, and let C be a sufficiently large number. Then by Lemma 4.1 and (4.2),

$$\begin{aligned} \|(4.5)\|_0 &\leqq \sum_{N=0}^{\infty} \|\mu_g\|_0 \left(\frac{b^*}{\operatorname{Re}\rho}+\frac{1}{C}\right)^N C^{2q} \exp\{c_3(Cb^{**}T)^{2q}\} \\ &= \|\mu_g\|_0 \left(1-\frac{b^*}{\operatorname{Re}\rho}-\frac{1}{C}\right)^{-1} C^{2q} \exp\{c_3(Cb^{**}T)^{2q}\}. \end{aligned} \tag{4.6}$$

For a general $\mathcal{F}^0$ tame function g, a similar calculation as (4.6) is obtained in an analogous way, and Theorem 4.2 follows. □

$Z(t, w)$ has the Markov property.

4.5. LEMMA. *For functions f and g in $\mathcal{F}^0(\boldsymbol{R}^d)$,*

$$\begin{aligned} &E_x[Z(t, w) f(w(t)) E_{w(t)}[Z(u, w) g(w(u))]] \\ &= E_x[Z(t+u, w) f(w(t)) g(w(t+u))]. \end{aligned}$$

PROOF. By (4.6) and Corollary 3.5 (ii), we can take the summation $\sum_{N=0}^{\infty} \sum_{\alpha(1)} \cdots \sum_{\alpha(N)}$ of the both sides of (3.12). Then, by Theorem 4.2, we can write down the terms in a suitable order to obtain the lemma. □

We shall decide the differential operator corresponding to $Z(t, w)$. Let $\mathcal{F}^{1,2q}$ be the set of all functions $g(t, x)$, $(t, x) \in [0, T] \times \boldsymbol{R}^d$, which satisfy the following.

(4.7, i) $g(t, x) \in \mathcal{F}^{2q}(\boldsymbol{R}^d)$ for each $t \in [0, T]$, and

$$\lim_{s \to t} \|g(s, \cdot)-g(t, \cdot)\|_{2q} = 0.$$

(4.7, ii) For each t, there is a function $g_t(t, x) \in \mathcal{F}^0(\boldsymbol{R}^d)$ such that

$$\begin{aligned} &\lim_{s \to t} \left\| g_t(t, \cdot) - \frac{g(s, \cdot)-g(t, \cdot)}{s-t} \right\|_0 = 0, \\ &\lim_{s \to t} \|g_t(t, \cdot)-g_t(s, \cdot)\|_0 = 0. \end{aligned}$$

4.6. THEOREM. *For a function $g(t, x)$ in $\mathcal{F}^{1,2q}$,*

$$\begin{aligned} &\lim_{u \to t} \frac{1}{u-t} E_x[Z(u, w) g(u, w(u)) - Z(t, w) g(t, w(t))] \\ &= E_x\left[Z(t, w)\left(\left(\frac{\partial}{\partial t}+A+B\right)g\right)(t, w(t))\right]. \end{aligned}$$

PROOF. Set

$$
\begin{aligned}
& E_x[Z(u, w)g(u, w(u))-Z(t, w)g(t, w(t))] \\
& = E_x[Z(u, w)(g(u, w(u))-g(t, w(u)))] \\
& \quad +E_x[(Z(u, w)-Z(t, w))g(t, w(u))] \\
& \quad +E_x[Z(t, w)(g(t, w(u))-g(t, w(t)))] \\
& \equiv K_1+K_2+K_3 .
\end{aligned}
$$

Step 1. From Lemma 4.5,

$$
\begin{aligned}
(4.8) \qquad & \left\| E.\left[Z(t, w)g_t(t, w(t))-\frac{K_1}{u-t}\right]\right\|_0 \\
& \leqq \|E.[Z(t, w)g_t(t, w(t))E_{w(t)}[1-Z(u-t, w)]]\|_0 \\
& \quad +\|E.[Z(t, w)E_{w(t)}[Z(u-t, w)(g_t(t, w(0))-g_t(t, w(u-t)))]]\|_0 \\
& \quad +\left\|E.\left[Z(t, w)E_{w(t)}\left[Z(u-t, w)\left(g_t(t, w(u-t))\right.\right.\right.\right. \\
& \qquad \left.\left.\left.\left.-\frac{g(u, w(u-t))-g(t, w(u-t))}{u-t}\right)\right]\right]\right\|_0 .
\end{aligned}
$$

Since (4.6) holds, (4.3) and (4.7) imply that the right hand side of (4.8) vanishes as $u\to t$.

Step 2. From Lemma 4.5 and (4.6),

$$
\begin{aligned}
& \left\|E.\left[Z(t, w)(B\cdot g)(t, w(t))-\frac{K_2}{u-t}\right]\right\|_0 \\
& = \left\|E.\left[Z(t, w)E_{w(t)}\left[(B\cdot g)(t, w(0))-\frac{(Z(u-t, w)-1)g(t, w(u-t))}{u-t}\right]\right]\right\|_0 \\
& \leqq c'\left\|E.\left[(B\cdot g)(t, w(0))-\frac{(Z(u-t, w)-1)g(t, w(u-t))}{u-t}\right]\right\|_0 \\
& \leqq c'\left\|E.\left[(B\cdot g)(t, w(0))-\frac{1}{u-t}\left\{\sum_\alpha \text{s-}\int_0^{u-t} b_\alpha(w(s))(dw(s))^{I(\alpha)}\right\}g(t, w(u-t))\right]\right\|_0 \\
& \quad +c'\left\|E.\left[\frac{1}{u-t}\sum_{N=2}^\infty\left\{\sum_{\alpha(1)}\cdots\sum_{\alpha(N)}\text{s-}\int_0^{u-t}(dw(s_1))^{I(\alpha(1))}\cdots\int_0^{s_{N-1}}(dw(s_N))^{I(\alpha(N))}\right.\right.\right. \\
& \qquad \left.\left.\left.\times\left(\prod_n b_{\alpha(n)}(w(s_n))\right)\right\}g(t, w(u-t))\right]\right\|_0 \\
& \equiv K_4+K_5 ,
\end{aligned}
$$

where

$$
c' = \left(1-\frac{b^*}{\operatorname{Re}\rho}-\frac{1}{C}\right)^{-1}C^{2q}\exp\{c_3(Cb^{**}T)^{2q}\} .
$$

A little modification of Corollary 3.4 implies that $\lim_{u\to t}K_4=0$.

On the other hand, by Lemma 4.1 and (3.7) for $N\geqq 2$,

$$\Big\| E.\Big[\Big\{\sum_{\alpha(1)} \cdots \sum_{\alpha(N)} \text{s-}\int_0^{u-t}(dw(s_1))^{I(\alpha(1))} \cdots \int_0^{s_{N-1}}(dw(s_N))^{I(\alpha(N))}$$
$$\times b_{\alpha(1)}(w(s_1)) \cdots b_{\alpha(N)}(w(s_N))\Big\} g(t,\ w(u-t))\Big]\Big\|_0$$
$$\leqq \sum_{\alpha(1)} \sum_{\alpha(2)} \int_0^{u-t} ds_1 \int_0^{s_1} ds_2 \int d|\mu_{g(t,\cdot)}| \int d|\nu_{\alpha(1)}|$$
$$\times \Big|\prod_k (H_k(1))^{\alpha_k^{(1)}}(H_k(2))^{\alpha_k^{(2)}}\Big| \exp\Big\{-(\mathrm{Re}\,\rho) \sum_k (H_k(2))^{2q}(s_1-s_2)\Big\}$$
$$\times\Big(\frac{b^*}{\mathrm{Re}\,\rho}+\frac{1}{C}\Big)^{N-2} C^{2q} \exp\{c_3(Cb^{**}T)^{2q}\},$$

where $\mu_{g(t,\cdot)}$ is the measure in $\mathcal{M}^{2q}(\boldsymbol{R}^d)$, which corresponds to $g(t, x)$. Thus by (4.4), $\lim_{u\to t} K_5=0$.

Step 3. By Lemma 4.5 and (4.6),

$$\Big\| E.\Big[Z(t,\ w)(A\cdot g)(t,\ w(t)) - \frac{K_3}{u-t}\Big]\Big\|_0$$
$$\leqq c'\Big\| E.\Big[(A\cdot g)(t,\ w(0)) - \frac{g(t,\ w(u-t)) - g(t,\ w(0))}{u-t}\Big]\Big\|_0.$$

Clearly the last term vanishes as $u\to t$. □

From (4.6), the series

$$E_x[g(w)] + \sum_{N\geqq 1} \sum_{|\alpha(1)|\leqq 2q} \cdots \sum_{|\alpha(N)|\leqq 2q} E_x\Big[\text{s-}\int_0^t (dw(s))^{I(\alpha)}$$
$$\times \int_0^s (dw(s_1))^{I(\alpha(1))} \cdots \int_0^{s_{N-1}} (dw(s_N))^{I(\alpha(N))}$$
$$\times b(w(s)) b_{\alpha(1)}(w(s_1)) \cdots b_{\alpha(N)}(w(s_N)) g(w)\Big] \tag{4.9}$$

converges for a particular $\mathcal{F}^\infty$ tame function $g(w)=g(w(u))$, $u\geqq t$, and, moreover, a slight adjustment of the argument in the proof of Theorem 4.2 guarantees that (4.9) also converges for each $\mathcal{F}^\infty$ tame function $g(w)$. We symbolically write (4.9) as

$$E_x\Big[\text{s-}\int_0^t b(w(s)) Z(s,\ w)(dw(s))^{I(\alpha)} g(w)\Big].$$

4.7. COROLLARY. *In the weak sense, $Z(t, w)$ solves the linear stochastic integral equation:*

$$Z(t,\ w) = 1 + \sum_{|\alpha|\leqq 2q} \text{s-}\int_0^t b_\alpha(w(s)) Z(s,\ w)(w(s))^{I(\alpha)}.$$

PROOF. A similar calculation as in the proof of Theorem 4.6 implies: for a particular $\mathcal{F}^\infty$ tame function $g(w)=g(w(u))$, $u\geqq t$,

$$(4.10)\qquad \frac{\partial}{\partial t}E_x[Z(t, w)g(w)] = E_x\Big[Z(t, w)\Big(\sum_{|\alpha|\leqq 2q} b_\alpha(w(s))\partial^\alpha g(w(u))\Big)\Big].$$

On the other hand, Corollary 3.4 and (4.6) yield

$$(4.11)\qquad \frac{\partial}{\partial t}E_x\Big[\Big(\sum_{|\alpha|\leqq 2q} \text{s-}\int_0^t b_\alpha(w(s))Z(s, w)(dw(s))^{I(\alpha)}\Big)g(w(u))\Big]$$
$$= \text{the right hand side of (4.10)}.$$

Here, by (1.2) and Lemma 4.5, (4.11) itself also holds for each $\mathcal{F}^\infty$ tame function g, and the proof of the corollary is complete. □

4.8. LEMMA. *Let* $\{b_\alpha^{(r)}\}_{r=1,2,\cdots}$ *and* $\{g^{(r)}\}_{r=1,2,\cdots}$ *be sequences in* $\mathcal{F}^0(\boldsymbol{R}^d)$ *such that* (4.4) *holds for each* r *and*

$$\lim_r \|b_\alpha - b_\alpha^{(r)}\|_0 = 0, \qquad \lim_r \|g - g^{(r)}\|_0 = 0.$$

Let $Z^{(r)}(t, w)$, $r=1, 2, \cdots$, *be the Girsanov densities in Definition* 4.3 *with* $b_\alpha = b_\alpha^{(r)}$, *respectively. Then,*

$$\lim_r \|E.[Z(t, w)g(w(t))] - E.[Z^{(r)}(t, w)g^{(r)}(w(t))]\|_0 = 0.$$

PROOF. Corollary 3.5 (i), combined with (4.6), imply Lemma 4.8. □

For an $\mathcal{F}^0$ tame function $f(w)=f(w(t_1), \cdots, w(t_N))$, $t_1 \leqq \cdots \leqq t_N$, set

$$\tilde{E}_x[f(w)] \equiv E_x[Z(t_N, w)f(w)], \qquad x \in \boldsymbol{R}^d.$$

Here, for a function $g(s, x)$ and an $\mathcal{F}^0(\boldsymbol{R}^d)$ tame function $f(w)$, we define

$$(4.12)\qquad \tilde{E}_x\Big[\Big(\int_0^t g(s, w(s))ds\Big)f(w)\Big] = \int_0^t \tilde{E}_x[g(s, w(s))f(w)]ds,$$

if the integral on the right hand side exists. (In the particular case $Z(t, w)\equiv 1$, (4.12) automatically holds, because Remark 3.6 (ii) and (3.7) derive that

$$E_x\Big[\Big(\int_0^t b(w(s))ds\Big)f(w(u))\Big] = E_x\Big[\text{s-}\int_0^t b(w(s))(dw(s))^{I(\mathbf{0})}f(w(u))\Big]$$
$$= \int_0^t E_x[b(w(s))f(w(u))]ds, \qquad t \leqq u,$$

where multi index $\mathbf{0}\equiv(0, \cdots, 0)$.)

The system of the expectations $\{\tilde{E}_x[\cdot]\,;\, x \in \boldsymbol{R}^d\}$ solves "martingale problem" for $(A+B)$:

4.9. THEOREM. *Let* $f(w)=f(w(t_1), \cdots, w(t_N), w(t))$, $t_1 \leqq \cdots \leqq t_N \leqq t$, *be an arbitrary* $\mathcal{F}^0$ *tame function. Then, for any function* g *in* $\mathcal{F}^{1,2q}$ *and* $u \geqq t$,

$$\tilde{E}_x\Big[\Big\{g(u,\ w(u))-\int_0^u\Big(\frac{\partial}{\partial t}+A+B\Big)g(s,\ w(s))ds\Big\}f(w)\Big]$$
$$=\tilde{E}_x\Big[\Big\{g(t,\ w(t))-\int_0^t\Big(\frac{\partial}{\partial t}+A+B\Big)g(s,\ w(s))ds\Big\}f(w)\Big]. \tag{4.13}$$

PROOF. Note that: in Lemma 4.5, $f(w(t))$ can be replaced by $f(w)$ in this theorem, by repeating the same argument as in the proof of Lemma 4.5. Then, from (4.12) and this extension of Lemma 4.5,

$$\begin{aligned}&\text{the left hand side of (4.13)}\\ &=E_x\Big[Z(t,\ w)f(w)E_{w(t)}\Big[Z(u-t,\ w)g(u,\ w(u-t))\\ &\qquad -\int_t^u Z(s-t,\ w)\Big(\frac{\partial}{\partial t}+A+B\Big)g(s,\ w(s-t))ds\Big]\Big]\\ &\quad -E_x\Big[\Big\{\int_0^t Z(t,\ w)\Big(\frac{\partial}{\partial t}+A+B\Big)g(s,\ w(s))ds\Big\}f(w)\Big].\end{aligned}$$

Since g is in $\mathcal{F}^{1,2q}$, Theorem 4.6 implies

$$\begin{aligned}&E_y\Big[Z(u-t,\ w)g(u,\ w(u-t))-\int_t^u Z(s-t,\ w)\Big(\frac{\partial}{\partial t}+A+B\Big)g(s,\ w(s-t))ds\Big]\\ &=g(t,\ y)\end{aligned}$$

for any $y\in \boldsymbol{R}^d$. Now the proof is complete. □

§ 5. A stochastic solution.

Let $\mathcal{F}^{0,0}$ be the set of all functions $u(t,\ x)$, $(t,\ x)\in[0,\ T]\times\boldsymbol{R}^d$, such that:

(5.1, i) $\quad u(t,\ x)\in\mathcal{F}^0(\boldsymbol{R}^d)\quad$ for each $t\in[0,\ T]$,

(5.1, ii) $\quad \lim_{s\to t}\|u(t,\ \cdot)-u(s,\ \cdot)\|_0=0\quad$ for each $t\in[0,\ T]$.

5.1. DEFINITION. *A stochastic solution* $W(t,\ x)$ of (0.1) is defined by

$$W(t,\ x)=E_x[Z(t,\ w)f(w(t))], \tag{5.2}$$

where $Z(t,\ w)$ is the Girsanov density in Definition 4.3.

Let $\mu_f(d\zeta)$ and $\nu_\alpha(d\xi)$'s be the measures in $\mathcal{M}^0(\boldsymbol{R}^d)$ corresponding to $f(x)$ and $b_\alpha(x)$'s, respectively. Then, by (3.7) and Definition 4.3, $W(t,\ x)$ can be written as

$$\begin{aligned}W(t,\ x)&=\int\mu_f(d\zeta)\exp\Big\{i\langle\zeta,\ x\rangle-\rho\sum_k\zeta_k^{2q}t\Big\}\\ &\quad+\sum_{N=1}^{\infty}\Big(\sum_{\alpha(1)}\cdots\sum_{\alpha(N)}\Big)\int\mu_f(d\zeta)\int\nu_{\alpha(1)}(d\xi^{(1)})\cdots\int\nu_{\alpha(N)}(ds^{(N)})\end{aligned} \tag{5.3}$$

$$\times\int_{t>s_1>\cdots>s_N>0} ds_1\cdots ds_N \exp\{i\langle\zeta+\xi^{(1)}+\cdots+\xi^{(N)}, x\rangle\}$$
$$\times\left[\prod_n \prod_k (iH_k(n))^{\alpha_k^{(n)}} \exp\{-\rho(H_k(n))^{2q}(s_{n-1}-s_n)\}\right]$$
$$\times\left[\prod_k \exp\{-\rho(H_k(N+1))^{2q}s_N\}\right]\Big),$$

where $H_k(n)$ are given in (2.5).

5.2. DEFINITION. A function $W(t, x)$ of $\mathcal{F}^{0,0}$ is a *wide sense solution* of (0.1), if there is a sequence of sets $\{W^{(n)}(t, x), f^{(n)}(x)\}_{n=1,2,\dots}$ in $\mathcal{F}^{1,2q}\times\mathcal{F}^{2q}(\boldsymbol{R}^d)$ which satisfies the following.

(i) For each n, $W^{(n)}$ is a classical solution of (0.1) with $f=f^{(n)}$,

(ii) $\lim_{n\to\infty}\|f^{(n)}-f\|_0=0$, and

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|W^{(n)}(t,\cdot)-W(t,\cdot)\|_0=0.$$

5.3. THEOREM. *If* (4.4) *holds for* b_α *in* $\mathcal{F}^0(\boldsymbol{R}^d)$, *then the stochastic solution* $W(t, x)$ *on* (5.2) *is well-defined for* f *in* $\mathcal{F}^0(\boldsymbol{R}^d)$ *and is a wide sense solution of* (0.1). *Moreover a wide sense solution of* (0.1) *is unique.*

5.4. REMARK. (4.4) is a sufficient condition, under which $(A+B)$ is strongly elliptic.

PROOF OF THEOREM 5.3. *Step* 1. Theorem 4.2 claims that (5.3) converges and that $W(t, x)$ satisfies (5.1, i). (3.9) and (4.3) imply

$$\lim_{t\to 0}\|W(t,\cdot)-f(\cdot)\|_0=0,$$

and they also imply (5.1, ii), by Lemma 4.5 and (4.6).

Step 2. From (5.3),

$$\|W(t,\cdot)\|_\kappa \leqq \int d|\mu_f|(1+|\zeta|)^\kappa \exp\Big\{-(\mathrm{Re}\rho)\sum_k \zeta_k^{2q}t\Big\} \tag{5.4}$$
$$+\sum_{N\geqq 1}\sum_{\alpha(1)}\cdots\sum_{\alpha(N)}\int d|\mu_f|\int d|\nu_{\alpha(1)}|\cdots\int d|\nu_{\alpha(N)}|$$
$$\times\int_{t>s_1>\cdots>s_N>0} ds_N\cdots ds_1$$
$$\times\left[\prod_n\prod_k |H_k(n)|^{\alpha_k^{(n)}}\exp\{-(\mathrm{Re}\,\rho)(H_k(n))^{2q}(s_{n-1}-s_n)\}\right]$$
$$\times(1+|\zeta+\xi^{(1)}+\cdots+\xi^{(N)}|)^\kappa$$
$$\times\left[\prod_k \exp\{-(\mathrm{Re}\,\rho)(\zeta_k+\xi_k^{(1)}+\cdots+\xi_k^{(N)})^{2q}s_N\}\right],$$

$$(5.5)\quad \left\|\frac{\partial W}{\partial t}(t,\cdot)\right\|_0 \leqq \int d|\mu_f||\rho||\zeta|^{2q}\exp\Big\{-(\mathrm{Re}\,\rho)\sum_k \zeta_k^{2q}t\Big\}$$
$$+\sum_{N\geqq 1}\sum_{\alpha(1)}\cdots\sum_{\alpha(N)}\int d|\mu_f|\int d|\nu_{\alpha(1)}|\cdots\int d|\nu_{\alpha(N)}|$$
$$\times\int_{t>s_1>\cdots>s_{N-1}>0} ds_1\cdots ds_{N-1}\Big[\prod_n\prod_k |H_k(n)|^{\alpha_k^{(n)}}\Big]$$
$$\times\Big(\int_0^{s_{N-1}} ds_N|\rho|\Big(\sum_k|\zeta_k|^{2q}\Big)\Big[\prod_n\prod_k \exp\{-(\mathrm{Re}\,\rho)(H_k(n))^{2q}(s_{n-1}-s_n)\}\Big]$$
$$\times\Big[\prod_k \exp\{-(\mathrm{Re}\,\rho)(H_k(N+1))^{2q}s_N\}\Big]$$
$$+\Big[\prod_{n=1}^{N-1}\prod_k \exp\{-(\mathrm{Re}\,\rho)(H_k(n+1))^{2q}(s_{n-1}-s_n)\}\Big]$$
$$\times\Big[\prod_k \exp\{-(\mathrm{Re}\,\rho)(H_k(N+1))^{2q}s_{N-1}\}\Big]\Big).$$

Now, if f is in $\mathcal{F}^{2q}(\boldsymbol{R}^d)$, then we see that $\|W(t,\cdot)\|_{2q}$ and $\|\partial W(t,\cdot)/\partial t\|_0$ are finite for each $t>0$, by applying the similar arguments as in the proof of Theorem 4.2 to (5.4) and (5.5). In fact, for a sufficiently large number C and a constant c,

$$\|W(t,\cdot)\|_{2q}\leqq \|f\|_{2q}+2^{2q-1}(\|f\|_0+\|f\|_{2q})$$
$$\times\Big(1-\frac{b^*}{\mathrm{Re}\,\rho}-\frac{1}{C}\Big)^{-1}C^{2q}\exp\{c(CTb^{**})^{2q}\},$$
$$\left\|\frac{\partial W}{\partial t}(t,\cdot)\right\|_0\leqq |\rho|\|f\|_{2q}+\Big(|\rho|\|f\|_{2q}+\sum_{|\alpha|\leqq 2q}\|\partial^\alpha f\|_{2q}\Big)$$
$$\times\Big(1-\frac{b^*}{\mathrm{Re}\,\rho}-\frac{1}{C}\Big)^{-1}C^{2q}\exp\{c(CTb^{**})^{2q}\}.$$

Thus we easily observe that W is in $\mathcal{F}^{1,2q}$.

Step 3. Assuming that f is in $\mathcal{F}^{2q}(\boldsymbol{R}^d)$, we shall prove that $W(t,x)$ is an $\mathcal{F}^{1,2q}$-class solution of (0.1) and the solution is unique within $\mathcal{F}^{1,2q}$.

Under the assumption, $W(t,x)$ is in $\mathcal{F}^{1,2q}$, as in Step 2, then Lemma 4.5 and Theorem 4.6 hold for $W(t,x)$. Now our statement in this step is verified, by repeating the proof of Theorem 7.9 in [**4**].

Step 4. For f in $\mathcal{F}^0(\boldsymbol{R}^d)$, it is easy to take a sequence $\{f^{(n)}\}$ in $\mathcal{F}^\infty(\boldsymbol{R}^d)$ such that $\lim_n\|f^{(n)}-f\|_0=0$. For each n, set $W^{(n)}(t,x)=E_x[Z(t,w)f^{(n)}(w(t))]$. From Step 3, $W^{(n)}$ is an $\mathcal{F}^{1,2q}$-class (and a classical) solution of (0.1) with $f=f^{(n)}$.

By a little modification of Lemma 4.8, $\lim_n\sup_{t\in[0,T]}\|W^{(n)}(t,\cdot)-W(t,\cdot)\|_0=0$, and W in (5.2) is a wide sense solution.

Step 5. We shall prove the uniqueness. Let $W'(t,x)$ be a wide sense solution of (0.1). Then there is a sequence of sets $\{W'^{(n)}, f'^{(n)}\}$ as in Definition 5.2. Since a solution of (0.1) is unique in $\mathcal{F}^{1,2q}$, as proved in Step 3, $W'^{(n)}$ must be represented by (5.2) with $f=f'^{(n)}$. Thus we have

$$\|W(t, \cdot)-W'^{(n)}(t, \cdot)\|_0 = \|E.[Z(t, w)f(w(t))]-E.[Z(t, w)f'^{(n)}(w(t))]\|_0.$$

By a slight extension of Lemma 4.8, $\sup_t\|W-W'\|_0=0$. □

We observe regularity of the stochastic solution:

5.5. COROLLARY. *Assume that* (4.4) *holds for* b_α, $|\alpha|\leqq 2q$, *in* $\mathcal{F}^0(\boldsymbol{R}^d)$. *If* f *is in* $\mathcal{F}^{2q}(\boldsymbol{R}^d)$, *then* $W(t, x)$ *is a classical solution of* (0.1).

PROOF. The corollary is clear by the proof of Theorem 5.3. □

Let γ be a number such that $0\leqq\gamma<1$, and set

$$Q(\gamma, \theta) = \frac{\theta^{1-\gamma}}{e(1-\gamma)(1-\theta)}+\left(\frac{1}{\theta}\right)^\gamma.$$

By a simple computation, we easily see: $\min_{0\leqq\theta\leqq 1}Q(\gamma, \theta)=Q(\gamma, \theta_*)\equiv Q_*(\gamma)\geqq 1$, where $0\leqq\theta_*<\sqrt{e}/(\sqrt{e}+1)$ is the non-negative solution of

$$\gamma\theta^2-(1-\gamma)\theta-\frac{\gamma}{e(1-\gamma)}(1-\theta)^2 = 0.$$

5.6. LEMMA. *For* $0\leqq\gamma<1$,

$$\sup_{y\geqq 0}\int_0^\tau ds\left(\frac{1}{\tau}\right)^\gamma y\exp\{-(\operatorname{Re}\rho)y(\tau-s)\} \leqq \left(\frac{1}{\tau}\right)^\gamma\frac{1}{\operatorname{Re}\rho}Q_*(\gamma). \tag{5.6}$$

PROOF. For $0\leqq\theta_0\leqq 1$,

$$\begin{aligned}&\text{the left hand side of (5.6)}\\ &= \left(\frac{1}{\tau}\right)^\gamma\left\{\int_0^{\theta_0}d\theta+\int_{\theta_0}^1 d\theta\right\}\left[\left(\frac{1}{\theta}\right)^\gamma(y\tau)\exp\{-(\operatorname{Re}\rho)y\tau(1-\theta)\}\right]\\ &\equiv L_1+L_2.\end{aligned}$$

Apply Lemma 3.1 to L_1 and carry out the integration in L_2, to obtain

$$L_1 \leqq \left(\frac{1}{\tau}\right)^\gamma\frac{1}{\operatorname{Re}\rho\, e(1-\gamma)}\frac{\theta_0^{1-\gamma}}{1-\theta_0},$$
$$L_2 \leqq \left(\frac{1}{\tau}\right)^\gamma\frac{1}{\operatorname{Re}\rho}\left(\frac{1}{\theta_0}\right)^\gamma.$$

Therefore,

$$\text{the left hand side of (5.6)} \leqq \left(\frac{1}{\tau}\right)^\gamma\frac{1}{\operatorname{Re}\rho}Q(\gamma, \theta_0).$$

Since $\min_{0\leqq\theta_0\leqq 1}Q(\gamma, \theta_0)=Q_*(\gamma)$, Lemma 5.6 has been proved. □

5.7. COROLLARY. *Assume that* $b_\alpha(x)$, $|\alpha|\leqq 2q$, *and* $f(x)$ *are in* $\mathcal{F}^0(\boldsymbol{R}^d)$, *and let* γ *be a number such that* $0\leqq\gamma<1$.

(i) *If*

$$\sum_{|\alpha|=2q} \|b_\alpha\|_0 Q_*(\gamma) < \operatorname{Re}\rho, \tag{5.7}$$

then $\|W(t,\cdot)\|_{2q\gamma}<\infty$ *for any* $t>0$.

(ii) *If* (5.7) *holds for* $1>\gamma\geqq 1/2$, *then* $W(t, x)$ *is a classical solution of* (0.1).

PROOF. *Step* 1. First we shall prove (i). In (5.4), we seek the bound of

$$\begin{aligned}\|K(f)\|_{2q\gamma} \equiv & \int d|\mu_f| \int d|\nu_{\alpha^{(1)}}| \cdots \int d|\nu_{\alpha^{(N)}}| \int_{t>s_1>\cdots>s_N>0} ds_1 \cdots ds_N \\ & \times \Big[\prod_k |H_k(N)|^{\alpha_k^{(N)}} \exp\{-(\operatorname{Re}\rho)(H_k(N))^{2q}(s_{N-1}-s_N)\}\Big] \\ & \times (1+|\zeta+\xi^{\alpha^{(1)}}+\cdots+\xi^{\alpha^{(N)}}|)^{2q\gamma} \Big[\prod_k \exp\{-(\operatorname{Re}\rho)(H_k(N+1))^{2q} s_N\}\Big],\end{aligned} \tag{5.8}$$

under the assumption (3.8). From Lemmas 3.1 and 5.6,

$$\begin{aligned}\|K(f)\|_{2q\gamma} \leqq & \|\mu_f\|_0 \|\nu_{\alpha^{(1)}}\|_0 \cdots \|\nu_{\alpha^{(N)}}\|_0 \Big(\frac{1}{\operatorname{Re}\rho}\Big)^L (c'+c''(Q_*(\gamma))^L) \\ & \times (c\max\{t, t^{1/2q}\})^{N-L} \frac{1}{[(N-L)/2q]!} \Big(\frac{1}{t}\Big)^\gamma,\end{aligned}$$

where c, c', c'' are constants. Now (i) is derived from (5.4) by a similar argument as before.

Step 2. We shall prove (ii). By (i), there is the measure $\mu_{W(s,\cdot)}$ in $\mathcal{M}^{2q\gamma}(\boldsymbol{R}^d)$, which corresponds to $W(s, x)$. From Lemma 4.5,

$$W(t+s, x) = E_x[Z(t, w)W(s, w(t))] = \{(5.3) \text{ for } \mu_f=\mu_{W(s,\cdot)}\}.$$

Here, under (3.8), the bound of $\|K(W(s,\cdot))\|_{2q}$ is obtained from Lemmas 3.1 and 5.6: for constants c, c', c'',

$$\begin{aligned}\|K(W(s,\cdot))\|_{2q} \leqq & \int d|\mu_{W(s,\cdot)}| \int d|\nu_{\alpha^{(1)}}| \cdots \int d|\nu_{\alpha^{(N)}}| \\ & \times \int_{t>s_1>\cdots>s_N>0} ds_1 \cdots ds_N \Big(\sum_k |\zeta_k|^{|\alpha^{(1)}|-2q\gamma}\Big)\Big(\sum_k |\zeta_k|^{2q\gamma}\Big) \\ & \times \Big[\prod_{n=2}^N \prod_k |H_k(n)|^{\alpha_k^{(n)}} \exp\{-(\operatorname{Re}\rho)(H_k(n))^{2q}(s_{n-1}-s_n)\}\Big] \\ & \times (1+|\zeta+\xi^{\alpha^{(1)}}+\cdots+\xi^{\alpha^{(N)}}|)^{2q} \\ & \times \Big[\prod_k \exp\{-(\operatorname{Re}\rho)(\zeta_k+\xi_k^{\alpha^{(1)}}+\cdots+\xi_k^{\alpha^{(N)}})^{2q} s_N\}\Big] \\ \leqq & \|\mu_{W(s,\cdot)}\|_{|\alpha^{(1)}|-2q\gamma} \|\nu_{\alpha^{(1)}}\|_0 \cdots \|\nu_{\alpha^{(N)}}\|_0 (c\max\{t, t^{1/2q}\})^{N-L-1} \\ & \times \Big(\frac{1}{\operatorname{Re}\rho}\Big)^{L+1} (c'+c''(Q_*(\gamma))^{L+1}) \frac{1}{[(N-L-1)/2q]!} \Big(\frac{1}{t}\Big)^\gamma,\end{aligned} \tag{5.9}$$

where $\|\mu_{W(s,\cdot)}\|_{|\alpha^{(1)}|-2q\gamma}<\infty$ by Step 1, since $|\alpha^{(1)}|=2q$ and $\gamma\geqq 1/2$. A similar estimation as (5.9) is proved without (3.8), and (5.4) implies that $\|W(t+s,\cdot)\|_{2q}<\infty$.

By an analogous argument as just said, we see that $\|\partial W(t+s, \cdot)/\partial t\|_0 < \infty$ for any t, $s > 0$, then the proof of Corollary 5.7 is complete. □

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Kunio NISHIOKA
Department of Mathematics
Tokyo Metropolitan University
Fukasawa, Setagaya-ku
Tokyo 158
Japan