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Limit theorems for point processes and their functionals

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1. Introduction.

The classical limit theorems for sums of independent random variables ([**6**]) have been extended in several directions. For instance, Skorohold ([**19**]) discussed functional limit theorems in which sums of independent random variables in a suitable time scale converge to a Lévy process, i.e., a process with independent increments which is continuous in probability. Further, these theorems have been extended to the case of sums of dependent random variables (see e.g. [**5**]). A unified approach to these problems has been given recently by several authors in the framework of semimartingales. Semimartingales extend the notion of Lévy processes and such basic processes as Wiener processes (i.e. Gaussian martingales) and Poisson point processes are simply characterized and naturally extended in the context of semimartingales.

The purpose of this paper is to discuss limit theorems in the framework of semimartingales represented by stochastic integrals of point processes: We discuss on the convergence of point processes and their functionals defined by stochastic integrals. Similar problem was discussed by several authors (e.g. [**5**], [**8**], [**10**], [**16**] and [**17**]), but a main difference is that, in our approach, we do not necessarily assume that the point processes are defined by jumps of semimartingales: Rather, we start with a sequence of point processes and their functionals represented by stochastic integrals and discuss the convergence of them. Our results, of course, overlap those of the above authors but we believe that our proofs are simpler in several points, and it should be remarked that not only Gaussian martingales and Poisson processes but also the general Lévy processes appear in our limit theorems. Also a merit of our approach seems to be in the point that it is useful to clarify the joint convergence of several processes related to a given sequence of point processes. For example, in the case of weighted sums of triangular arrays of random variables, it seems more natural to start with the point processes defined by the original arrays rather than those defined by the weighted sums; we can then consider different weighted sums

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of the arrays at the same time (see Example 7.4 and [13]).

In section 2 we review some basic facts on semimartingales and point processes. In section 3 we summarize the notion of Skorohod's function spaces and convergence of stochastic processes related to them. In section 4, a central limit theorem is given as the convergence of a class of semimartingales to a Gaussian martingale. In section 5 we discuss the convergence of point processes to Poisson point processes. In section 6 the convergence of stochastic integrals based on point processes is studied and, combining these results with those of section 4, we obtain a main theorem (Theorem 6.6) for the convergence of a class of semimartingales to Lévy processes. Several applications will be discussed in later sections.

2. A summary on point processes and semimartingales.

The purpose of this section is to recall some basic facts on semimartingales and point processes. For details [7] can be consulted: We do not restrict ourselves to point processes of the class (QL) as in [7] but necessary modifications are almost obvious.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $\boldsymbol{F}=(\mathcal{F}_t)$, $t\in[0, \infty)$, be a right-continuous family of sub-σ-fields of $\mathcal{F}$ each containing all P-null sets. Such a family $\boldsymbol{F}$ is called a *filtration* on $(\Omega, \mathcal{F}, P)$. We assume that the readers are familiar with such basic notions as $(\mathcal{F}_t)$-adapted processes, $(\mathcal{F}_t)$-predictable processes, $(\mathcal{F}_t)$-(local) martingales, $(\mathcal{F}_t)$-stopping times, etc. (cf. [4] or [7]). We denote by $\mathcal{M}^2(\boldsymbol{F}, P)$, $\mathcal{M}^2_{\mathrm{loc}}(\boldsymbol{F}, P)$ and $\mathcal{M}^c_{\mathrm{loc}}(\boldsymbol{F}, P)$ the spaces of all square-integrable $(\mathcal{F}_t)$-martingales (i.e., all $(\mathcal{F}_t)$-martingales such that $E[M_t^2]<\infty$ for all $t\in[0, \infty)$), all locally square-integrable martingales and all continuous local martingales (which are necessarily locally square-integrable), respectively. For M belonging to these spaces we always assume that $M_0=0$ and that $t\mapsto M_t$ is right-continuous almost surely (abbreviated as *a.s.*). For $M, M'\in\mathcal{M}^2_{\mathrm{loc}}(\boldsymbol{F}, P)$, there exists a unique $(\mathcal{F}_t)$-predictable process $A=(A_t)$ with the following properties: $A_0=0$, $t\mapsto A_t$ is of bounded variation on each finite interval a.s. and $M_tM'_t-A_t$ is a local $(\mathcal{F}_t)$-martingale. This process is denoted by $\langle M, M'\rangle$. Also Meyer introduced the process $[M, M']$: It is defined by

$$[M, M']_t = \langle M^c, M'^c\rangle_t+ \sum_{s\leq t} \Delta M_s\cdot\Delta M'_s \tag{2.1}$$

where M^c is a continuous martingale part of M and $\Delta M_s=M_s-M_{s-}$.

Let $M^i\in\mathcal{M}^c_{\mathrm{loc}}(\boldsymbol{F}, P)$, $i=1, 2, \cdots, d$, such that $\langle M^i, M^j\rangle_t=\phi_{ij}(t)$ are continuous *deterministic* processes, $i, j=1, 2, \cdots, d$. Then, for $t>s\geqq 0$, $M_t-M_s=(M_t^i-M_s^i)_{i=1}^d$ is independent of $\mathcal{F}_s$ and Gaussian distributed, i.e.,

$$(2.2)\qquad E\Big[\exp\Big(\sqrt{-1}\sum_{i=1}^{d}\lambda^i(M_t^i-M_s^i)\Big)\Big|\mathcal{F}_s\Big]$$

$$=\exp\Big\{-\frac{1}{2}\sum_{i,j=1}^{d}(\phi_{ij}(t)-\phi_{ij}(s))\lambda^i\lambda^j\Big\},\qquad (\lambda^i)\in\boldsymbol{R}^d.$$

Hence M_t is a d-dimensional continuous Gaussian process with independent increments. Such a process is called a d-dimensional *Gaussian martingale.*

Let $(X, \mathcal{B}_X)$ be a measurable space. Though the case we mostly discuss in this paper is when X is a locally compact Hausdorff space with a countable open base and $\mathcal{B}_X=\mathcal{B}(X)$ is the topological σ-field of X, we give here a general definition. By a *point function with values in* X or simply a *point function on* X, we mean a function p defined on a countable subset $D_p\subset[0, \infty)$ taking values in X,

$$p\ :\ t\in D_p\longmapsto p(t)\in X.$$

p defines a measure N_p on $[0, \infty)\times X$ with values in $\{0, 1, \cdots, \infty\}$ by

$$N_p(E)=\#\{t\in D_p:(t, p(t))\in E\}.$$

N_p is called the *counting measure associated with* p. It is obvious that p can be recovered from its counting measure N_p. Let Π_X be the totality of point functions on X and $\mathcal{B}(\Pi_X)$ be the smallest σ-field on Π_X with respect to which the map $p\mapsto N_p(E)$ is measurable for every $E\in\mathcal{B}([0, \infty))\times\mathcal{B}_X$. Random elements of the space $(\Pi_X, \mathcal{B}(\Pi_X))$, i.e., a $(\Pi_X, \mathcal{B}(\Pi_X))$-valued random variable defined on a probability space is called a *point process taking values in* X, or a *point process on* X, simply. Then the associated counting measure N_p is a random point measure on $[0, \infty)\times X$.

Let $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)$ be as above and we consider point processes defined on this probability space. A point process p on X is called $(\mathcal{F}_t)$-*adapted* if, for every $B\in\mathcal{B}_X$, $t\mapsto N_p([0, t]\times B)$ is $(\mathcal{F}_t)$-adapted. It is called σ-*finite* if $U_n\in\mathcal{B}_X$, $n=1, 2, \cdots$, exist such that $U_n\subset U_{n+1}$, $\bigcup_n U_n=X$ and with probability one $N_p([0, t]\times U_n)<\infty$ for every $n=1, 2, \cdots$ and $t>0$. In this case we can find for every $n=1, 2, \cdots$, an increasing sequence of $(\mathcal{F}_t)$-stopping times $\tau_k^{(n)}$ such that $\lim_{k\to\infty}\tau_k^{(n)}=\infty$ a.s. and

$$(2.3)\qquad E[N_p([0, t\wedge\tau_k^{(n)}]\times U_n)]<\infty$$

for every $t>0$ and $n, k=1, 2, \cdots$. In the case when X is a locally compact Hausdorff space with a countable open base, it is always understood that the above definition of σ-finiteness is referred to $\{U_n\}$ which is a *compact exhaustion* of X.

Let p be a σ-finite $(\mathcal{F}_t)$-adapted point process on X. We say that p *possesses the compensator* $\hat{N}_p$ if a non-negative random measure $\hat{N}_p(E)$ on $[0, \infty)\times X$ exists, i.e., $\hat{N}_p(E)$, $E\in\mathcal{B}([0, \infty))\times\mathcal{B}_X$, is $\bar{\boldsymbol{R}}_+=[0, \infty]$-valued random variable and, with probability one, $E\mapsto\hat{N}_p(E)$ is a measure on $\{[0, \infty)\times X, \mathscr{B}([0, \infty))\times\mathscr{B}_X\}$, such that the following hold:

(i) $t\mapsto\hat{N}_p([0, t]\times U)$ is $(\mathscr{F}_t)$-predictable for every $U\in\mathscr{B}_X$,

(ii) if U_n, $n=1, 2, \cdots$, are those subsets in the definition of σ-finiteness of p, then $\hat{N}_p([0, t]\times U_n)<\infty$ for all $t>0$ and $n=1, 2, \cdots$, a.s.,

(iii) $t\mapsto N_p([0, t]\times(U_n\cap B))-\hat{N}_p([0, t]\times(U_n\cap B))$ is a local $(\mathscr{F}_t)$-martingale for every $n=1, 2, \cdots$ and $B\in\mathscr{B}_X$.

$\hat{N}_p$ is uniquely determined (up to an obvious equivalence) from N_p and $\hat{N}_p(\{s\}\times B)\leqq 1$ for every $B\in\mathscr{B}_X$ and $s\geqq 0$. In [7] we considered the case $\hat{N}_p(\{s\}\times B)\equiv 0$ exclusively and called such a class of point processes as class (QL). The existence of compensators is assured fairly generally: In particular, if X is a locally compact Hausdorff space with a countable open base, the compensators exist for every $(\mathscr{F}_t)$-adapted σ-finite point process on X.

Suppose that p posseses the compensator $\hat{N}_p$. We set

$$\tilde{N}_p([0, t]\times B)=N_p([0, t]\times B)-\hat{N}_p([0, t]\times B).$$

If $B\subset U_n$ for some n, then $t\mapsto\tilde{N}_p([0, t]\times B)$ is a local martingale and actually it is an element of $\mathscr{M}^2_{\mathrm{loc}}(\boldsymbol{F}, P)$. We can show by the same but obviously modified arguments as in [7] (page 61) that

$$\text{(2.4)}\qquad \begin{aligned}&\langle\tilde{N}_p([0, t]\times B), \tilde{N}_p([0, t]\times B')\rangle\\ &\quad=\hat{N}_p([0, t]\times(B\cap B'))-\sum_{s\leqq t}\hat{N}_p(\{s\}\times B)\cdot\hat{N}_p(\{s\}\times B'),\end{aligned}$$

if $B, B'\in\mathscr{B}([0, \infty))\times\mathscr{B}_X$ and if $B, B'\subset U_n$ for some n.

A real function defined on $[0, \infty)\times X\times\Omega$ is called $(\mathscr{F}_t)$-predictable if the mapping $(t, x, \omega)\mapsto f(t, x, \omega)$ is $\mathcal{S}/\mathscr{B}(\boldsymbol{R})$-measurable where $\mathcal{S}$ is the smallest σ-field on $[0, \infty)\times X\times\Omega$ with respect to which all g having the following properties are measurable:

(i) for each $t\geqq 0$, $(x, \omega)\mapsto g(t, x, \omega)$ is $\mathscr{B}_X\times\mathscr{F}_t/\mathscr{B}(\boldsymbol{R})$-measurable,

(ii) for each (x, ω), $t\mapsto g(t, x, \omega)$ is left-continuous.

For a given $(\mathscr{F}_t)$-adapted, σ-finite point process p on X possessing the compensator $\hat{N}_p$, we introduce the following classes:

$$\Phi_p(\boldsymbol{F}, P)=\Big\{f(t, x, \omega)\ :\ f \text{ is } (\mathscr{F}_t)\text{-predictable and for each } t>0,\ \int_0^{t+}\int_X|f(s, x, \omega)|N_p(ds\,dx)<\infty \text{ a.a. } \omega\Big\},$$

$$\Phi_p^1(\boldsymbol{F}, P)=\Big\{f(t, x, \omega)\ :\ f \text{ is } (\mathscr{F}_t)\text{-predictable and for every } t>0,\ E\Big[\int_0^{t+}\int_X|f(s, x, \cdot)|\hat{N}_p(ds\,dx)\Big]<\infty\Big\},$$

$$\Phi_p^2(\boldsymbol{F}, P)=\Big\{f(t, x, \omega)\ :\ f \text{ is } (\mathcal{F}_t)\text{-predictable and for every } t>0,\ E\Big[\int_0^{t+}\int_X |f(s, x, \cdot)|^2\hat{N}_p(ds\,dx)\Big]<\infty\Big\}$$

and

$$\Phi_p^{2,\mathrm{loc}}(\boldsymbol{F}, P)=\{f(t, x, \omega)\ :\ f \text{ is } (\mathcal{F}_t)\text{-predictable and there exists a sequence of stopping times } \sigma_n \text{ such that } \sigma_n\uparrow\infty \text{ a.s. and } I_{[0,\sigma_n]}(t)f(t, x, \omega)\in\Phi_p^2(\boldsymbol{F}, P),\ n=1, 2, \cdots\}.$$

These classes are denoted simply by Φ_p, Φ_p^1, Φ_p^2 and $\Phi_p^{2,\mathrm{loc}}$ if there is no danger of confusions.

First for $f\in\Phi_p$, we define $\int_0^{t+}\int_X f(s, x, \omega)N_p(ds\,dx)$ ω-wise as the usual Lebesgue-Stieltjes integral and this is clearly equal to the absolutely convergent sum $\sum f(s, p(s), \omega)$ where the summation runs over $s\leqq t$, $s\in D_p$. Next, let $f\in\Phi_p^1$. Then we have

$$E\Big[\int_0^{t+}\int_X |f(s, x, \omega)|N_p(ds\,dx)\Big]=E\Big[\int_0^{t+}\int_X |f(s, x, \omega)|\hat{N}_p(ds\,dx)\Big].$$

This implies, in particular, that $\Phi_p^1\subset\Phi_p$. We set

$$\begin{aligned}(2.5)\qquad &\int_0^{t+}\int_X f(s, x, \omega)\tilde{N}_p(ds\,dx)\\ &=\int_0^{t+}\int_X f(s, x, \omega)N_p(ds\,dx)-\int_0^{t+}\int_X f(s, x, \omega)\hat{N}_p(ds\,dx).\end{aligned}$$

If, furthermore, $f\in\Phi_p^1\cap\Phi_p^2$ then we can show that $t\mapsto\int_0^{t+}\int_X f(s, x, \omega)\tilde{N}_p(ds\,dx)$ $\in\mathcal{M}^2(\boldsymbol{F}, P)$ and that

$$\begin{aligned}(2.6)\qquad &\Big\langle\int_0^{t+}\int_X f(s, x, \omega)\tilde{N}_p(ds\,dx)\Big\rangle\\ &=\int_0^{t+}\int_X f(s, x, \omega)^2\hat{N}_p(ds\,dx)-\sum_{s\leqq t}\Big\{\int_X f(s, x, \omega)\hat{N}_p(\{s\}\times dx)\Big\}^2.\end{aligned}$$

If $f\in\Phi_p^2$, we set

$$f_{n,k}(s, x, \omega)=I_{(-n,n)}(f(s, x, \omega))I_{U_n}(x)I_{[0,\tau_k^{(n)}]}(s)f(s, x, \omega)$$

where $U_n\in\mathcal{B}_X$ are those subsets of X in the definition of σ-finiteness and $\tau_k^{(n)}$ are stopping times satisfying (2.3). Then it is easy to see that $f_{n,k}\in\Phi_p^1\cap\Phi_p^2$ and

$$E\Big[\int_0^{t+}\int_X |f_{n,k}(s, x, \omega)-f_{n',k'}(s, x, \omega)|^2\hat{N}_p(ds\,dx)\Big]\longrightarrow 0\quad\text{as}\quad k, k'\to\infty\quad\text{and}\quad n, n'\to\infty.$$

Since this expectation dominates

$$E\left[\left\{\int_0^{t+}\int_X f_{n,k}(s, x, \omega)\tilde{N}_p(ds\,dx)-\int_0^{t+}\int_X f_{n',k'}(s, x, \omega)\tilde{N}_p(ds\,dx)\right\}^2\right],$$

we see that there exists unique $M\in\mathcal{M}^2(\boldsymbol{F}, P)$ such that

$$E\left[\left\{M(t)-\int_0^{t+}\int_X f_{n,k}(s, x, \omega)\tilde{N}_p(ds\,dx)\right\}^2\right]\longrightarrow 0,$$

as $k, n\to\infty$. It is easy to see that M is uniquely determined from f indifferently to a particular choice of U_n and $\tau_k^{(n)}$. This M is denoted by $\int_0^{t+}\int_X f(s, x, \omega)$ $\tilde{N}_p(ds\,dx)$. Finally, if $f\in\Phi_p^{2,\mathrm{loc}}$ then $\int_0^{t+}\int_X f(s, x, \omega)\tilde{N}_p(ds\,dx)$ is defined to be the unique element $M\in\mathcal{M}_{\mathrm{loc}}^2(\boldsymbol{F}, P)$ such that

$$M(t\wedge\sigma_n)=\int_0^{t+}\int_X I_{[0,\sigma_n]}(s)f(s, x, \omega)\tilde{N}_p(ds\,dx), \qquad n\geqq 1,$$

where $\{\sigma_n\}$ is a family of stopping times in the definition of $\Phi_p^{2,\mathrm{loc}}$.

For $f, g\in\Phi_p^{2,\mathrm{loc}}$, it holds

$$\text{(2.7)}\qquad \left\langle\int_0^{t+}\int_X f(s, x, \omega)\tilde{N}_p(ds\,dx),\ \int_0^{t+}\int_X g(s, x, \omega)\tilde{N}_p(ds\,dx)\right\rangle$$
$$=\int_0^{t+}\int_X f(s, x, \omega)g(s, x, \omega)\hat{N}_p(ds\,dx)$$
$$-\sum_{s\leqq t}\left\{\int_X f(s, x, \omega)\hat{N}_p(\{s\}\times dx)\right\}\left\{\int_X g(s, x, \omega)\hat{N}_p(\{s\}\times dx)\right\}.$$

The most important class of point processes is of course that of *Poisson point processes*: Generally, a point process p on X is called a Poisson point process if the following are satisfied;
(i) if $E_1, E_2, \cdots, E_n\in\mathcal{B}([0, \infty))\times\mathcal{B}_X$ are disjoint then $N_p(E_1), N_p(E_2), \cdots, N_p(E_n)$ are independent,
(ii) for $E\in\mathcal{B}([0, \infty))\times\mathcal{B}_X$, $N_p(E)$ is Poisson distributed, the case $N_p(E)=\infty$ a.s. being allowed as a Poisson random variable with infinite expectation.

An $(\mathcal{F}_t)$-adapted Poisson point process is called *$(\mathcal{F}_t)$-Poisson point process* if, for every $s\geqq 0$, the family $N_p((s, t]\times B)$, $t\geqq s$, $B\in\mathcal{B}_X$ is independent of $\mathcal{F}_s$. Let p be an $(\mathcal{F}_t)$-Poisson point process which is σ-finite. Then

$$\text{(2.8)}\qquad \nu_p(E)=E[N_p(E)]$$

defines a σ-finite measure on $\{[0, \infty)\times X, \mathcal{B}([0, \infty))\times\mathcal{B}_X\}$ in the sense that there exist $U_n\in\mathcal{B}_X$, $n\geqq 1$, such that $U_n\subset U_{n+1}$, $\bigcup_n U_n=X$ and $\nu_p([0, t]\times U_n)<\infty$ for all $n\geqq 1$, $t>0$. ν_p satisfies

$$\text{(2.9)}\qquad \nu_p(\{s\}\times X)=0 \qquad \text{for all}\quad s\geqq 0.$$

p possesses the compensator $\hat{N}_p$ which coincides with the deterministic measure ν_p. *This property characterizes a Poisson point process.* Namely, if an $(\mathcal{F}_t)$-adapted σ-finite point process p possesses the compensator which is a deterministic measure on $\{[0, \infty)\times X, \mathcal{B}([0, \infty))\times \mathcal{B}_X\}$ satisfying (2.9) then p is an $(\mathcal{F}_t)$-Poisson point process. Furthermore, given ν on $\{[0, \infty)\times X, \mathcal{B}([0, \infty))\times \mathcal{B}_X\}$ which is σ-finite in the above sense and satisfies the condition (2.9) we can construct a σ-finite $(\mathcal{F}_t)$-Poisson point process with compensator $\hat{N}_p=\nu$ on some probability space $(\Omega, \mathcal{F}, P)$ with $(\mathcal{F}_t)$.

Martingale characterization for Gaussian martingales and Poisson point processes stated above can be combined together in the following form: Let $M(t)=(M^1(t), \cdots, M^d(t))$ and $p_1, p_2, \cdots, p_n$ be given on $(\Omega, \mathcal{F}, P)$ with $(\mathcal{F}_t)$ where $M^i\in\mathcal{M}_{\mathrm{loc}}^c$ such that $\langle M^i, M^j\rangle_t=\phi_{ij}(t)$ are continuous *deterministic* processes $i, j=1, 2, \cdots, d$ and where $p_1, p_2, \cdots, p_n$ are σ-finite $(\mathcal{F}_t)$-point processes on $X_1, X_2, \cdots, X_n$, respectively, possessing compensators $\hat{N}_{p_1}, \hat{N}_{p_2}, \cdots, \hat{N}_{p_n}$ which are *deterministic* σ-finite measures (in the above sense) on $\{[0, \infty)\times X_i, \mathcal{B}([0, \infty))\times \mathcal{B}_{X_i}\}$ satisfying $\hat{N}_{p_i}(\{s\}\times X_i)=0$, $s\geqq 0$, for $i=1, 2, \cdots, n$. Furthermore, we assume that the domains D_{p_i} are mutually disjoint a. s. Then $M(t)$ is a d-dimensional Gaussian martingale and p_i are $(\mathcal{F}_t)$-Poisson point processes on X_i, respectively such that $\{M(\cdot), p_1, \cdots, p_n\}$ are mutually independent. In the succeeding sections this characterization will play the key role.

3. Skorohod's function space.

The purpose of this section is to put on record notations and elementary facts on Skorohod's function space for the later reference.

By $D([0, \infty):\boldsymbol{R}^d)$ $(d\geqq 1)$ we denote the space of all right-continuous $\boldsymbol{R}^d$-valued functions on $[0, \infty)$ having left limits. We endow this space with the J_1-topology (see [**15**]). By $D([0, \infty):\boldsymbol{R})^d$ we denote the product space $D([0, \infty):\boldsymbol{R})\times\cdots\times D([0, \infty):\boldsymbol{R})$ which is of course endowed with the product topology (i. e., the convergence in this space is defined as that of every component). Similarly we can define $D([0, \infty):\boldsymbol{R}^n)\times D([0, \infty):\boldsymbol{R}^m)$ in the same manner. The reader should notice that $D([0, \infty):\boldsymbol{R}^n)\times D([0, \infty):\boldsymbol{R}^m)$ may be identified with $D([0, \infty):\boldsymbol{R}^{n+m})$ as a set but the topology of the first is weaker than that of the latter.

The convergence in law of random elements of these spaces will be denoted by $X_n\xrightarrow{\mathcal{D}}X$. This notation will also be used to express the weak convergence of the laws of random elements of any other topological spaces. When we need to emphasize the space, we write, for example, $X_n\xrightarrow{\mathcal{D}}X$ in $D([0, \infty):\boldsymbol{R}^d)$, etc. By $X_n\xrightarrow{\mathcal{D}_f}X$ we denote the convergence of all finite-dimensional marginal distributions, and by $\xi_n\xrightarrow{P}\xi$ we denote the convergence in probability when ξ_n and

ξ are $\boldsymbol{R}^d$-valued random variables. Thus, $X_n(t)\xrightarrow{P}X(t)$, $t\geqq 0$ means that

$$P[|X_n(t)-X(t)|\geqq\varepsilon]\longrightarrow 0 \qquad \text{for every} \quad \varepsilon>0,\ t\geqq 0.$$

The following four lemmas are easy to prove and we omit the proofs (cf. section 4 of [**20**]).

LEMMA 3.1. *Let x_n, $x\in D([0,\infty):\boldsymbol{R})$ and suppose that, for every n, $x_n(t)$ is nondecreasing in t. If $x_n(t)\to x(t)$ for every $t\geqq 0$ and if $x(t)$ is continuous then the convergence is uniform on every finite interval.*

LEMMA 3.2. *Let x_n, x, y_n and $y\in D([0,\infty):\boldsymbol{R}^d)$.*
(i) *If $(x_n, y_n)\to(x, y)$ in $D([0,\infty):\boldsymbol{R}^d)\times D([0,\infty):\boldsymbol{R}^d)$ and if y is continuous then $(x_n, y_n)\to(x, y)$ in $D([0,\infty):\boldsymbol{R}^{2d})$.*
(ii) *If $(x_n, y_n)\to(x, y)$ in $D([0,\infty):\boldsymbol{R}^{2d})$ then*

$$x_n+y_n\longrightarrow x+y \qquad \text{in} \quad D([0,\infty):\boldsymbol{R}^d). \tag{3.1}$$

LEMMA 3.3. *Let $X_n=(X_n(t))_{t\geqq 0}$ and $X=(X(t))_{t\geqq 0}$ be stochastic processes with sample paths in $D([0,\infty):\boldsymbol{R})$, and suppose that X_n and X are nondecreasing in t a.s. If X has continuous sample paths with probability one and if $X_n\xrightarrow{\mathcal{D}_f}X$, then $X_n\xrightarrow{\mathcal{D}}X$ in $D([0,\infty):\boldsymbol{R})$. Especially, if X is a deterministic, continuous function then $X_n(t)\xrightarrow{P}X(t)$, $t\geqq 0$ implies that $X_n\xrightarrow{\mathcal{D}}X$.*

LEMMA 3.4. *Let X_n, X and Y_n be stochastic processes with sample paths in $D([0,\infty):\boldsymbol{R}^d)$ and let ϕ be an $\boldsymbol{R}^d$-valued continuous function. If $X_n\xrightarrow{\mathcal{D}}X$ and $Y_n\xrightarrow{\mathcal{D}}\phi$, then $X_n+Y_n\xrightarrow{\mathcal{D}}X+\phi$.*

Lemma 3.4 is of course an easy consequence of Lemma 3.2.

Recently several authors have given useful criterions for the tightness of a family of martingales (see Lemma 6 of [**16**], also [**17**]). The next lemma is a version which is suited for our later use.

LEMMA 3.5. *Let $M_n\in\mathcal{M}^2_{\mathrm{loc}}$, $n\geqq 1$ and let*

$$\langle M_n\rangle_t\xrightarrow{P}\phi(t), \qquad t\geqq 0 \tag{3.2}$$

where $\phi(t)$ is a continuous deterministic function. Then the family $\{M_n\}_n$ is tight in $D([0,\infty):\boldsymbol{R})$. Furthermore, if in addition, there exists $C>0$ such that $P[\sup_t|\Delta M_n(t)|\leqq C]=1$, $n\geqq 1$, then any limit process M of $\{M_n\}_n$ is a square-integrable martingale such that $\langle M\rangle_t=\phi(t)$ a.s.

PROOF. Since the first half is Lemma 6 of [**16**], we will prove only the latter half. Clearly it suffices to consider on every finite time-interval $[0, T]$, $T>0$. We assume for the moment that there exists $C=C(T)>0$ such that $\langle M_n\rangle_T \leqq C$, a.s., $n=1, 2, \cdots$. By Lemma 3.6, which we will state later, we have

$$\sup_n E[M_n(t)^4] < \infty, \qquad 0\leqq t\leqq T. \tag{3.3}$$

Now let M be any limit process of $\{M_n\}$. We need to prove that $M(t)$ and $M(t)^2-\phi(t)$ are martingales. Let $0\leqq t_1<\cdots<t_k\leqq s<t\leqq T$ $(k\geqq 1)$ and let H be a bounded continuous function on $\boldsymbol{R}^k$. Since $M_n\in\mathcal{M}^2$, we have that

$$E[(M_n(t)-M_n(s))H(M_n(t_1), \cdots, M_n(t_k))] = 0. \tag{3.4}$$

Let T_p consist of those t in $[0, T]$ for which $P[M(t)\neq M(t-)]=0$. $[0, T]\setminus T_p$ is at most countable (see page 124 of [**1**]). If $t_1, \cdots, t_k, s, t\in T_p$, then keeping (3.3) in mind we have from the well-known continuity theorem (see Theorem 5.1 and page 124 of [**1**]) that

$$E[(M(t)-M(s))H(M(t_1), \cdots, M(t_k))] = 0. \tag{3.5}$$

Since T_p is dense in $[0, T]$, we easily see that (3.5) holds for all $0\leqq t_1<\cdots<t_k \leqq s<t\leqq T$. Thus we have that $M(t)$ is a martingale and in a similar way we can show that $M(t)^2-\phi(t)$ is also a martingale. It is a standard argument to drop the assumption that $\langle M_n\rangle_T\leqq C$: We define $\sigma_n=\inf\{t\leqq T : \langle M_n\rangle_t\geqq\phi(T)+1\}$ $(\sigma_n=T$ if $\{\ \}=\emptyset)$ and put $M_n'(t)=M_n(t\wedge\sigma_n)$. Since $P[M_n'\neq M_n]=P[\sigma_n<T]\leqq P[\langle M_n\rangle_T\geqq\phi(T)+1]\to 0$, we have that M is a limit process of $\{M_n'\}$ (cf. Theorem 4.1 of [**1**]). Since we can apply the previous argument to $\{M_n'\}$, we have the assertion.

LEMMA 3.6. *Let $M\in\mathcal{M}^2_{\mathrm{loc}}(\boldsymbol{F}, P)$ and suppose that $\langle M\rangle_T\leqq C_1$ and $\sup_{0\leqq t\leqq T}|\Delta M(t)| \leqq C_2$ a.s. Then, for $0<\lambda<1/\{4(\sqrt{C_1}+C_2)\}$, it holds that*

$$E[\exp\{\lambda \sup_{0\leqq t\leqq T}|M(t)|\}] \leqq 1/\{1-4\lambda(\sqrt{C_1}+C_2)\}.$$

PROOF. Let $0\leqq\tau\leqq\sigma$ be $(\mathcal{F}_t)$-stopping times. Then we have

$$\begin{aligned} &E[|M(t\wedge\sigma)-M(t\wedge\tau-)| \,/\, \mathcal{F}_{t\wedge\tau}] \\ &\leqq |\Delta M(t\wedge\tau)|+E[|M(t\wedge\sigma)-M(t\wedge\tau)| \,/\, \mathcal{F}_{t\wedge\tau}] \\ &\leqq C_2+E[(M(t\wedge\sigma)-M(t\wedge\tau))^2 \,/\, \mathcal{F}_{t\wedge\tau}]^{1/2} \\ &\leqq C_2+\sqrt{C_1}. \end{aligned}$$

Therefore, we have the assertion (see page 193 of [**4**]).

With a slight modification of the proof of Lemma 3.5 we obtain

LEMMA 3.7. *Let $M_n\in\mathcal{M}^2_{\mathrm{loc}}(\boldsymbol{F}^n, P^n)$, $n=1, 2, \cdots$ satisfy* (3.2) *and we assume* *that there exists* $C>0$ *such that* $\sup_{t\geq 0}|\Delta M_n(t)|\leqq C$, *a.s. Let* X_n, $n=1, 2, \cdots$ *be* $(\mathcal{F}_t^n)$*-adapted random elements of* $D([0, \infty):\boldsymbol{R}^d)$ *converging in law to* X. *Then* $\{(M_n, X_n)\}_n$ *is tight in* $D([0, \infty):\boldsymbol{R})\times D([0, \infty):\boldsymbol{R}^d)$ *and furthermore any limit process* $(\tilde{M}, \tilde{X})$ *satisfies the following.*

(i) $\tilde{X}$ *is identical in law to* X.

(ii) $\tilde{M}$ *is a square-integrable martingale such that* $\langle\tilde{M}\rangle=\phi$ *with respect to the filtration generated by* $(\tilde{M}, \tilde{X})$.

4. A central limit theorem.

The aim of this section is to give a central limit theorem for local martingales of the form

$$X_n(t) = M_n(t)+\int_0^{t+}\int_X f_n(s, x, \omega)\tilde{N}_{p_n}(ds\,dx)$$

where $M_n\in\mathcal{M}_{\mathrm{loc}}^c(\boldsymbol{F}^n, P^n)$ and $f_n\in\Phi_{p_n}^{2,\mathrm{loc}}(\boldsymbol{F}^n, P^n)$, $n\geqq 1$. (The underlying probability space may depend on n.)

THEOREM 4.1. *Let* M *be a Gaussian martingale with quadratic characteristic* $\langle M\rangle$ *and suppose that the following two conditions are satisfied.*

(A) *For every* $T>0$, *there exist positive constants* $a_n\to 0$ *such that*

$$\sup_{\substack{0\leq t\leq T\\ x\in X}}|f_n(t, x, \omega)|\leqq a_n, \qquad a.s.$$

(B) $\langle X_n\rangle_t\xrightarrow{P}\langle M\rangle_t$, $t\geqq 0$.

Then

$$X_n\xrightarrow{\mathcal{D}} M, \qquad n\to\infty.$$

PROOF. This theorem may be reduced to the result of [**16**]. However, for our later use, we will give another simple proof. By assumption (A) we have

$$\sup_{0\leq t\leq T}|\Delta X_n(t)|\leqq 2a_n, \qquad \text{a.s.} \tag{4.1}$$

Therefore, applying Lemma 3.5 we have from (B) that $\{X_n\}_n$ is tight and that any limit process X^* is a square-integrable martingale such that $\langle X^*\rangle=\langle M\rangle$. Thus if X^* has continuous paths with probability one, we can conclude that X^* and M are identical in law (see section 2), which proves that $X_n\xrightarrow{\mathcal{D}}M$. To see that X^* is continuous a.s., we need only to recall that the maximum discontinuity is a continuous functional with respect to the Skorohod J_1-topology: The continuity of the paths of X^* is clear from (4.1) because $a_n\to 0$.

We next consider the multi-dimensional case. Let $M_n^i\in\mathcal{M}_{\mathrm{loc}}^c$, $f_n^i\in\Phi_{p_n}^{2,\mathrm{loc}}$

($i=1, 2, \cdots, d$), $n=1, 2, \cdots$, and define d-dimensional stochastic processes $X_n=(X_n^1, \cdots, X_n^d)$, $n=1, 2, \cdots$, by

$$X_n^i(t) = M_n^i(t)+\int_0^{t+}\int_X f_n^i(s, x, \omega)\tilde{N}_{p_n}(ds\,dx), \qquad i = 1, 2, \cdots, d. \tag{4.2}$$

THEOREM 4.2. *Let $M=(M^1, \cdots, M^d)$ be a Gaussian martingale with quadratic characteristic $\langle M^i, M^j\rangle$, and assume the following conditions.*
(A′) $f_n=(f_n^1, \cdots, f_n^d)$ *satisfies* (A) *in Theorem* 4.1.
(B′) $\langle X_n^i, X_n^j\rangle_t \xrightarrow{P} \langle M^i, M^j\rangle_t$, $1\leqq i, j\leqq d$, $t\geqq 0$.
Then,

$$X_n \xrightarrow{\mathscr{D}} M \quad \text{in} \quad D([0, \infty): \boldsymbol{R}^d).$$

PROOF. Applying Theorem 4.1 to each component, we see that $\{X_n\}_n$ is tight (see Lemma 3.2 (i)) and every limit process has continuous paths a.s. Let X be any limit process. With a slight modification of the proof of Theorem 4.1, we easily see that X^i ($i=1, 2, \cdots, d$) and $X^iX^j-\langle M^i, M^j\rangle$ ($1\leqq i, j\leqq d$) are martingales with respect to the filtration generated by $X=(X^i)$. This proves that X and M are identical in law, which proves the theorem.

5. Convergence to Poisson point processes.

Let Y be a locally compact Hausdorff space with a countable open basis. By $C_K(Y)$ we denote the space of all continuous real valued functions defined on Y vanishing outside a compact subset of Y. $\mathfrak{M}(Y)$ denotes the totality of non-negative Radon measures and is endowed with the vague topology: $\mu_n\in\mathfrak{M}(Y)$ converges to $\mu\in\mathfrak{M}(Y)$ if and only if $\int f(x)\mu_n(dx)\to\int f(x)\mu(dx)$ for every $f\in C_K(Y)$. It is well known that $H\subset\mathfrak{M}(Y)$ is relatively compact if and only if $\sup\left\{\left|\int f\mu\right| : \mu\in H\right\}<\infty$ for all $f\in C_K(Y)$. Let $\mathfrak{N}(Y)$ be the totality of Radon measures with values in $\{0, 1, \cdots\}\cup\{\infty\}$. $\mathfrak{N}(Y)$ is a closed subset of $\mathfrak{M}(Y)$. A measurable mapping ξ from a probability space to $(\mathfrak{M}(Y), \mathscr{B}(\mathfrak{M}))$, where $\mathscr{B}(\mathfrak{M})$ is the topological σ-field, is called a random measure. The convergence in law of random measures is defined as usual: ξ_n converges to ξ in law, which we denote by $\xi_n\xrightarrow{\mathscr{D}}\xi$ in $\mathfrak{M}(Y)$, if $P\circ\xi_n^{-1}$ converges weakly to $P\circ\xi^{-1}$. It is well known that $\xi_n\xrightarrow{\mathscr{D}}\xi$ holds if and only if $\int f\xi_n\xrightarrow{\mathscr{D}}\int f\xi$ for every $f\in C_K(Y)$, which condition is also equivalent to

$$(\xi_n(A_1), \cdots, \xi_n(A_d)) \xrightarrow{\mathscr{D}} (\xi(A_1), \cdots, \xi(A_d))$$

for $d\geqq 1$ and all Borel sets $A_1, \cdots, A_d$ of Y contained in a compact subset satisfying $P[\xi(\partial A_j)=0]=1$, $j=1, 2, \cdots, d$.

It should be remarked that since $\mathfrak{M}(Y)$ is a Polish space, by Skorohod's theorem we can realize the convergence in law by an almost sure convergence on some probability space without changing the laws of the random measures. So far we explained only the notations and facts we need in this paper. For details and proofs we refer to Jagers [**9**] (see also [**11**]).

Hereafter we will consider the case where $Y=[0, \infty)\times X$, X being a locally compact Hausdorff space with a countable open basis. When there is no danger of confusion, we will often drop Y from $\mathfrak{M}(Y)$, $\mathfrak{N}(Y)$, $C_K(Y)$ etc. and write simply as $\mathfrak{M}$, $\mathfrak{N}$, C_K etc. Let p be a point process taking values in X. Then the counting measure N_p is a random element of $\mathfrak{N}$. A random element of $\mathfrak{N}$ is also often called a point process but in this paper we will not adopt this terminology because it is not compatible with that in section 2. In fact a point process in our sense (defined in section 2) corresponds to a random element of

$$\mathfrak{N}_0 = \{\xi\in\mathfrak{N} : \xi(\{s\}\times X)=0 \text{ or } 1 \text{ for all } s\geqq 0\}.$$

Indeed, we clearly have $P[N_p\in\mathfrak{N}_0]=1$ for any point process p and conversely, a random measure ξ satisfying $P[\xi\in\mathfrak{N}_0]=1$ may easily be identified with a counting measure of a point process. It should be noticed that $\mathfrak{N}$ is closed in $\mathfrak{M}$ but $\mathfrak{N}_0$ is not.

THEOREM 5.1. *Let p_n, $n=1, 2, \cdots$, be $\boldsymbol{F}^n=(\mathcal{F}_t^n)$-point processes on X and let $\mu\in\mathfrak{M}$ be a deterministic measure such that $\mu(\{t\}\times X)=0$ for all $t\geqq 0$. (The underlying probability space may depend on n.) Suppose that*

$$\hat{N}_{p_n}(dt\,dx) \xrightarrow{\mathcal{D}} \mu(dt\,dx) \quad \text{in } \mathfrak{M}, \qquad \text{as } n\to\infty. \tag{5.1}$$

Then

$$N_{p_n}(dt\,dx) \xrightarrow{\mathcal{D}} N_p(dt\,dx) \quad \text{in } \mathfrak{M}, \qquad \text{as } n\to\infty \tag{5.2}$$

where p is the Poisson point process with compensator μ.

REMARK. A necessary and sufficient condition for (5.1) is

$$\int_0^\infty\int_X f(s, x)\hat{N}_{p_n}(ds\,dx) \xrightarrow{P} \int_0^\infty\int_X f(s, x)\mu(ds\,dx) \tag{5.3}$$

for every $f\in C_K$.

For simple point processes, the result was obtained by T. Brown [**2**, **3**] and Kabanov-Liptser-Shiryayev [**10**], and the discrete case is due to Durrett-Resnick [**5**]. Combining these results with Rényi's theorem [**18**], we can easily prove Theorem 5.1. In this sense our theorem is essentially due to the above authors.

However, in the succeeding sections we will need the next theorem which treats the joint convergence of Theorems 4.2 and 5.1, and its proof does not seem to be carried out using the idea of the above authors. Thus we will use a quite different (and natural) method to prove the following theorem which includes Theorem 5.1.

THEOREM 5.2. *We assume all conditions in Theorems* 4.2 *and* 5.1 *with respect to filtrations* $\boldsymbol{F}^n=(\mathcal{F}_t^n)$, $n=1, 2, \cdots$. *Then,*

$$(X_n, N_{p_n}) \xrightarrow{\mathcal{D}} (M^*, N_{p^*}) \quad in \quad D([0, \infty): \boldsymbol{R}^d)\times\mathfrak{M},$$

where M^* *and* p^* *are mutually independent and identical in law to* M *and* p *in Theorems* 4.2 *and* 5.1, *respectively.*

The idea of the proof is as follows: We first prove that $\{N_{p_n}\}_n$ is tight in $\mathfrak{M}$ and any limit measure is in fact the counting measure of a suitable point process. Then by choosing a subsequence we can assume that $(X_n, N_{p_n}) \xrightarrow{\mathcal{D}} (\tilde{X}, N_p)$ for some $(\tilde{X}, p)$. To prove the theorem it suffices to show that $\tilde{X}$ is a Gaussian martingale and that $\hat{N}_p=\mu$ with respect to the filtration generated by $\tilde{X}$ and p (see the characterization of Gaussian martingales and Poisson point processes stated in section 2). Here are the details of the proof.

LEMMA 5.3. *Let* $f\in C_K$ *and define*

$$A_n(t) = \int_0^{t+}\int_X f(s, x)\hat{N}_{p_n}(ds\,dx),$$

$$Z_n(t) = \int_0^{t+}\int_X f(s, x)N_{p_n}(ds\,dx)$$

and

$$W_n(t) = \int_0^{t+}\int_X f(s, x)\tilde{N}_{p_n}(ds\,dx) \quad (=Z_n(t)-A_n(t)).$$

Under the assumption of Theorem 5.1, $\{A_n\}_n$, $\{Z_n\}_n$ *and* $\{W_n\}_n$ *are tight in* $D([0, \infty): \boldsymbol{R})$. *Furthermore, any limit process* W *of* $\{W_n\}_n$ *is a square-integrable martingale such that*

$$\langle W\rangle_t = \int_0^t\int_X f(s, x)^2\mu(ds\,dx).$$

PROOF. The tightness of $\{A_n\}_n$ is obvious because A_n converges in law to $A(t)=\int_0^t\int_X f(s, x)\mu(ds\,dx)$. Indeed, if $f\geqq 0$ then this may easily be checked using Lemma 3.3. To drop the condition $f\geqq 0$, consider $f=f^+-f^-$ and apply Lemma 3.4. We next prove the tightness of $\{W_n\}_n$. By Lemma 3.5 it suffices to show that $\langle W_n\rangle_t$ converges to a continuous function. Observe that

$$\sum_{0\leq s\leq t}\left(\int f(s,\ x)\hat{N}_{p_n}(\{s\}\times dx)\right)^2\leqq \max_{s\leq t} \Delta\bar{A}_n(s)\cdot\bar{A}_n(t) \tag{5.4}$$

where $\bar{A}_n(t)=\int_0^{t+}\int|f|\hat{N}_{p_n}$. Since $\bar{A}_n(t)\xrightarrow{\mathscr{D}}\int_0^t\int|f|\mu(ds\,dx)$, which is continuous, we have that the right-hand side of (5.4) converges in probability to 0. Therefore, we see that

$$\begin{aligned}\langle W_n\rangle_t &= \int_0^{t+}\int f^2\hat{N}_{p_n}(ds\,dx)-\sum_{s\leq t}\left(\int f\hat{N}_{p_n}(\{s\}\times dx)\right)^2\\ &\xrightarrow{P}\int_0^t\int f^2\mu(ds\,dx).\end{aligned}$$

Thus we have the tightness of $\{W_n\}_n$. The latter half of the assertion of Lemma 5.3 is also proved by Lemma 3.5. Since $Z_n=A_n+W_n$, the tightness of $\{Z_n\}_n$ may be reduced to that of $\{A_n\}_n$ and $\{W_n\}_n$ (see Lemma 3.4).

We are now ready to prove Theorem 5.1 and 5.2.

PROOF OF THEOREMS 5.1 AND 5.2. As we mentioned before, the tightness in $\mathfrak{M}$ of $\{N_{p_n}\}_n$ is equivalent to that of the family of random variables $\left\{\int_0^\infty\int_X f(s,\ x)\,N_{p_n}(ds\,dx)\right\}_n$ for every $f\in C_K$. However, the latter is clear from Lemma 5.3. Now let $\xi(ds\,dx)$ be any limit of $\{N_{p_n}\}_n$. Since $\mathfrak{N}$ is closed, it holds that $\xi\in\mathfrak{N}$. However, we need to show that ξ is the counting measure of a point process (i.e., $\xi\in\mathfrak{N}_0$ a.s.). To this end it suffices to show that $\int_0^{t+}\int_X f(s,\ x)\xi(ds\,dx)$ has no discontinuities greater than 1 a.s. for every $f\in C_K$ satisfying $0\leqq f\leqq 1$. Since we assumed that $N_{p_{n'}}\xrightarrow{\mathscr{D}}\xi$ in $\mathfrak{M}$, it is easy to see that

$$\begin{aligned}Z_{n'}(t) &= \int_0^{t+}\int_X f(s,\ x)N_{p_{n'}}(ds\,dx)\\ &\xrightarrow{\mathscr{D}_f} Z(t)=\int_0^{t+}\int_X f(s,\ x)\xi(ds\,dx).\end{aligned}$$

Since we have the tightness of $\{Z_n\}_n$ by Lemma 5.3, this proves that $Z_{n'}\xrightarrow{\mathscr{D}}Z$. Keeping in mind that $\Delta Z_n(t)\leqq 1$ for every $t\geqq 0$, a.s., we have that $\Delta Z(t)\leqq 1$ for every $t\geqq 0$ a.s., which proves that $\xi\in\mathfrak{N}_0$. Therefore there exists a unique point process p such that $N_p=\xi$; in particular, $D_p=\{t\geqq 0:\xi(\{t\}\times X)=1\}$. To see that p is a Poisson point process it suffices to show that the compensator is μ. Since $\hat{N}_{p_n}\xrightarrow{\mathscr{D}}\mu$ and $N_{p_{n'}}\xrightarrow{\mathscr{D}}N_p$, it is almost obvious that $\hat{N}_p=\mu$. Indeed, as we have seen in the above, for any $f\in C_K$, we have that

(5.5)
$$\int_0^{t+}\int f(s, x)N_{p_{n'}}(ds\,dx) \xrightarrow{\mathcal{D}} \int_0^{t+}\int f(s, x)N_p(ds\,dx).$$

Since
$$\int_0^{t+}\int f(s, x)\hat{N}_{p_n}(ds\,dx) \xrightarrow{\mathcal{D}} \int_0^{t}\int f(s, x)\mu(ds\,dx),$$

we have from Lemma 3.4 and (5.5) that

(5.6)
$$\int_0^{t+}\int_X f(s, x)\tilde{N}_{p_{n'}}(ds\,dx)$$
$$\xrightarrow{\mathcal{D}} \int_0^{t+}\int_X f(s, x)N_p(ds\,dx)-\int_0^{t}\int_X f(s, x)\,\mu(ds\,dx).$$

By Lemma 5.3 the right-hand side of (5.6) is a martingale, which implies that p possesses the deterministic compensator μ. This proves Theorem 5.1. The proof of Theorem 5.2 can be carried out in a similar way: By Theorems 4.2 and 5.1 we have that X_n converges in law to a continuous process M and N_{p_n} converges in law to N_p. Thus we have the tightness of $\{(X_n, N_{p_n})\}_n$ in $D([0, \infty): \boldsymbol{R}^d)\times\mathfrak{M}$. Let (M^*, N_{p^*}) be any limit point. We need to prove that M^{*i}, $M^{*i}(t)M^{*j}(t)-\langle M^i, M^j\rangle_t$ and $N_{p^*}([0, t]\times E)-\mu([0, t]\times E)$ are martingales with respect to the filtration generated by (M^*, p^*). But this may easily be checked by repeating the proof of Theorems 4.2 and 5.1 simultaneously (but use Lemma 3.7 in place of Lemma 3.5).

6. Convergence of stochastic integrals.

Throughout this section we assume the conditions of Theorem 5.1. Hence we have $N_{p_n}\xrightarrow{\mathcal{D}}N_p$ (Theorem 5.1) and that $\int_0^{t+}\int fN_{p_n}$ and $\int_0^{t+}\int_0 f\tilde{N}_{p_n}$ converge in law to $\int_0^{t+}\int fN_p$ and $\int_0^{t+}\int f\tilde{N}_p$ respectively, provided that $f(s, x)\in C_K$. In this section we study the convergence of stochastic integrals $\int_0^{t+}\int f_nN_{p_n}$ and $\int_0^{t+}\int f_n\tilde{N}_{p_n}$ where the integrand f_n does not necessarily have a compact support and may depend on n.

NOTATION. Let $f_n(t, x)$ and $f(t, x)$ be $\boldsymbol{R}^d$-valued measurable functions on $([0, \infty)\times X, \mathcal{B}([0, \infty))\times\mathcal{B}_X)$ and let $\nu\in\mathfrak{M}$. We say that f_n *converges continuously* to f (ν-a.e.) and write "$f_n\xrightarrow{\text{c.c.}}f$ (ν-a.e.)" if and only if there exists a ν-null set $E\in\mathcal{B}([0, \infty))\times\mathcal{B}_X$ such that, if $(t, x)\notin E$ then $f_n(t_n, x_n)\to f(t, x)$ whenever $(t_n, x_n)\to(t, x)$.

Clearly, if $f_n(t, x)$ converges uniformly to a continuous $f(t, x)$ then $f_n\xrightarrow{\text{c.c.}}f$ (ν-a.e.) for every $\nu\in\mathfrak{M}$. The following fact is well known and easily proved (see e.g. [**11**] page 94 A7.3).

LEMMA 6.1. *Let ν_n, $\nu \in \mathfrak{M}$ and let f_n and f be $\boldsymbol{R}^d$-valued measurable functions on $[0, \infty)\times X$. Assume that*

(i) $f_n \xrightarrow{\text{c.c.}} f$ (ν-a.e.),

(ii) $\nu_n \longrightarrow \nu$ *in* $\mathfrak{M}$,

(iii) *there exists $C>0$ such that for every n, $|f_n| \leqq C$ (ν_n-a.e.)*

and

(iv) *f_n and f vanish identically outside a common compact set $K \subset [0, \infty)\times X$.*

Then,

$$\int_0^\infty \int_X f_n(s, x)\nu_n(ds\,dx) \longrightarrow \int_0^\infty \int_X f(s, x)\nu(ds\,dx) \qquad \text{in } \boldsymbol{R}^d.$$

Lemma 6.1 may be strengthened as follows.

LEMMA 6.2. *In addition to the conditions of Lemma 6.1, assume that ν is continuous in t; i.e.,*

(v) $\nu(\{t\}\times X)=0$ *for every* $t \geqq 0$.

Then,

$$\int_0^{t+} \int_X f_n(s, x)\nu_n(ds\,dx) \longrightarrow \int_0^t \int_X f(s, x)\nu(ds\,dx) \tag{6.1}$$

in $D([0, \infty): \boldsymbol{R}^d)$.

PROOF. Clearly it suffices to consider the case $d=1$, and keeping Lemma 3.2 in mind we can also assume that $f_n \geqq 0$ (for the general case consider f_n^+ and f_n^-). Now by the previous lemma we have that (6.1) holds for every fixed $t \geqq 0$, which combined with Lemma 3.1 proves our assertion.

When we consider $\int_0^{t+}\int f N_{p_n}$, we need to modify Lemma 6.2 a little because N_p does not satisfy the continuity condition (v) except in the trivial case. Thus we prepare

LEMMA 6.3. *Let ν_n, $\nu \in \mathfrak{N}_0$ and assume* (i), (ii) *and* (iv) *of Lemma* 6.1 (*we may drop* (iii)). *If $\nu(\{0\}\times X)=0$ then*

$$\int_0^{t+} \int_X f_n(s, x)\nu_n(ds\,dx) \longrightarrow \int_0^{t+} \int_X f(s, x)\nu(ds\,dx)$$

in $D([0, \infty): \boldsymbol{R}^d)$.

PROOF. All necessary idea is found in Jagers [**9**]: Without loss of generality we may assume that K in (iv) satisfies that $\nu(\partial K)=0$. Therefore, by (ii) we have that $\nu_n(K)\to\nu(K)$. Thus we may assume that $\nu_n(K)=\nu(K)$ $(=r$, say) for all sufficiently large n since ν_n and ν are integral valued. The restrictions ν_n^* and ν^* of ν_n and ν to K may be expressed as follows:

$$\nu_n^*(ds\,dx)=\sum_{i=1}^{r}\delta_{(t_i^n,\,x_i^n)}(ds\,dx)\,,$$

$$\nu(ds\,dx)=\sum_{i=1}^{r}\delta_{(t_i,\,x_i)}(ds\,dx)\,,$$

where $\delta_{(t,u)}(ds\,dx)$ is the Dirac measure at (t, u). Since ν_n^* converges weakly to ν^*, we may assume that $(t_i^n, x_i^n)\to(t_i, x_i)$, $n\to\infty$ $(i=1, 2, \cdots, r)$. Therefore if we number t_i's so that $0<t_1<t_2<\cdots<t_r$ then it holds that $0<t_1^n<t_2^n<\cdots<t_r^n$ for all sufficiently large n. Now observe that $x_n(t)=\int_0^{t+}\int f\nu_n$ and $x(t)=\int_0^{t+}\int f\nu$ may be expressed as $x_n(t)=\sum_{t_i^n\le t}f_n(t_i^n, x_i^n)$ and $x(t)=\sum_{t_i\le t}f(t_i, x_i)$, respectively. Let us define polygons as follows: let $\lambda_n(0)=0$, $\lambda_n(t_i)=t_i^n$ and $\lambda_n(t)=t$ if $t\geqq t_r+1$, and $\lambda_n(t)$ be linear on intervals $[0, t_1]$, $[t_1, t_2]$, $\cdots$, $[t_r, t_r+1]$. Since $t_i^n\to t_i$ as $n\to\infty$ $(i=1, 2, \cdots, r)$, we have that $\lambda_n(t)$ converges to $\lambda(t)\equiv t$ uniformly. To prove our lemma it suffices to show that $x_n(\lambda_n(t))\to x(t)$ uniformly for $t\geqq 0$. To this end observe that $x_n(\lambda_n(t))=\sum_{t_i^n\le\lambda_n(t)}f_n(t_i^n, x_i^n)=\sum_{t_i\le t}f_n(t_i^n, x_i^n)$. Thus $x_n(\lambda_n(t))$, $n=1, 2, \cdots$ are step functions with common time of discontinuities. Since $(t_i^n, x_i^n)\to(t_i, x_i)$, we have from the assumption (i) that $f_n(t_i^n, x_i^n)\to f(t_i, x_i)$, which combined with the above fact proves that $x_n(\lambda_n(t))\to x(t)$ uniformly for $t\geqq 0$.

NOTATION. Let $f(s, x, \omega)=(f^1(s, x, \omega), \cdots, f^d(s, x, \omega))$ be $\boldsymbol{R}^d$-valued function on $[0, \infty)\times X\times\Omega$. We write $f\in\Phi_p$ (or Φ_p^2) if and only if $f^i\in\Phi_p$ (or Φ_p^2, respectively), $i=1, \cdots, d$. $\int_0^{t+}\int_X fN_p$ is defined to be $\left(\int_0^{t+}\int_X f^1N_p, \cdots, \int_0^{t+}\int_X f^dN_p\right)$, if $f\in\Phi_p$. $\int_0^{t+}\int_X f\tilde{N}_p$ is defined in a similar way if $f\in\Phi_p^2$.

PROPOSITION 6.4. *Let f_n, g_n, f and g be $\boldsymbol{R}^d$-valued measurable functions on $[0, \infty)\times X$ vanishing outside a common compact set $K\subset[0, \infty)\times X$. Under the condition of Theorem 5.1, if $f_n\xrightarrow{c.c.}f$ (μ-a.e.) and $g_n\xrightarrow{c.c.}g$ (μ-a.e.) and if f_n is uniformly bounded, then*

$$\left(\int_0^{t+}\int_X f_n(s, x)\tilde{N}_{p_n}(ds\,dx),\ \int_0^{t+}\int_X g_n(s, x)N_{p_n}(ds\,dx)\right)$$

$$\xrightarrow{\mathcal{D}}\left(\int_0^{t+}\int_X f(s, x)\tilde{N}_p(ds\,dx),\ \int_0^{t+}\int_X g(s, x)N_p(ds\,dx)\right),$$

in $D([0, \infty):\boldsymbol{R}^{2d})$.

The assumption that f_n is uniformly bounded may be dropped if

$$(*)\qquad \int_0^{t+}\int_X f_n(s,\,x)\hat{N}_{p_n}(ds\,dx) \xrightarrow{\mathcal{D}} \int_0^t\int_X f(s,\,x)\mu(ds\,dx)\,.$$

PROOF. As we mentioned in section 5, the convergence of (5.2) may be realized by an almost sure convergence on some suitable probability space. Thus we may and do assume that (5.2) holds a.s. Since μ-null set is also N_p-null set with probability one, it follows from our assumption that $(f_n,\,g_n) \xrightarrow{\text{c.c.}} (f,\,g)$ (N_p-a.s.) with probability one. Therefore, applying Lemma 6.3 we see that

$$\Big(\int_0^{t+}\int_X f_n N_{p_n},\ \int_0^{t+}\int_X g_n N_{p_n}\Big) \longrightarrow \Big(\int_0^{t+}\int_X f N_p,\ \int_0^{t+}\int_X g N_p\Big), \qquad \text{a.s.}$$

By Lemma 6.2 we also have

$$\int_0^{t+}\int_X f_n \hat{N}_{p_n} \longrightarrow \int_0^t\int_X f\hat{N}_p\,, \qquad \text{a.s.}$$

Therefore, combining these two with the definition $\tilde{N}_{p_n}=N_{p_n}-\hat{N}_{p_n}$ we have that

$$\Big(\int_0^{t+}\int_X f_n \tilde{N}_{p_n},\ \int_0^{t+}\int_X g_n N_{p_n}\Big) \longrightarrow \Big(\int_0^{t+}\int_X f \tilde{N}_p,\ \int_0^{t+}\int_X g N_p\Big), \qquad \text{a.s.,}$$

(see Lemma 3.2 (i) and (ii)).

We next relax the assumption that f_n and g_n, $n\geqq 1$, vanish outside a common compact set. Let $K_1 \Subset K_2 \Subset \cdots$ be a compact exhaustion of X.

THEOREM 6.5. *Let f_n, g_n, f and g be $\boldsymbol{R}^d$-valued measurable functions on $[0,\,\infty)\times X$ satisfying that* (i) $f_n\in\Phi^2_{p_n}$, (ii) $g_n\in\Phi_{p_n}$, (iii) $\{f_n\}_n$ *is uniformly bounded. We assume that the conditions of Theorem* 5.1 *are satisfied and let $(f_n,\,g_n) \xrightarrow{\text{c.c.}} (f,\,g)$ (μ-a.e.) as $n\to\infty$ where $f\in\Phi^2_p$ and $g\in\Phi_p$. If for every $\varepsilon>0$, $T>0$,*

$$\lim_{k\to\infty}\ \limsup_{n\to\infty}\ P\Big[\int_0^{T+}\int_{X\setminus K_k} |f_n|^2 \hat{N}_{p_n}(ds\,dx) \geqq \varepsilon\Big] = 0 \tag{6.2}$$

and

$$\lim_{k\to\infty}\ \limsup_{n\to\infty}\ P\Big[\int_0^{T+}\int_{X\setminus K_k} |g_n| \hat{N}_{p_n}(ds\,dx) \geqq \varepsilon\Big] = 0\,, \tag{6.3}$$

then

$$\begin{aligned}&\Big(\int_0^{t+}\int_X f_n(s,\,x)\tilde{N}_{p_n}(ds\,dx),\ \int_0^{t+}\int_X g_n(s,\,x)N_{p_n}(ds\,dx),\ N_{p_n}\Big)\\ &\xrightarrow{\mathcal{D}} \Big(\int_0^{t+}\int_X f(s,\,x)\tilde{N}_p(ds\,dx),\ \int_0^{t+}\int_X g(s,\,x)N_p(ds\,dx),\ N_p\Big)\end{aligned} \tag{6.4}$$

in $D([0, \infty): \boldsymbol{R}^{2d}) \times \mathfrak{M}$.

The assumption (iii) *may be dropped if for every* $k \geqq 1$,

$$(**) \qquad \int_0^{t+}\int_{K_k} f_n(s, x)\hat{N}_{p_n}(ds\,dx) \xrightarrow{\mathcal{D}} \int_0^t\int_{K_k} f(s, x)\mu(ds\,dx).$$

PROOF. Let $\phi_k(x)$, $k=1, 2, \cdots$ be continuous functions such that $I_{K_k}(x) \leqq \phi_k(x) \leqq I_{K_{k+1}}(x)$, and define

$$W_n^k(t) = \int_0^{t+}\int_X \phi_k(x) f_n(s, x)\tilde{N}_{p_n}(ds\,dx),$$

$$W^k(t) = \int_0^{t+}\int_X \phi_k(x) f(s, x)\tilde{N}_p(ds\,dx),$$

$$Z_n^k(t) = \int_0^{t+}\int_X \phi_k(x) g_n(s, x) N_{p_n}(ds\,dx),$$

and

$$Z^k(t) = \int_0^{t+}\int_X \phi_k(x) g(s, x) N_p(ds\,dx), \qquad k, n \geqq 1.$$

For every fixed $k \geqq 1$, we have from Proposition 6.4 that as $n \to \infty$,

$$(W_n^k(t), Z_n^k(t)) \xrightarrow{\mathcal{D}} (W^k(t), Z^k(t)) \quad \text{in} \quad D([0, \infty): \boldsymbol{R}^{2d}).$$

Therefore, to prove that $(W_n, Z_n) \xrightarrow{\mathcal{D}} (W, Z)$ it suffices to show the following (see Billingsley [**1**] Theorem 4.2).

$$(6.5) \qquad \lim_{k\to\infty} \limsup_{n\to\infty} P\Big[\sup_{0\leqq t\leqq T} |W_n^k(t) - W_n(t)| \geqq \varepsilon\Big] = 0$$

and

$$(6.6) \qquad \lim_{k\to\infty} \limsup_{n\to\infty} P\Big[\sup_{0\leqq t\leqq T} |Z_n^k(t) - Z_n(t)| \geqq \varepsilon\Big] = 0.$$

However, (6.6) is immediate from (6.3), while (6.5) follows from (6.2) by the Lenglart inequality ([**14**]) (see also Corollary of Lemma 1 of [**16**]).

REMARK. (6.4) implies, for example,

$$(6.7) \qquad \int_0^{t+}\int f_n \tilde{N}_{p_n} + \int_0^{t+}\int g_n N_{p_n} \xrightarrow{\mathcal{D}} \int_0^{t+}\int f\tilde{N}_p + \int_0^{t+}\int g N_p \quad \text{in} \quad D([0, \infty): \boldsymbol{R}^d).$$

(See Lemma 3.2 (ii).)

In (6.4) (or (6.7)) the limiting processes are Lévy processes without Gaussian part. But if we combine Theorems 4.2 and 6.5 we have the following theorem where the limiting processes have both Gaussian and Poisson parts.

THEOREM 6.6. *We assume all the conditions of Theorem* 5.1. *Let* f_n, g_n, f *and* g *be as in Theorem* 6.5 *but we drop* (6.2); *instead we assume that for every* $T>0$

$$\lim_{k\to\infty}\limsup_{n\to\infty}\sup\{|f_n(t,x)| : 0\leqq t\leqq T,\ x\notin K_k\}=0. \tag{6.8}$$

Let $M_n^i\in\mathcal{M}_{\mathrm{loc}}^c$ *and define* $X_n=(X_n^1,\cdots,X_n^d)$ *and* $X_{n,k}=(X_{n,k}^1,\cdots,X_{n,k}^d)$ $(n,k\geqq 1)$ *by*

$$X_n^i(t)=M_n^i(t)+\int_0^{t+}\int_X f_n^i\tilde{N}_{p_n}+\int_0^{t+}\int_X g_n^i N_{p_n}$$

and

$$X_{n,k}^i(t)=M_n^i(t)+\int_0^{t+}\int_{X\setminus K_k} f_n^i\tilde{N}_{p_n}.$$

If there exists a continuous $\boldsymbol{R}^d\otimes\boldsymbol{R}^d$*-valued function* $\phi(t)=(\phi^{ij}(t))_{i,j}$ *such that for every* $t\geqq 0$ *and* $\delta>0$,

$$\lim_{k\to\infty}\limsup_{n\to\infty}P[|\langle X_{n,k}^i, X_{n,k}^j\rangle_t-\phi^{ij}(t)|\geqq\delta]=0, \tag{6.9}$$

then

$$X_n(t)\xrightarrow{\mathcal{D}}M(t)+\int_0^{t+}\int_X f\tilde{N}_p+\int_0^{t+}\int_X gN_p$$

in $D([0,\infty),\boldsymbol{R}^d)$, *where* p *is a Poisson point process possessing compensator* μ *and* M *is a Gaussian martingale independent of* p *such that* $\langle M^i, M^j\rangle=\phi^{ij}$.

The assumption (iii) (*in Theorem* 6.5) *may be replaced by* (**).

PROOF. Choose $1\leqq k(1)\leqq k(2)\leqq\cdots\to\infty$ and let $f_n^{(1)}(t,x)=f_n(t,x)I_{K_{k(n)}}(x)$, $f_n^{(2)}(t,x)=f_n(t,x)-f_n^{(1)}(t,x)$, $n\geqq 1$. Define

$$W_n(t)=M_n(t)+\int_0^{t+}\int_X f_n^{(2)}\tilde{N}_{p_n},$$

$$Y_n(t)=\int_0^{t+}\int_X f_n^{(1)}\tilde{N}_{p_n},$$

and

$$Z_n(t)=\int_0^{t+}\int_X g_nN_{p_n},\qquad n\geqq 1.$$

Notice that we have by definition $X_n(t)=W_n(t)+Y_n(t)+Z_n(t)$. Our idea of the proof is to apply the central limit theorem to W_n and Theorem 6.5 to (Y_n, Z_n) and then consider the joint convergence of (W_n, Y_n, Z_n) using Theorem 5.2. Now as we will see later, if $k(n)$ tends to infinity slowly enough, we can assume that, for every $T>0$, (6.10) $$\sup\{|f_n^{(2)}(t, x)| : 0 \leqq t \leqq T, \ x \in X\} \longrightarrow 0,$$

(6.11) $$\langle W_n^i, W_n^j\rangle_t \xrightarrow{P} \phi^{ij}(t), \qquad t \geqq 0, \quad 1 \leqq i, j \leqq d,$$

and

(6.12) $$\lim_{m\to\infty} \limsup_{n\to\infty} P\left[\int_0^T\int_{X\setminus K_m} |f_n^{(1)}|^2 \hat{N}_{p_n} \geqq \varepsilon\right] = 0, \qquad \text{for all} \quad \varepsilon > 0.$$

By Theorem 4.2, we have from (6.10) and (6.11) that $W_n \xrightarrow{\mathscr{D}} M$. In fact we also have the independence of M and p: By Theorem 5.2 we have that $(W_n, N_{p_n}) \xrightarrow{\mathscr{D}} (M, N_p)$ where M and p are independent. Therefore, keeping in mind that $\int_0^{t+}\int f\tilde{N}_p$ and $\int_0^{t+}\int gN_p$ are functionals of p, we have the assertion of the theorem if we show that $(Y_n, Z_n) \xrightarrow{\mathscr{D}} \left(\int_0^{t+}\int f\tilde{N}_p, \int_0^{t+}\int gN_p\right)$ (cf. Lemma 3.2 (ii)). However, this is done in Theorem 6.5. ($f_n^{(1)}$ plays the role of f_n in Theorem 6.5.) Indeed, (6.2) is satisfied by (6.12) while other assumptions of Theorem 6.5 are also satisfied by assumption. Now let us return to the proof of (6.10)-(6.12). Let us consider (6.12) first. Let $\phi_m(x)$ be as in the proof of Theorem 6.5 and put

$$\xi_i^{(n)} = \int_0^{T+}\int \{\phi_{i+1}(x) - \phi_i(x)\} |f_n|^2 \hat{N}_{p_n}(ds\,dx), \qquad i, n \geqq 1,$$

and

$$a_i = \int_0^T\int \{\phi_{i+1}(x) - \phi_i(x)\} |f|^2 \hat{N}_p(ds\,dx), \qquad i \geqq 1.$$

Observe that, if $m \leqq k(n)$ then

(6.13) $$\begin{aligned} &\int_0^{T+}\int_{X\setminus K_{m+1}} |f_n^{(1)}(s, x)|^2 \hat{N}_{p_n}(ds\,dx) \\ &\leqq \int_0^{T+}\int_X \{\phi_{k(n)+1} - \phi_m\} |f_n|^2 \hat{N}_{p_n} = \sum_{i=m}^{k(n)} \xi_i^{(n)}. \end{aligned}$$

Thus (6.12) may be reduced to

(6.14) $$\lim_{m\to\infty} \limsup_{n\to\infty} P\left[\sum_{i=m}^{k(n)} \xi_i^{(n)} \geqq \varepsilon\right] = 0.$$

To see that (6.14) (hence (6.12)) holds for any $\{k(n)\}_n$ tending to infinity slowly enough, notice that $\sum a_i \leqq \int_0^T\int |f|^2 \mu(ds\,dx) < \infty$ (recall that $f \in \Phi_p^2$). Thus the assertion is established by the next lemma.

LEMMA 6.7. *Let a_i, $i=1, 2, \cdots$ be nonnegative numbers such that $\sum a_i < \infty$, and let $\xi_i^{(n)}$, $i, n \geqq 1$ be nonnegative random variables satisfying*

(6.15) $$\xi_i^{(n)} \xrightarrow{P} a_i \quad \text{as} \quad n \to \infty$$

for every $i \geqq 1$. *Then there exist* $k(1) \leqq k(2) \leqq \cdots \to \infty$ *such that*

(6.16) $$\lim_{m\to\infty} \limsup_{n\to\infty} P\left[\sum_{i=m}^{k(n)} \xi_i^{(n)} \geqq \delta\right] = 0, \qquad \textit{for every} \quad \delta > 0$$

and

(6.17) $$\sum_{i=1}^{k(n)} \xi_i^{(n)} \xrightarrow{P} \sum_{i=1}^{\infty} a_i, \qquad \text{as} \quad n \to \infty.$$

PROOF. By (6.15) we can choose $\tau(1) < \tau(2) < \cdots$ so that

(6.18) $$P[|\xi_i^{(n)} - a_i| > 2^{-i}] \leqq 2^{-i} \qquad \text{for all} \quad n \geqq \tau(i).$$

Now define $k(n) = j$ iff $\tau(j) \leqq n < \tau(j+1)$. Since $j \leqq k(n)$ implies that $n \geqq \tau(j)$, we see from (6.18) that

(6.19) $$P[\xi_j^{(n)} > 2^{-j} + a_j] \leqq 2^{-j} \qquad \text{for all} \quad j \leqq k(n).$$

For any given $\delta > 0$, it holds that $\sum_{j=m}^{\infty} (2^{-j} + a_j) < \delta$ for all sufficiently large m. Therefore, we obtain that

$$\begin{aligned} P\left[\sum_{j=m}^{k(n)} \xi_j^{(n)} > \delta\right] &\leqq P\left[\sum_{j=m}^{k(n)} \xi_j^{(n)} > \sum_{j=m}^{k(n)} (2^{-j} + a_j)\right] \\ &\leqq \sum_{j=m}^{k(n)} P[\xi_j^{(n)} > 2^{-j} + a_j] \leqq 2^{1-m}, \end{aligned}$$

for all sufficiently large m by (6.19). Thus we have (6.16). (6.17) follows from (6.15) and (6.16) (see Theorem 4.2 of [**1**]).

We now return to the proof of Theorem 6.6 and prove that (6.10) and (6.11) are automatically satisfied for $\{k(n)\}_n$ chosen in the above: (6.10) is obvious by (6.8). To see (6.11), observe that

$$\begin{aligned} &|\langle W_n^i, W_n^j \rangle_t - \langle X_{n,m}^i, X_{n,m}^j \rangle_t| \\ &\leqq |\langle W_n^i, W_n^j - X_{n,m}^j \rangle_t| + |\langle W_n^i - X_{n,m}^i, X_{n,m}^j \rangle_t| \\ &\leqq \langle W_n^i \rangle_t^{1/2} \langle W_n^j - X_{n,m}^j \rangle_t^{1/2} + \langle X_{n,m}^j \rangle_t^{1/2} \langle W_n^i - X_{n,m}^i \rangle_t^{1/2} \\ &\leqq 2\left(\sum_{q=1}^{k(n)} \xi_q^{(n)}\right)^{1/2} \left(\sum_{q=m-1}^{k(n)} \xi_q^{(n)}\right)^{1/2}, \qquad \text{if} \quad k(n) \geqq m > 1. \end{aligned}$$

(Since the extreme left-hand side does not depend on M_n, we assume that $M_n \equiv 0$.) Therefore, by (6.16) and (6.17) we easily see that for every $\delta > 0$,

(6.20) $$\lim_{m\to\infty} \limsup_{n\to\infty} P[|\langle W_n^i, W_n^j \rangle_t - \langle X_{n,m}^i, X_{n,m}^j \rangle_t| \geqq \delta] = 0.$$

Combining (6.20) and (6.9) we obtain that for every $\delta>0$,

$$\begin{aligned}&\lim_{n\to\infty} P[|\langle W_n^i, W_n^j\rangle_t-\phi^{ij}(t)|\geqq\delta]\\&\leqq \lim_{m\to\infty}\limsup_{n\to\infty} P\Big[|\langle W_n^i, W_n^j\rangle_t-\langle X_{n,m}^i, X_{n,m}^j\rangle_t|\geqq\frac{\delta}{2}\Big]\\&\quad+\lim_{m\to\infty}\limsup_{n\to\infty} P\Big[|\langle X_{n,m}^i, X_{n,m}^j\rangle_t-\phi^{ij}(t)|\geqq\frac{\delta}{2}\Big]=0.\end{aligned}$$

Thus we have (6.11) and hence the proof of Theorem 6.6 is complete.

7. Discrete case.

In this section we will consider discrete point processes associated with triangular arrays of random variables which are dependent in general, and we will rewrite the results of the previous sections. Some of the results overlap those of Durrett-Resnick [**5**].

For simplicity we will consider only the case where the limiting Poisson point processes are temporally homogeneous, which condition is not essential. Let $(\Omega_n, \mathcal{F}^n, P_n)$, $n=1, 2, \cdots$ be probability spaces as before and let $(\mathcal{F}_i^n)_{i=1}^\infty$ be an increasing family of sub-σ-fields of $\mathcal{F}^n$. For real $t\geqq 0$, we define $\boldsymbol{F}^n=(\mathcal{F}_t^n)$ by $\mathcal{F}_t^n=\mathcal{F}_{[t]}^n$. Suppose $\{\xi_{ni}\}_{i,n=1}^\infty$ is a triangular array of $(\mathcal{F}_i^n)$-adapted real-valued random variables. We define discrete point processes p_n, $n\geqq 1$ as follows.

$$X=[-\infty, 0)\cup(0, \infty],$$

$$D_{p_n}=\{i/n : i=1, 2, \cdots, \xi_{ni}\neq 0\},$$

$$p_n(i/n)=\xi_{ni} \qquad \text{if} \quad i/n\in D_{p_n}.$$

The compensators are given by

$$\hat{N}_{p_n}([0, t]\times E)=\sum_{i\leqq nt} P_n[\xi_{ni}\in E/\mathcal{F}_{i-1}^n], \qquad E\in\mathcal{B}(X).$$

Let $\nu(dx)$ be a Borel measure on $(-\infty, \infty)\setminus\{0\}$ such that $\int_{|x|>\varepsilon}\nu(dx)<\infty$ for every $\varepsilon>0$. Note that $\nu(dx)$ may be considered as a Radon measure on X $(=[-\infty, 0)\cup(0, \infty])$ by putting $\nu(\{-\infty\})=\nu(\{\infty\})=0$. By rewriting Theorem 5.1 we have a result of [**5**]:

THEOREM 7.1. *If, for every $t>0$ and every continuity point x of $\nu(dx)$,*

$$\sum_{k\leqq nt} P_n[\xi_{nk}>x / \mathcal{F}_{k-1}^n] \xrightarrow{P} t\nu(x, \infty) \qquad \text{if} \quad x>0, \tag{7.1}$$

$$\sum_{k\leqq nt} P_n[\xi_{nk}<x / \mathcal{F}_{k-1}^n] \xrightarrow{P} t\nu(-\infty, x] \qquad \text{if} \quad x<0, \tag{7.2}$$

then $N_{p_n} \xrightarrow{\mathcal{D}} N_p$, *where* p *is a Poisson point process with compensator* $dt\,\nu(dx)$.

We next rewrite Theorem 6.6. Let $f_n(t, x)$, $f(t, x)$, $g_n(t, x)$ and $g(t, x)$ be measurable functions on $[0, \infty)\times \boldsymbol{R}$ satisfying the following four conditions.

(7.3) $\quad f_n(t, 0) = g_n(t, 0) = 0\,, \qquad t \geqq 0\,,\ n \geqq 1\,.$

(7.4) $\quad \int_0^t\int_{|x|>0} \{f(s, x)^2 + |g(s, x)| I(|x| \leqq 1)\}\nu(dx)ds < \infty\,, \qquad t \geqq 0\,.$

(7.5) $\quad$ There exists $C>0$ such that $|f_n(t, x)| \leqq C,\ t \geqq 0\,,\ x \in \boldsymbol{R}\,,\ n \geqq 1\,.$

(7.6) $\quad \lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} \sup\{|f_n(t, x)| : 0 \leqq t \leqq T,\ |x| \leqq \varepsilon\} = 0\,, \qquad$ for every $T>0$.

The condition (7.3) corresponds to that we are considering point processes with values in $X=[-\infty, 0)\cup(0, \infty]$. It should be noticed that (7.4) implies that $f \in \Phi_p^2$ and $g \in \Phi_p$. It also follows from (7.5) that $f_n \in \Phi_{p_n}^2$. Notice that $g_n \in \Phi_{p_n}$ is always true. (7.6) corresponds to (6.8).

THEOREM 7.2. *Suppose that* (7.1) *through* (7.6) *are satisfied and we further assume the following three conditions.*

(7.7) $\quad (f_n, g_n) \xrightarrow{c.c.} (f, g) \qquad (dt\,\nu(dx)\text{-}a.e.)\,.$

(7.8) $\quad$ *There exists* $\sigma^2 \geqq 0$ *such that*

$$\lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} P_n[|\sum_{i\leqq nt}(E[f_n(i/n, \xi_{ni}^\varepsilon)^2 / \mathcal{F}_{i-1}^n] - \{E[f_n(i/n, \xi_{ni}^\varepsilon) / \mathcal{F}_{i-1}^n]\}^2) - \sigma^2 t| \geqq \delta] = 0$$

for every $t \geqq 0$, $\delta>0$, *where* $\xi_{ni}^\varepsilon = \xi_{ni} I(|\xi_{ni}| \leqq \varepsilon)$.

(7.9) $$\lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} P_n[\sum_{i\leqq nt} E[|g_n(i/n, \xi_{ni}^\varepsilon)| / \mathcal{F}_{i-1}^n] \geqq \delta] = 0$$

for every $t \geqq 0$, $\delta>0$.

Then, putting $A_n(t) = \sum_{i\leqq nt} E[f_n(i/n, \xi_{ni}) / \mathcal{F}_{i-1}^n]$, *we have*

$$\Big(\sum_{i\leqq nt}\{f_n(i/n, \xi_{ni}) + g_n(i/n, \xi_{ni})\} - A_n(t),\ N_{p_n}\Big) \xrightarrow{\mathcal{D}} \Big(\sigma B(t) + \int_0^{t+}\int f\tilde{N}_p + \int_0^{t+}\int g N_p,\ N_p\Big),$$

where p *is the same as in Theorem* 7.1 *and where* B *is a standard Brownian motion independent of* p.

The assumption (7.5) *may be dropped if* $f_n \in \Phi_{p_n}^{2,\mathrm{loc}}$ *and if*

$$\sum_{i\leq nt} E[f_n(i/n,\ \xi_{ni})I(|\xi_{ni}|>\varepsilon_k)\ /\ \mathcal{F}_{i-1}^n] \xrightarrow{\mathcal{D}} \int_0^t \int_{|x|>\varepsilon_k} f(s,\ x)\,ds\,\nu(dx), \qquad n\to\infty \tag{7.10}$$

for some ε_k (>0) *tending to* 0.

REMARKS 7.3. (i) Multi-dimensional cases can be considered in a similar way.

(ii) If, in addition, $A_n(t)\xrightarrow{\mathcal{D}}a(t)$ in $D([0,\infty):\boldsymbol{R})$ (or in $D([0,\infty):\boldsymbol{R}^d)$ in the multi-dimensional case), then we have the convergence of $\sum_{i\leq nt}\{f_n(i/n,\xi_{ni})+g_n(i/n,\xi_{ni})\}$ itself.

(iii) [5] considers the following case: $f_n(t,\ x)=f(t,\ x)=xI(|x|<\tau)$, $g_n(t,\ x)=g(t,\ x)=xI(|x|\geqq\tau)$, where τ and $-\tau$ are continuity points of $\nu(dx)$. In this case (7.3) through (7.9) except (7.8) are automatically satisfied if $\int\min\{1,\ x^2\}\nu(dx)<\infty$.

EXAMPLE 7.4. Let ξ_1, ξ_2, $\cdots$ be nonnegative, independent, identically distributed random variables such that

$$\lim_{x\to\infty} xP[\xi_1>x]=1\,.$$

Then for any $1<\alpha_1<\cdots<\alpha_d$ $(d\geqq 1)$ and $0<\beta<1/2$, we have

$$\Big(n^{-\alpha_1}\sum_{i\leq nt}\xi_i^{\alpha_1},\ \cdots,\ n^{-\alpha_d}\sum_{i\leq nt}\xi_i^{\alpha_d},\ n^{-1/2}\sum_{i\leq nt}(\xi_i^\beta-E[\xi_1^\beta])\Big)$$
$$\xrightarrow{\mathcal{D}}\Big(\int_0^{t+}\int_{x>0}x^{\alpha_1}N_p,\ \cdots,\ \int_0^{t+}\int_{x>0}x^{\alpha_d}N_p,\ \sigma B(t)\Big),$$

where p is a Poisson point process on $\boldsymbol{R}\setminus\{0\}$ with compensator $I(x>0)(1/x^2)dsdx$ and where B is a standard Brownian motion independent of p, σ^2 being the variance of ξ_1^β. The convergence holds in $D([0,\infty):\boldsymbol{R})^{d+1}$ (in fact in $D([0,\infty):\boldsymbol{R}^{d+1})$). It should be remarked that the assertion may be restated as follows.

$$\Big(\int_0^{t+}\int x^{\alpha_1}N_{p_n},\ \cdots,\ \int_0^{t+}\int x^{\alpha_d}N_{p_n},\ n^{\beta-1/2}\int_0^{t+}\int x^\beta\tilde{N}_{p_n}\Big)$$
$$\xrightarrow{\mathcal{D}}\Big(\int_0^{t+}\int x^{\alpha_1}N_p,\ \cdots,\ \int_0^{t+}\int x^{\alpha_d}N_p,\ \sigma B(t)\Big).$$

The proof of this example can easily be carried out by checking the conditions of Theorem 7.2. (Notice that it suffices to consider the convergence of each component.)

8. Asymptotic independence of point processes.

Let X be of the form $X_1\times\cdots\times X_d$, each of $\{X_i\}_{i=1}^d$ being a locally compact Hausdorff space with a countable open base. X is of course endowed with the product topology and we denote by π_i the projection from X to X_i: $\pi_i(x_1,\cdots,x_d)=x_i$. Let p be a point process with values in X. We denote by $\pi_i p$ the i^{th} component of p, i.e., $D_{\pi_i p}=D_p$ and $(\pi_i p)(t)=\pi_i(p(t))$, $t\in D_p$. The compensator of $\pi_i p$ is given as follows using the compensator of p:

$$\hat{N}_{\pi_i p}([0,t]\times E)=\hat{N}_p([0,t]\times\pi_i^{-1}(E)),\qquad E\in\mathcal{B}(X_i).$$

In this section we will consider the asymptotic independence of components of point processes $\{p_n\}_n$ with values in X.

THEOREM 8.1. *Assume that for every i $(1\leqq i\leqq d)$, $N_{\pi_i p_n}$ and $\hat{N}_{\pi_i p_n}$ are Radon measures on $[0,\infty)\times X_i$ a.s., and satisfy*

$$\hat{N}_{\pi_i p_n}(dt\,dx)\xrightarrow{\mathcal{D}}\mu_i(dt\,dx),\qquad n\to\infty \tag{8.1}$$

where μ_i is a deterministic Radon measure on $[0,\infty)\times X_i$ continuous in t. If

$$\hat{N}_{p_n}([0,t]\times(\pi_i^{-1}(K_i)\cap\pi_j^{-1}(K_j)))\xrightarrow{P}0,\qquad t\geqq 0 \tag{8.2}$$

for all compact sets $K_i\subset X_i$, $K_j\subset X_j$, $i\neq j$, then

$$\{N_{\pi_1 p_n},\cdots,N_{\pi_d p_n}\}\xrightarrow{\mathcal{D}}\{N_{p^1},\cdots,N_{p^d}\},\qquad n\to\infty \tag{8.3}$$

in $\mathfrak{M}([0,\infty)\times X_1)\times\cdots\times\mathfrak{M}([0,\infty)\times X_d)$ where $p^1,\cdots,p^d$ are mutually independent Poisson point processes such that $\hat{N}_{p^i}=\mu_i$, $i=1,\cdots,d$.

PROOF. The convergence of each component is immediate from Theorem 5.1 and therefore it remains to prove the independence of $p^1,\cdots,p^d$. Let $\bar{X}_i=X_i\cup\{\Delta_i\}$ be one-point compactification of X_i, $i=1,\cdots,d$, and let $Z=\bar{X}_1\times\cdots\times\bar{X}_d\setminus\{\Delta\}$ where $\Delta=(\Delta_1,\cdots,\Delta_d)$. Since X is a subset of Z, p_n may be regarded as a point process with values in Z. Of course we have to check that N_{p_n} and $\hat{N}_{p_n}$ are Radon measures (a.e.) not only on $[0,\infty)\times X$ but also on $[0,\infty)\times Z$. But this can easily be seen because we assumed that $N_{\pi_i p_n}$ and $\hat{N}_{\pi_i p_n}$ are Radon measures (a.s.) on $[0,\infty)\times X_i$, $i=1,2,\cdots,d$. Now let $F_i=\{x\in Z:\pi_j x=\Delta_j,\ j\neq i\}$. Notice that $F_1,F_2,\cdots,F_d$ are mutually disjoint. By (8.1) and (8.2) we have that

$$\hat{N}_{p_n}\xrightarrow{\mathcal{D}}\Gamma\quad\text{in}\quad\mathfrak{M}([0,\infty)\times Z) \tag{8.4}$$

where $\Gamma\in\mathfrak{M}([0,\infty)\times Z)$ is concentrated on $[0,\infty)\times\bigcup_{i=1}^d F_i$ and satisfies that

$$\Gamma([0,t]\times\tilde{E}^i)=\mu_i([0,t]\times E),\qquad E\in\mathcal{B}(X_i),\ 1\leqq i\leqq d.$$

Here, $\tilde{E}^i$ denotes the set $\{x\in Z : x_i\in E,\ x_j=\Delta_j \text{ for all } j\neq i\}$ $(\subset F_i)$. By Theorem 5.1 we have from (8.4) that

$$N_{p_n} \xrightarrow{\mathcal{D}} N_p \quad \text{in } \mathfrak{M}([0,\infty)\times Z), \tag{8.5}$$

where p is the Poisson point process with $\hat{N}_p=\Gamma$. Now let $f_i\in C_K([0,\infty)\times X_i)$, $i=1, 2, \cdots, d$, and define a function $f:[0,\infty)\times Z\to \boldsymbol{R}^d$, $f(t, x)=(f_1(t, \pi_1 x), \cdots, f_d(t, \pi_d x))$, $x\in Z$. Here, we set $f_i(t, \Delta_i)=0$, $i=1, 2, \cdots, d$. Note that f has a compact support in $[0,\infty)\times Z$ (but does not in $[0,\infty)\times X$ except the trivial case). Therefore, by (8.5) we have that

$$\int_0^\infty\int_Z f(t, x)N_{p_n}(dt\,dx) \xrightarrow{\mathcal{D}} \int_0^\infty\int_Z f(t, x)N_p(dt\,dx),$$

which may be written as

$$\begin{aligned}&\left(\int_0^\infty\int_{X_1} f_1(t, u)N_{\pi_1 p_n}(dt\,du),\ \cdots,\ \int_0^\infty\int_{X_d} f_d(t, u)N_{\pi_d p_n}(dt\,du)\right)\\ &\xrightarrow{\mathcal{D}} \left(\int_0^\infty\int_Z f_1(t, x_1)N_p(dt\,dx),\ \cdots,\ \int_0^\infty\int_Z f_d(t, x_d)N_p(dt\,dx)\right).\end{aligned} \tag{8.6}$$

Since $\hat{N}_p$ and hence N_p are concentrated on $\bigcup_{i=1}^d F_i$, we see that

$$\begin{aligned}\int_0^\infty\int_Z f_i(t, x_i)N_p(dt\,dx) &= \sum_{j=1}^d \int_0^\infty\int_{F_j} f_i(t, x_i)N_p(dt\,dx)\\ &= \int_0^\infty\int_{F_i} f(t, x_i)N_p(dt\,dx).\end{aligned}$$

Keeping in mind that $\{F_i\}_{i=1}^d$ are disjoint, we conclude that $\left\{\int_0^\infty\int_Z f_i(t, x_i) N_p(dt\,dx)\right\}_{i=1}^d$ are mutually independent. Thus (8.6) implies the asymptotic independence of $\{\pi_i p_n\}_{i=1}^d$ as $n\to\infty$, which completes the proof of the theorem.

As an example of Theorem 8.1, let us consider the convergence of point processes defined from a sequence of independent, identically distributed random variables.

Let $\{\xi_i\}_{i=1}^\infty$ be a sequence of i.i.d. (independent, identically distributed) random variables which are uniformly distributed over $[0, 1]$. For each a $(0\leqq a\leqq 1)$, we define point processes p_n^a, $n=1, 2, \cdots$ as follows.

$$D_{p_n^a}=\left\{\frac{k}{n} : k=1, 2, \cdots,\ \xi_k\neq a\right\}, \qquad p_n^a\left(\frac{k}{n}\right)=n(\xi_k-a).$$

The compensator of p_n is given by

$$\begin{aligned}\hat{N}_{p_n^a}([0, t]\times dx) &= [nt]P(n(\xi_1-a)\in dx)\\ &= I(-na\leqq x\leqq n(1-a))\frac{[nt]}{n}dx.\end{aligned}$$

Thus we obtain that

$$\hat{N}_{p_n^a} \xrightarrow{\mathcal{D}} \chi^a(x)dt\,dx \quad \text{in} \quad \mathfrak{M}([0, \infty)\times \boldsymbol{R}), \tag{8.7}$$

where $\chi^a(x)=I_{[0,\infty)}(x)$, 1 or $I_{(-\infty,0]}(x)$ according as $a=0$, $0<a<1$ or $a=1$. Hence it follows from Theorem 5.1 that, for every $0\leqq a\leqq 1$,

$$N_{p_n^a} \xrightarrow{\mathcal{D}} N_{p^a} \quad \text{in} \quad \mathfrak{M}([0, \infty)\times \boldsymbol{R}), \qquad n\to\infty \tag{8.8}$$

where p^a is a Poisson point process with compensator $\chi^a(x)dt\,dx$. We now consider the joint convergence of $\{N_{p_n^a}\}_a$.

THEOREM 8.2. *Let* $0=a_1<\cdots<a_d=1$ $(d\geqq 2)$ *and let* $p_n^i=p_n^{a_i}$ $(i=1, \cdots, d)$, $n\geqq 1$. *Then*

$$\{N_{p_n^1}, \cdots, N_{p_n^d}\} \xrightarrow{\mathcal{D}} \{N_{p^1}, \cdots, N_{p^d}\}, \qquad \text{as} \quad n\to\infty$$

in $\mathfrak{M}([0, \infty)\times \boldsymbol{R})\times\cdots\times\mathfrak{M}([0, \infty)\times \boldsymbol{R})$, *where* $\{p^i\}_{i=1}^d$ *are mutually independent Poisson point processes such that* $\hat{N}_{p^i}(dt\,dx)=\chi^{a_i}(x)dt\,dx$, $i=1, \cdots, d$.

PROOF. Define $D_{p_n}=\{i/n : i=1, 2, \cdots\}$, $p_n(t)=(p_n^1(t), \cdots, p_n^d(t))$, $t\in D_{p_n}$. Let us now apply Theorem 8.1. The condition (8.1) is clearly satisfied by (8.7). To check (8.2), notice that

$$\hat{N}_{p_n}([0, t]\times E) = [nt]P((n(\xi_1-a_1), \cdots, n(\xi_1-a_d))\in E).$$

Therefore, for any $\alpha, \beta, \gamma, \delta$ $(\alpha<\beta, \gamma<\delta)$, it holds that

$$\begin{aligned}&\hat{N}_{p_n}([0, t]\times(\pi_i^{-1}([\alpha, \beta])\cap\pi_j^{-1}([\gamma, \delta]))\\&= [nt]P\Big(\xi_1\in\Big[\frac{\alpha}{n}+a_i, \frac{\beta}{n}+a_i\Big]\cap\Big[\frac{\gamma}{n}+a_j, \frac{\delta}{n}+a_j\Big]\Big).\end{aligned}$$

This vanishes identically for all sufficiently large n provided that $a_i\neq a_j$. Thus we have (8.2) and the proof of the theorem is complete.

By considering $\min\{p_n^0(s): s\leqq t, s\in D_{p_n}\}$ and $\max\{p_n^1(s): s\leqq t, s\in D_{p_n}\}$, $t\geqq 0$ $(n\geqq 1)$, we have from Theorem 8.2 the following.

COROLLARY.

$$(n\min_{k\leq nt}\xi_k,\ n\max_{k\leq nt}\xi_k-n) \xrightarrow{\mathcal{D}} (X_0(t), X_1(t)) \qquad \text{as} \quad n\to\infty$$

in $D([\varepsilon, \infty): \boldsymbol{R}^2)$ *for every* $\varepsilon>0$, *whher* X_0 *and* X_1 *are mutually independent and identical in law to* $\min\{p^0(s): s\leqq t, s\in D_{p^0}\}$ *and* $\max\{p^1(s): s\leqq t, s\in D_{p^1}\}$, *respectively*.

9. Sums, maxima and minima of random variables.

In this section we will treat some applications of the results of the previous section. Let $\{\xi_{ni}\}$ be as in section 7 and let $\mu(dx)$ be an infinite Borel measure on (x_0, ∞) $(x_0 \geqq -\infty)$ such that $\mu(x_1, \infty) < \infty$ for all $x_1 > x_0$. Thus if we put $\mu(\{\infty\})=0$ then μ is a Radon measure on $(x_0, \infty]$. Assume that for some $a_n > 0$, $b_n \in \boldsymbol{R}$ it holds that

$$\sum_{i \leqq nt} P_n[a_n \xi_{ni} + b_n > x \,/\, \mathcal{F}_{i-1}^n] \xrightarrow{P} t\mu(x, \infty), \qquad t \geqq 0 \tag{9.1}$$

for any continuity point x $(>x_0)$ of μ. Then as we have seen in Theorem 7.1, the point process p_n defined by $a_n\xi_{ni}+b_n$ converges to a Poisson point process p^0 possessing the compensator $dt\,\mu(dx)$. Thus as an easy consequence we obtain that

$$M_n(t) \equiv a_n(\max_{i \leqq nt} \xi_{ni}) + b_n \xrightarrow{\mathcal{D}} M(t) \qquad \text{in} \quad D([\varepsilon, \infty) : \boldsymbol{R}) \tag{9.2}$$

for every $\varepsilon > 0$, where $M(t) = \max\{p^0(s) : s \leqq t,\ s \in D_{p^0}\}$. This fact was first pointed out by Durrett-Resnick [5]. Let us further assume that there exists an infinite Borel measure ν on $(-\infty, x_1)$ $(x_1 \leqq \infty)$ such that $\nu(-\infty, x) < \infty$ for all $x < x_1$ and that for some $c_n > 0$, $d_n \in \boldsymbol{R}$,

$$\sum_{i \leqq nt} P_n[c_n \xi_{ni} + d_n < x \,/\, \mathcal{F}_{i-1}^n] \xrightarrow{P} t\nu(-\infty, x), \qquad t \geqq 0 \tag{9.3}$$

at all continuity points of $\nu(dx)$. Since $\min \xi_{ni} = -(\max(-\xi_{ni}))$, we have from (9.2) that

$$m_n(t) \equiv c_n(\min_{i \leqq nt} \xi_{ni}) + d_n \xrightarrow{\mathcal{D}} m(t) \qquad \text{in} \quad D([\varepsilon, \infty) : \boldsymbol{R})$$

for every $\varepsilon > 0$, where $m(t) = \min\{p^1(s) : s \leqq t,\ s \in D_{p^1}\}$, p^1 being a Poisson point process possessing compensator $dt\,\nu(dx)$.

We next consider the joint convergence of $(M_n(t), m_n(t))$. The next theorem is a generalization of Corollary of Theorem 8.2.

THEOREM 9.1. *Let ν, μ, M_n, M, m_n and m be as in the above and assume that* (9.1) *and* (9.3) *hold. Then*

$$(M_n(t), m_n(t)) \xrightarrow{\mathcal{D}} (\tilde{M}(t), \tilde{m}(t)) \qquad \textit{in} \quad D([0, \infty) : \boldsymbol{R}^2),$$

where $\tilde{M}$ and $\tilde{m}$ are mutually independent and are identical in law to M and m, respectively.

PROOF. Since we have already seen the convergence of each component, the only thing to be proved is the independence of $\tilde{M}$ and $\tilde{m}$. To this end we will see the asymptotic independence of the components of the point process defined from $\{(a_n\xi_{ni}+b_n, c_n\xi_{ni}+d_n): i=1, 2, \cdots\}$. By Theorem 8.1 it suffices to show that

$$\sum_{i\leqq nt} P[a_n\xi_{ni}+b_n\geqq x,\ c_n\xi_{ni}+d_n\leqq y \,/\, \mathcal{F}_{i-1}^n] \xrightarrow{P} 0\,, \tag{9.4}$$

for every $x>x_0$, $y<x_1$ and $t\geqq 0$. (Notice that $[x, \infty]$ $(x>x_0)$ is a compact subset of $(x_0, \infty]$.) (9.4) may be restated as

$$\sum_{i\leqq nt} P[(x-b_n)/a_n\leqq\xi_{ni}\leqq(y-d_n)/c_n \,/\, \mathcal{F}_{i-1}^n] \xrightarrow{P} 0\,. \tag{9.5}$$

However, (9.5) is in fact obvious since $(x-b_n)/a_n>(y-d_n)/c_n$ for all sufficiently large n. Indeed, if $(x-b_n)/a_n\leqq(y-d_n)/c_n$ it follows that

$$[nt]\leqq\sum_{i\leqq nt}\{P[\xi_{ni}\geqq(x-b_n)/a_n \,/\, \mathcal{F}_{i-1}^n]+P[\xi_{ni}\leqq(y-d_n)/c_n \,/\, \mathcal{F}_{i-1}^n]\}.$$

However, the right-hand side converges to $t\{\mu(x, \infty)+\nu(-\infty, y)\}$ $(<\infty)$ while the left side diverges as $n\to\infty$. Thus we see that $(x-b_n)/a_n\leqq(y-c_n)/d_n$ occurs at most finitely many times, which completes the proof of the theorem.

We next consider the joint convergence of the sums and the maxima. In the rest of this section we will assume the assumptions of Theorem 7.1 as well as

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty} P_n[|\sum_{i\leqq nt}\{E[(\xi_{ni}^\varepsilon)^2 \,/\, \mathcal{F}_{i-1}^n]-(E[\xi_{ni}^\varepsilon \,/\, \mathcal{F}_{i-1}^n])^2\}-\sigma^2 t|\geqq\delta]=0\,, \quad t\geqq 0,\ \delta>0, \tag{9.6}$$

for some $\sigma\geqq 0$, where $\xi_{ni}^\varepsilon=\xi_{ni}I(|\xi_{ni}|\leqq\varepsilon)$. We then have (see Remark 7.3 (iii)), if $\int\min(1, x^2)\,\nu(dx)<\infty$,

$$\sum_{i\leqq nt}\xi_{ni}-A_n(t) \xrightarrow{\mathcal{D}} \sigma B(t)+\int_0^{t+}\int_{|x|\leqq\tau} x\tilde{N}_p+\int_0^{t+}\int_{|x|>\tau} xN_p\,, \tag{9.7}$$

where $A_n(t)=\sum_{i\leqq nt}E[\xi_{ni}^\tau \,/\, \mathcal{F}_{i-1}^n]$, p is a Poisson point process with compensator $dt\nu(dx)$ and B is a standard Brownian motion independent of p. (Here we choose $\tau\,(>0)$ so that τ and $-\tau$ are continuity points of $\nu(dx)$.) In view of the results of section 6 we have

THEOREM 9.2. *Let ν be an infinite Borel measure such that $\int\min(1, x^2)\nu(dx)<\infty$. If* (7.1) *and* (7.2) *hold, then*

$$(\sum_{i\leqq nt}\xi_{ni}-A_n(t),\ \max_{i\leqq nt}\xi_{ni}) \xrightarrow{\mathcal{D}} (X(t), Y(t)) \quad \text{in} \quad D([0, \infty): \boldsymbol{R}^2)$$

as $n\to\infty$, where X is the right-hand side of (9.7) *and $Y(t)=\sup\{p(s):\ s\leqq t,\ s\in D_p\}$.*

The assumption that $\nu(\boldsymbol{R}\setminus\{0\})=\infty$ is not essential and may be removed with the understanding that $p(s)=0$ if $s\notin D_p$. In any case it should be remarked that Y is a *functional* of X. Indeed, the point process p coincides with that of the discontinuities of X and hence $Y(t)=\max\{\Delta X(s)\vee 0: s\leqq t\}$. In the special case when $\nu(0, \infty)=0$, we have that $Y(t)$ vanishes identically a.s. However, in this case, it could happen that (9.1) holds for suitably chosen $a_n>0$ and b_n (and therefore $a_n\max_{i\leq nt}\xi_{ni}+b_n$ has nontrivial limiting processes). In such a case the limit process of the maxima is no longer a functional of that of the sums and in fact we have

THEOREM 9.3. *Let ν be a Borel measure on $\boldsymbol{R}\setminus\{0\}$ such that $\nu(0, \infty)=0$, $\int_{-\infty}^{0}\min(1, x^2)\,\nu(dx)<\infty$ and let μ be a Borel measure on (x_0, ∞) $(x_0\geqq-\infty)$ such that $\mu(x, \infty)<\infty$ for every $x>x_0$. Under the assumption of Theorem* 7.1, *if* (9.1) *and* (9.6) *hold for some $a_n>0$, $b_n\in\boldsymbol{R}$ and $\sigma^2\geqq 0$, then*

$$\Big(\sum_{i\leq nt}\xi_{ni}-A_n(t),\ a_n\max_{i\leq nt}\xi_{ni}+b_n\Big)\xrightarrow{\mathcal{D}}(X(t), Y(t)), \qquad in\quad D([\varepsilon, \infty):\boldsymbol{R}^2)$$

for every $\varepsilon>0$, where X and Y are mutually independent and are identical in law to the right-hand side of (9.7) *and M in* (9.2), *respectively.*

PROOF. For simplicity we assume that $x_0=-\infty$. (Other cases may be treated in a similar way with a slight modification.) Let p_n be the point process on $(\boldsymbol{R}\setminus\{0\})\times(x_0, \infty]$ defined by $p_n(i/n)=(\xi_{ni}, a_n\xi_{ni}+b_n)$, $i=1, 2, \cdots (n\geqq 1)$. Notice that $\sum_{i\leq nt}\xi_{ni}-A(t)$ is a functional of the first component of p_n while $a_n\max_{i\leq nt}\xi_{ni}+b_n$ is that of the second component. Thus to have the asymptotic independence of the sums and the maxima, we can apply Theorem 8.1 and it suffices to show that for every $\alpha>0$, $\beta<0$, $\gamma>x_0$,

$$\sum_{i\leq nt}P[\xi_{ni}\in[\alpha, \infty),\ a_n\xi_{ni}+b_n\in[\gamma, \infty)\ /\ \mathcal{F}_{i-1}^n]\xrightarrow{P}0 \tag{9.8}$$

and

$$\sum_{i\leq nt}P[\xi_{ni}\in(-\infty, \beta],\ a_n\xi_{ni}+b_n\in[\gamma, \infty)\ /\ \mathcal{F}_{i-1}^n]\xrightarrow{P}0\,. \tag{9.9}$$

However, (9.8) is obvious from the assumption (7.1) and $\nu(0, \infty)=0$:

$$\sum_{i\leq nt}P[\xi_{ni}\in[\alpha, \infty)\ /\ \mathcal{F}_{i-1}^n]\xrightarrow{P}t\,\nu(\alpha, \infty)=0\,.$$

To see (9.9) it suffices to show that $\beta<(\gamma-b_n)/a_n$ holds for all sufficiently large n, which may be easily checked using the idea of the proof of Theorem 9.1.

Theorem 9.3 is an extension of the result in [12], where the case of i.i.d. is considered.

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