J. Math. Soc. Japan
Vol. 37, No. 4, 1985

Stable vector bundles of rank 2 on a 3-dimensional rational scroll

By Toshio HOSOH and Sadao ISHIMURA

(Received March 19, 1984)

Introduction. Let (X, H) be a couple of a $\boldsymbol{P}^2$-bundle over $\boldsymbol{P}^1$ and a very ample divisor on it. We say that (X, H) is a 3-dimensional rational scroll if the H-degree of a fibre is one (Definition (1.2)). In this paper we investigate moduli of some families of stable vector bundles of rank 2 on a 3-dimensional rational scroll.

In §1, we prove a main tool of this paper (Theorem (1.5)). In §2 and §3, particular families are treated. One family forms a projective space (Theorem (2.2)) and another family forms a complement of a dual 3-dimensional rational scroll (Theorem (3.19)).

§1. Preliminary.

(1.1) Let k be an algebraically closed field of arbitrary characteristic and X be a $\boldsymbol{P}^2$-bundle over $\boldsymbol{P}^1$ defined over k. There are integers $a \leqq b \leqq 0$ such that for the vector bundle $\mathcal{V}=\mathcal{O}_{P^1}(a) \oplus \mathcal{O}_{P^1}(b) \oplus \mathcal{O}_{P^1}$ on $\boldsymbol{P}^1$, X is isomorphic to $\boldsymbol{P}(\mathcal{V})$.

Let π be the projection of X to $\boldsymbol{P}^1$. Let D be a divisor on X such that $\pi_* \mathcal{O}_X(D) \simeq \mathcal{V}$ and F be a fibre of π. For an integer $q \geqq 1-a$, the divisor $H=D+qF$ is very ample and the intersection number $(F \cdot H^2)=1$.

DEFINITION (1.2). The couple (X, H) is called a 3-dimensional rational scroll.

DEFINITION (1.3). Let $\mathcal{E}$ be a vector bundle of rank 2 on a 3-dimensional rational scroll (X, H). $\mathcal{E}$ is stable if for any invertible subsheaf $\mathcal{L}$ of $\mathcal{E}$, the inequality

$$(C_1(\mathcal{L}) \cdot H^2) < (C_1(\mathcal{E}) \cdot H^2)/2$$

holds.

(1.4) Fix a 3-dimensional rational scroll (X, H) as above and define the integer $p=(D \cdot H^2)=2q+a+b$. Note that $p \geqq 2$ because $p=(q+a)+(q+b)$ and $q \geqq 1-a$. For integers α, x and y, let $\boldsymbol{M}(\alpha; x, y)$ be the set of all stable vector bundles of rank 2 on (X, H) with fixed Chern classes $C_1=-\alpha D+(\alpha p+1)F$ and $C_2=xD^2+yD \cdot F$. In this section we prove the following theorem which is a main tool of this paper.

THEOREM (1.5). *If* $\alpha>0$ *and* $x\leqq 0$ *then for any* $\mathcal{E}$ *in* $\boldsymbol{M}(\alpha\,;x,\,y)$, *there exist integers* $l\geqq 0$ *and* m *such that* $\mathcal{E}(-lD-mF)$ *has a nonzero section whose scheme of zeros has codimension* $\geqq 2$ *and the following inequalities hold*

$$\begin{aligned} y&\geqq l(\alpha p+1)-(\alpha+2l)m-b\{l(l+\alpha)+x\}\\ &\geqq l(\alpha p+1)-(\alpha+2l)m \qquad (1.5.1)\\ &\geqq 2l(l+\alpha)p+l\,. \end{aligned}$$

REMARK (1.6). Let $\mathcal{E}$ be a stable vector bundle of rank 2 on X. Then $\mathcal{E}\otimes\mathcal{L}$ is in $\boldsymbol{M}(\alpha\,;x,\,y)$ for some triple $(\alpha\,;x,\,y)$ $(\alpha>0,\ x\leqq 0)$ and a line bundle $\mathcal{L}$ if and only if $(C_1(\mathcal{E})\cdot H^2)$ is odd and $(\Delta(\mathcal{E})\cdot F)$ is positive, where $\Delta(\mathcal{E})$ is the cycle $-C_2(\mathcal{E}nd(\mathcal{E}))=C_1(\mathcal{E})^2-4C_2(\mathcal{E})$.

PROOF. Assume that $\mathcal{E}\otimes\mathcal{L}$ is in $\boldsymbol{M}(\alpha\,;x,\,y)$ for some $\alpha>0$, $x\leqq 0$, y and a line bundle $\mathcal{L}$. Then

$$(C_1(\mathcal{E})\cdot H^2)=(-\alpha D+(\alpha p+1)F\cdot H^2)-2(C_1(\mathcal{L})\cdot H^2)\equiv 1 \pmod 2$$

and

$$(\Delta(\mathcal{E})\cdot F)=(\Delta(\mathcal{E}\otimes\mathcal{L})\cdot F)=\alpha^2-4x>0\,.$$

Conversely, assume that $(C_1(\mathcal{E})\cdot H^2)$ is odd and $(\Delta(\mathcal{E})\cdot F)$ is positive. Since $(C_1(\mathcal{E})\cdot H^2)$ is odd, replacing $\mathcal{E}$ by $\mathcal{E}\otimes\mathcal{L}$ for suitable line bundle $\mathcal{L}$, we may assume $C_1(\mathcal{E})=-D+(p+1)F$ (or $-2D+(2p+1)F$). If $C_2(\mathcal{E})=xD^2+yD\cdot F$ then $(\Delta(\mathcal{E})\cdot F)=1-4x$ (or $4-4x$ respectively). Therefore we have $x\leqq 0$.

LEMMA (1.7). *Let* $\mathcal{E}$ *be a vector bundle of rank* 2 *on* $\boldsymbol{P}^2$. *If* $C_2(\mathcal{E})\leqq 0$ *then* $H^0(\boldsymbol{P}^2,\,\mathcal{E})\neq(0)$.

PROOF. First notice that $\mathcal{E}$ is not simple since $\Delta(\mathcal{E})=C_1(\mathcal{E})^2-4C_2(\mathcal{E})\geqq 0$ ([**3**] Corollary 4.3.1). Since $C_1(\mathcal{E}^\vee)=-C_1(\mathcal{E})$, $C_2(\mathcal{E}^\vee)=C_2(\mathcal{E})$ and $\mathcal{E}^\vee=\mathcal{E}(-C_1(\mathcal{E}))$, we may assume $C_1(\mathcal{E})\leqq 0$ and $C_2(\mathcal{E})\leqq 0$. Let n be the least integer such that $H^0(\boldsymbol{P}^2,\,\mathcal{E}(n))\neq(0)$. We have to show that $n\leqq 0$. Take a nonzero section s of $\mathcal{E}(n)$. Then the scheme of zeros of s represents the second Chern class of $\mathcal{E}(n)$. So we see that

$$C_2(\mathcal{E}(n))=n^2+nC_1(\mathcal{E})+C_2(\mathcal{E})\geqq 0\,.$$

If n were positive, we would have $2n>-C_1(\mathcal{E})$. Hence $C_1(\mathcal{E}(n))=2n+C_1(\mathcal{E})>0$. This implies, however, that $\mathcal{E}$ is simple ([**5**] Proposition (4.1)). This is a contradiction. Thus we have $n\leqq 0$.

PROPOSITION (1.8). *Let* $\mathcal{E}$ *be a vector bundle of rank* 2 *on* X. *If* $(C_2(\mathcal{E})\cdot F)\leqq 0$ *then there is a line bundle* $\mathcal{L}=\mathcal{O}_X(lD+mF)$ *such that* $l\geqq 0$ *and* $\mathcal{E}\otimes\mathcal{L}^\vee$ *has a* *nonzero section whose scheme of zeros has codimension* $\geqq 2$.

PROOF. Since $(C_2(\mathcal{E})\cdot F)\leqq 0$, by Lemma (1.7) we have $\pi_*\mathcal{E}\neq 0$. $\pi_*\mathcal{E}$ is torsion free because so is $\mathcal{E}$. Let $\mathcal{L}'$ be an invertible subsheaf of $\pi_*\mathcal{E}$ such that the composition

$$\pi^*\mathcal{L}' \longrightarrow \pi^*\pi_*\mathcal{E} \longrightarrow \mathcal{E}$$

is not zero. This morphism defines an elements s of $\mathrm{Hom}(\pi^*\mathcal{L}', \mathcal{E}) = H^0(X, \pi^*\mathcal{L}'^{\vee}\otimes\mathcal{E})$. Let Y be the scheme of zeros of s. Let A be the maximal effective divisor contained in Y. s is regarded as a section of $\pi^*\mathcal{L}'^{\vee}\otimes\mathcal{E}(-A)$ and its scheme of zeros has codimension $\geqq 2$. Then $\mathcal{L}=\pi^*\mathcal{L}'\otimes\mathcal{O}_X(A)$ is a desired line bundle.

LEMMA (1.9). *Let $\mathcal{E}$ be a vector bundle of rank 2 on X with $C_2(\mathcal{E})=xD^2+yD\cdot F$. If $\mathcal{E}$ has a nonzero section such that its scheme of zeros has codimension $\geqq 2$ then $x\geqq 0$ and $bx+y\geqq 0$.*

PROOF. Let Y be the scheme of zeros of the section. Then Y represents the second Chern class of $\mathcal{E}$. The complete linear systems $|F|$ and $|D-aF|$ are base point free. Thus we see that $(C_2(\mathcal{E})\cdot F)=x\geqq 0$ and $(C_2(\mathcal{E})\cdot D-aF) = bx+y\geqq 0$.

(1.10) PROOF OF THEOREM (1.5). Let $\mathcal{E}$ be as in Theorem (1.5). Then we have $(C_2(\mathcal{E})\cdot F)=x\leqq 0$. By Proposition (1.8) there are integers $l\geqq 0$ and m such that $\mathcal{E}(-lD-mF)$ has a nonzero section whose scheme of zeros has codimension $\geqq 2$. The section makes the line bundle $\mathcal{O}_X(lD+mF)$ a subsheaf of $\mathcal{E}$ and then by the stability we have

$$(lD+mF\cdot H^2)=lp+m<(C_1(\mathcal{E})\cdot H^2)/2=1/2.$$

Hence $lp+m\leqq 0$. The second Chern class of $\mathcal{E}(-lD-mF)$ is

$$\{l(l+\alpha)+x\}D^2+\{y+(\alpha+2l)m-l(\alpha p+1)\}D\cdot F.$$

By Lemma (1.9) we have $l(l+\alpha)+x\geqq 0$ and

$$b\{l(l+\alpha)+x\}+y+(\alpha+2l)m-l(\alpha p+1)\geqq 0.$$

Using the inequality $lp+m\leqq 0$, we get

$$\begin{aligned} y &\geqq l(\alpha p+1)-(\alpha+2l)m-b\{l(l+\alpha)+x\} \\ &\geqq l(\alpha p+1)-(\alpha+2l)m \\ &\geqq 2l(l+\alpha)p+l. \end{aligned}$$

§ 2. The moduli of $\boldsymbol{M}(\alpha; 0, 0)$.

In this section, we investigate the moduli of $\boldsymbol{M}(\alpha; 0, 0)$.

PROPOSITION (2.1). *Let α be a positive integer and $\mathcal{E}$ be a vector bundle of rank 2 on X with $C_1(\mathcal{E})=-\alpha D+(\alpha p+1)F$ and $C_2(\mathcal{E})=0$. Then $\mathcal{E}$ is stable if and only if $\mathcal{E}$ is obtained from a non-trivial extension*

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X(-\alpha D+(\alpha p+1)F) \longrightarrow 0.$$

PROOF. If $\mathcal{E}$ is in $\boldsymbol{M}(\alpha; 0, 0)$ then (1.5.1) says that

$$\begin{aligned} 0 &\geqq l(\alpha p+1)-(\alpha+2l)m \\ &\geqq 2l(l+\alpha)p+l. \end{aligned}$$

Therefore $l=m=0$. Hence $\mathcal{E}$ has a nonzero section whose scheme of zeros has codimension $\geqq 2$. Since $C_2(\mathcal{E})=0$, the section makes $\mathcal{O}_X$ a line subbundle of $\mathcal{E}$. Thus we get an extension;

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X(-\alpha D+(\alpha p+1)F) \longrightarrow 0.$$

This is non-trivial because $\mathcal{E}$ is stable.

Conversely, assume that there is a non-trivial extension

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X(-\alpha D+(\alpha p+1)F) \longrightarrow 0.$$

Since $(C_1(\mathcal{E})\cdot H^2)=1$, we have to show that for every invertible subsheaf $\mathcal{L}$ of $\mathcal{E}$, $(C_1(\mathcal{L})\cdot H^2)\leqq 0$ holds. If contrary there were an invertible subsheaf $\mathcal{L}$ of $\mathcal{E}$ such that $(C_1(\mathcal{L})\cdot H^2)\geqq 1$ then $\mathcal{L}$ would not be contained in $\mathcal{O}_X$. Therefore the composition

$$\mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X(-\alpha D+(\alpha p+1)F)$$

is not zero. Comparing H-degrees, we see that the morphism $\mathcal{L}\rightarrow \mathcal{O}_X(-\alpha D+(\alpha p+1)F)$ is an isomorphism. This is a contradiction.

THEOREM (2.2). *The moduli of $\boldsymbol{M}(\alpha; 0, 0)$ is the projective space $\boldsymbol{P}(H^1(X, \mathcal{O}_X(\alpha D-(\alpha p+1)F))^\vee)$ and has a universal family.*

PROOF. Denote by $\mathcal{M}$ the line bundle $\mathcal{O}_X(-\alpha D+(\alpha p+1)F)$ and W the cohomology group $H^1(X, \mathcal{M}^\vee)$. For $\mathcal{E}$ in $\boldsymbol{M}(\alpha; 0, 0)$, taking a choice of a non-trivial extension

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{M} \longrightarrow 0$$

is equivalent to taking a choice of a nonzero element of $H^0(X, \mathcal{E})\simeq k$. So the set $\boldsymbol{M}(\alpha; 0, 0)$ is parametrized by the projective space $\boldsymbol{P}=\boldsymbol{P}(W^\vee)$. Let ρ and σ be the projections $X\times\boldsymbol{P}\rightarrow X$ and $X\times\boldsymbol{P}\rightarrow\boldsymbol{P}$. A universal quotient morphism

$W^{\vee}\otimes\mathcal{O}_P\to\mathcal{O}_P(1)$ defines an element ξ of $H^0(\boldsymbol{P}, W\otimes\mathcal{O}_P(1))\simeq H^1(X\times\boldsymbol{P}, \rho^*\mathcal{M}^{\vee}\otimes\sigma^*\mathcal{O}_P(1))$. ξ provides us with an extension

$$0\longrightarrow\sigma^*\mathcal{O}_P(1)\longrightarrow\tilde{\mathcal{E}}\longrightarrow\rho^*\mathcal{M}\longrightarrow 0. \tag{2.2.1}$$

We claim that $\tilde{\mathcal{E}}$ is a universal family of $\boldsymbol{M}(\alpha\,;0,0)$. Assume that there are a k-scheme T and a vector bundle $\mathcal{R}$ on $X\times T$ such that for every closed point t of T, $\mathcal{R}_t=\mathcal{R}\otimes k(t)$ is in $\boldsymbol{M}(\alpha\,;0,0)$. Let ρ' and σ' be the projections $X\times T\to X$ and $X\times T\to T$. Since $H^1(X, \mathcal{R}_t)=0$ for all $t\in T$, $\sigma'_*(\mathcal{R})$ is locally free ([**1**] EGA III, 7.7 and 7.8). By this and Proposition (2.1), $\sigma'^*\sigma'_*\mathcal{R}$ is a line sub-bundle of $\mathcal{R}$ and the cokernel of $\sigma'^*\sigma'_*\mathcal{R}\to\mathcal{R}$ is isomorphic to the line bundle $\sigma'^*\mathcal{L}\otimes\rho'^*\mathcal{M}$ for a line bundle $\mathcal{L}$ on T. Put $\mathcal{Q}=\mathcal{L}^{\vee}\otimes\sigma'_*\mathcal{R}$. Then there is an extension

$$0\longrightarrow\sigma'^*\mathcal{Q}\longrightarrow\mathcal{R}\otimes\sigma'^*\mathcal{L}^{\vee}\longrightarrow\rho'^*\mathcal{M}\longrightarrow 0. \tag{2.2.2}$$

This corresponds to an element of $H^1(X\times T, \rho'^*\mathcal{M}^{\vee}\otimes\sigma'^*\mathcal{Q})\simeq H^0(T, W\otimes\mathcal{Q})$ which can be regarded as a morphism $W^{\vee}\otimes\mathcal{O}_T\to\mathcal{Q}$. By Proposition (2.1), this is surjective at every closed point of T therefore actually surjective by Nakayama's lemma. By the universality of $\boldsymbol{P}$ there is a morphism $g:T\to\boldsymbol{P}$ such that $\mathcal{Q}\simeq g^*\mathcal{O}_P(1)$ and the pull back of the extension (2.2.1) by the morphism $\mathrm{id}\times g$ is equivalent to (2.2.2). Thus we get $\mathcal{R}\otimes\sigma'^*\mathcal{L}^{\vee}\simeq(\mathrm{id}\times g)^*\tilde{\mathcal{E}}$. This shows that $\boldsymbol{P}$ is the fine moduli space of $\boldsymbol{M}(\alpha\,;0,0)$ and $\tilde{\mathcal{E}}$ is a universal family.

§ 3. The moduli of $\boldsymbol{M}(1\,;0,1)$.

In this section, we investigate the moduli of $\boldsymbol{M}(1\,;0,1)$.

PROPOSITION (3.1). *If $\mathcal{E}$ is in $\boldsymbol{M}(1\,;0,1)$ then there is a non-trivial extension*

$$0\longrightarrow\mathcal{O}_X\longrightarrow\mathcal{E}(F)\longrightarrow\mathcal{O}_X(-D+(p+3)F)\longrightarrow 0.$$

PROOF. In this case, (1.5.1) says that

$$\begin{aligned}1&\geqq l(p+1)-(1+2l)m\\&\geqq 2l(l+1)p+l.\end{aligned}$$

From these we deduce that $l=0$ and $m=0$ or $m=-1$. Assume $m=0$. $\mathcal{E}$ has a nonzero section whose scheme of zeros has codimension $\geqq 2$. Let Y be the scheme of zeros of the section. There is an exact sequence

$$0\longrightarrow\mathcal{O}_X\longrightarrow\mathcal{E}\longrightarrow\mathcal{I}_Y\otimes\det\mathcal{E}\longrightarrow 0,$$

where $\mathcal{I}_Y$ is the sheaf of ideals of Y in X. Since $C_2(\mathcal{E})=D\cdot F$, Y is a line in a fibre of π. Denote by $\mathcal{N}_{Y/X}$ the normal bundle of Y in X then $\mathcal{E}\otimes\mathcal{O}_Y\simeq\mathcal{N}_{Y/X}$ ([**4**] Chapter II, 5.1). On one hand, we have that $\det\mathcal{E}\otimes\mathcal{O}_Y\simeq\mathcal{O}_Y(-1)$. On the other hand, $\det \mathcal{N}_{Y/X}$ is obviously isomorphic to $\mathcal{O}_Y(1)$. Thus we see that this is not the case. We have, therefore, that $m=-1$ and hence $\mathcal{E}(F)$ has a nonzero section whose scheme of zeros has codimension $\geqq 2$. Since $C_2(\mathcal{E}(F))=0$, the section gives rise to a line subbundle $\mathcal{O}_X$ of $\mathcal{E}(F)$ and we get an extension;

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}(F) \longrightarrow \mathcal{O}_X(-D+(p+3)F) \longrightarrow 0.$$

This extension is not trivial because of the stability of $\mathcal{E}$.

PROPOSITION (3.2). *Let $\mathcal{E}$ be a vector bundle of rank 2 on X. Then the following conditions are equivalent to each other.*

(1) *There is a non-trivial extension*

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}(F) \longrightarrow \mathcal{O}_X(-D+(p+3)F) \longrightarrow 0$$

and $\mathcal{E}$ is not stable.

(2) *There are a line Y in a fibre of π and an exact sequence*

$$0 \longrightarrow \mathcal{O}_X(-D+(p+1)F) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_Y \longrightarrow 0.$$

PROOF. (1)$\Rightarrow$(2). Let $\mathcal{L}$ be an invertible subsheaf of $\mathcal{E}(F)$ such that

$$(C_1(\mathcal{L})\cdot H^2) \geqq (C_1(\mathcal{E}(F))\cdot H^2)/2=3/2.$$

$\mathcal{L}$ is not contained in $\mathcal{O}_X$ so that the composition

$$\mathcal{L} \longrightarrow \mathcal{E}(F) \longrightarrow \mathcal{O}_X(-D+(p+3)F)$$

is a nonzero morphism and not an isomorphism because the extension is non trivial. This shows that if $\mathcal{L}=\mathcal{O}_X(lD+mF)$ then $l\leqq -1$, $m\leqq p+3$ and $lp+m=2$. We see that $l=-1$ and $m=p+2$. Then $\mathcal{E}(F)\otimes\mathcal{L}^{\vee}=\mathcal{E}(D-(p+1)F)$ has a nonzero section whose scheme of zeros Y has codimension $\geqq 2$. So we have

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}(D-(p+1)F) \longrightarrow \mathcal{O}_X(D-(p+1)F)\otimes\mathcal{I}_Y \longrightarrow 0.$$

Since $C_2(\mathcal{E}(D-(p+1)F))=D\cdot F$, we see that Y is a line in a fibre of π. Tensoring the above sequence with the line bundle $\mathcal{O}_X(-D+(p+1)F)$, we get a desired exact sequence.

(2)$\Rightarrow$(1). Tensoring the given exact sequence with the line bundle $\mathcal{O}_X(F)$, we have

$$(3.2.1) \qquad 0 \longrightarrow \mathcal{O}_X(-D+(p+2)F) \longrightarrow \mathcal{E}(F) \longrightarrow \mathcal{O}_X(F)\otimes\mathcal{I}_Y \longrightarrow 0.$$

Since $H^i(X, \mathcal{O}_X(-D+(p+2)F))=(0)$ $(i=0, 1)$,

$$H^0(X, \mathcal{E}(F)) \simeq H^0(X, \mathcal{O}_X(F)\otimes\mathcal{I}_Y) \simeq k.$$

Let s be a nonzero section of $\mathcal{E}(F)$ and Z be the scheme of zeros of s. By

(3.2.1), we see that $H^0(X, \mathcal{E}(F-A))=(0)$ for any positive divisor A. Hence Z is of codimension $\geqq 2$. Z must be empty because of $C_2(\mathcal{E}(F))=0$. Thus we get an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}(F) \longrightarrow \mathcal{O}_X(-D+(p+3)F) \longrightarrow 0. \tag{3.2.2}$$

If $\mathcal{E}(F)\simeq\mathcal{O}_X\oplus\mathcal{O}_X(-D+(p+3)F)$ then the cokernel of the morphism $\mathcal{O}_X(-D+(p+2)F)\to\mathcal{E}(F)$ in (3.2.1) contains the torsion subsheaf $\mathcal{O}_F(-1)$. This is impossible and hence (3.2.2) does not split.

REMARK (3.3). Let $\mathcal{L}=\mathcal{O}_X(-D+(p+1)F)$ and Y be a line in a fibre of π, then $H^1(X, \mathcal{L})=H^2(X, \mathcal{L})=(0)$ and $\det \mathcal{N}_{Y/X}\simeq\mathcal{L}^\vee\otimes\mathcal{O}_Y$. Hence $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{J}_Y, L)\simeq\mathcal{O}_Y$ and $\mathrm{Ext}^1(\mathcal{J}_Y, \mathcal{L})\simeq H^0(Y, \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{J}_Y, \mathcal{L}))\simeq k$ so that the set of all isomorphism classes of such vector bundles as in Proposition (3.2) is in one to one correspondence with the set of all lines in fibres of π ([2] Remark 1.1.1).

(3.4) Put $\tilde{X}=\boldsymbol{P}(\mathcal{V}^\vee)$. Let $\tilde{\pi}$ be the projection of $\tilde{X}$ to $\boldsymbol{P}^1$, $\tilde{D}$ be a divisor on $\tilde{X}$ such that $\tilde{\pi}_*\mathcal{O}_{\tilde{X}}(\tilde{D})\simeq V^\vee$ and $\tilde{F}$ be a fibre of $\tilde{\pi}$. Denote by Ω the relative differential sheaf of $\pi: X\to\boldsymbol{P}^1$. There is an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow (\pi^*\mathcal{V}^\vee)(D) \longrightarrow \Omega^\vee \longrightarrow 0. \tag{3.4.1}$$

From this we get

$$\begin{array}{ccc} \tilde{X}\underset{\boldsymbol{P}^1}{\times}X & \simeq & \boldsymbol{P}((\pi^*\mathcal{V}^\vee)(D)) \\ \cup & & \cup \\ \tilde{Y} & \simeq & \boldsymbol{P}(\Omega^\vee) \end{array} .$$

$\tilde{X}$ parametrizes all the lines of fibres of π and the morphism

$$\tilde{Y} \hookrightarrow \tilde{X}\underset{\boldsymbol{P}^1}{\times}X \longrightarrow \tilde{X}$$

is the universal family of the lines. Now we consider the following diagram

$$\begin{tikzcd} & \tilde{X}\times X \arrow[dl, "\tilde{q}"'] \arrow[d] \arrow[dr, "q"] & \supset & \tilde{X}\underset{\boldsymbol{P}^1}{\times}X\supset\tilde{Y} \arrow[d, "f"] \\ \tilde{X} & \boldsymbol{P}^1\times\boldsymbol{P}^1 & X \quad \supset & \Delta_{\boldsymbol{P}^1} \end{tikzcd} \tag{3.4.2}$$

where $\Delta_{\boldsymbol{P}^1}$ is the diagonal of $\boldsymbol{P}^1\times\boldsymbol{P}^1$.

PROPOSITION (3.5). *Denote by $\tilde{\mathcal{L}}$ the line bundle*

$$\mathcal{O}_{\tilde{X}\times X}(-\tilde{X}\times D+(p+1)\tilde{X}\times F-\tilde{D}\times X-(p+3)\tilde{F}\times X).$$

Then

$$\tilde{\mathcal{I}}^{\vee}\otimes\mathcal{O}_{Y}\simeq\det\mathcal{N}_{\tilde{Y}/\tilde{X}\times X}$$

and

$$H^1(\tilde{X}\times X,\ \tilde{\mathcal{I}})=H^2(\tilde{X}\times X,\ \tilde{\mathcal{I}})=0.$$

PROOF. As a divisor on $\tilde{X}\times X$, $\tilde{X}\times_{P^1}X$ is linearly equivalent to $\tilde{X}\times F+\tilde{F}\times X$. Thus we have $\tilde{X}\times_{P^1}X|_{\tilde{X}\times_{P^1}X}\sim 2f^{-1}(x)$ $(x\in\Delta_{P^1}(k))$. By the exact sequence (3.4.1), we see that $\tilde{Y}$ is linearly equivalent to $(\tilde{D}\times X+\tilde{X}\times D)|_{\tilde{X}\times_{P^1}X}$ on $\tilde{X}\times_{P^1}X$. Therefore we have

$$\begin{aligned}\det\mathcal{N}_{\tilde{Y}/\tilde{X}\times X}&\simeq\mathcal{N}_{\tilde{Y}/\tilde{X}\times_{P^1}X}\otimes\mathcal{N}_{\tilde{X}\times_{P^1}X/\tilde{X}\times X}\\&\simeq\mathcal{O}_{\tilde{Y}}((\tilde{D}\times X+\tilde{X}\times D)|_{\tilde{Y}}+2f^{-1}(x)|_{\tilde{Y}})\\&\simeq\tilde{\mathcal{I}}^{\vee}\otimes\mathcal{O}_{\tilde{Y}}.\end{aligned}$$

The vanishing of cohomology groups is straightforward.

(3.6) As in Remark (3.3), there is a vector bundle Q on $\tilde{X}\times X$ defined by an exact sequence

$$0\longrightarrow\tilde{\mathcal{I}}\longrightarrow Q\longrightarrow\mathcal{J}_{\tilde{Y}}\longrightarrow 0.$$

By Proposition (3.2) and Remark (3.3), Q is a family of vector bundles on X which parametrizes the set of all the isomorphic classes of such vector bundles as in Proposition (3.2).

PROPOSITION (3.7). *There is an exact sequence*

$$0\longrightarrow\tilde{q}^*\mathcal{O}_{\tilde{X}}(\tilde{D}+(p+1)\tilde{F})\longrightarrow Q'\longrightarrow q^*\mathcal{O}_X(-D+(p+3)F)\longrightarrow 0 \tag{3.7.1}$$

on $\tilde{X}\times X$, *where* $Q'=Q(\tilde{X}\times F+\tilde{D}\times X+(p+2)\tilde{F}\times X)$.

PROOF. By Proposition (3.2), $\tilde{q}_*Q(\tilde{X}\times F)$ is an invertible sheaf on $\tilde{X}$ so that $\tilde{q}^*\tilde{q}_*Q(\tilde{X}\times F)$ is a line subbundle of $Q(\tilde{X}\times F)$. Put $\mathcal{L}=\tilde{q}^*\tilde{q}_*Q(\tilde{X}\times F)$. We claim that $\mathcal{L}\simeq\mathcal{O}_{\tilde{X}\times X}(-\tilde{F}\times X)$. Since $\mathcal{L}$ is a line subbundle of $Q(\tilde{X}\times F)$, we have $C_2(Q(\tilde{X}\times F)\otimes\mathcal{L}^{\vee})=0$. If $C_1(\mathcal{L})=x\tilde{D}\times X+y\tilde{F}\times X$ then

$$\begin{aligned}C_2(Q(\tilde{X}\times F)\otimes\mathcal{L}^{\vee})&=(\tilde{X}\times F-C_1(\mathcal{L}))\cdot(\tilde{X}\times F-C_1(\mathcal{L})+C_1(Q))+C_2(Q)\\&=(x^2+x)(\tilde{D}^2)\times X-x(p+3)\tilde{D}\times F+x\tilde{D}\times D\\&\quad-(y+1)(p+3)\tilde{F}\times F+(y+1)\tilde{F}\times D\\&\quad+(2xy+y+x(p+3)+1)(\tilde{D}\cdot\tilde{F})\times X.\end{aligned}$$

$C_2(Q(\tilde{X}\times F)\otimes\mathcal{L}^{\vee})=0$ implies $x=0$ and $y=-1$. So we have $\mathcal{L}\simeq\mathcal{O}_{\tilde{X}\times X}(-\tilde{F}\times X)$ and get an exact sequence

$$0\longrightarrow\mathcal{O}_{\tilde{X}\times X}\longrightarrow Q(\tilde{X}\times F+\tilde{F}\times X)\longrightarrow\det Q(\tilde{X}\times F+\tilde{F}\times X)\longrightarrow 0.$$

By tensoring this sequence with the line bundle $\tilde{q}^*\mathcal{O}_{\tilde{X}}(\tilde{D}+(p+1)\tilde{F})$, the desired exact sequence is obtained.

(3.8) When we regard the exact sequence (3.7.1) as an element of

$$H^1(\tilde{X}\times X,\ \tilde{q}^*\mathcal{O}_{\tilde{X}}(\tilde{D}+(p+1)\tilde{F})\otimes q^*\mathcal{O}_X(D-(p+3)F))$$
$$\simeq H^0(\tilde{X},\ H^1(X,\ \mathcal{O}_X(D-(p+3)F))\otimes\mathcal{O}_{\tilde{X}}(\tilde{D}+(p+1)\tilde{F})),$$

this defines a morphism

$$\eta\ :\ H^1(X,\ \mathcal{O}_X(D-(p+3)F))^{\vee}\otimes\mathcal{O}_{\tilde{X}}\longrightarrow\mathcal{O}_{\tilde{X}}(\tilde{D}+(p+1)\tilde{F}).$$

By Proposition (3.2), for any closed point x of $\tilde{X}$, $\eta(x)$ is a nonzero element of $H^1(X,\ \mathcal{O}_X(D-(p+3)F))$. This means that η is surjective at any closed point of $\tilde{X}$. Therefore η is surjective by Nakayama's lemma and hence it defines a morphism

$$\Psi\ :\ \tilde{X}\longrightarrow \boldsymbol{P}=\boldsymbol{P}(H^1(X,\ \mathcal{O}_X(D-(p+3)F))^{\vee})$$

such that $\Psi^*\mathcal{O}_{\boldsymbol{P}}(1)\simeq\mathcal{O}_{\tilde{X}}(\tilde{D}+(p+1)\tilde{F})$.

REMARK (3.9). The cohomology groups $H^1(X,\ \mathcal{O}_X(D-(p+3)F))$ and $H^0(\tilde{X},\ \mathcal{O}_{\tilde{X}}(\tilde{D}+(p+1)\tilde{F}))$ are dual to each other.

PROOF.

$$\begin{aligned}&H^1(X,\ \mathcal{O}_X(D-(p+3)F))\\ \simeq\ &H^1(\boldsymbol{P}^1,\ \pi_*\mathcal{O}_X(D-(p+3)F))\\ \simeq\ &H^1(\boldsymbol{P}^1,\ \mathcal{V}(-(p+3)))\\ \simeq\ &H^0(\boldsymbol{P}^1,\ \mathcal{V}^{\vee}(p+1))^{\vee}\\ \simeq\ &H^0(\tilde{X},\ \mathcal{O}_{\tilde{X}}(\tilde{D}+(p+1)\tilde{F}))^{\vee}\end{aligned}$$

(3.10) Let ξ be a nonzero element of $H^1(X,\ \mathcal{O}_X(D-(p+3)F))$ and $\mathcal{E}$ be the vector bundle which is defined by the extension

$$(3.10.1)\qquad 0\longrightarrow\mathcal{O}_X\longrightarrow\mathcal{E}(F)\longrightarrow\mathcal{O}_X(-D+(p+3)F))\longrightarrow 0$$

corresponding to ξ.

LEMMA (3.11). *$\mathcal{E}$ is not stable if and only if $H^0(X,\ \mathcal{E}^{\vee})\neq(0)$.*

PROOF. If $H^0(X,\ \mathcal{E}^{\vee})\neq(0)$, then $\mathcal{E}^{\vee}$ is not stable because of $(C_1(\mathcal{E}^{\vee})\cdot H^2)=-1$. On the contrary, if $\mathcal{E}$ is not stable then by Proposition (3.2), there is an exact sequence

$$0\longrightarrow\mathcal{O}_X(-D+(p+1)F)\longrightarrow\mathcal{E}\longrightarrow\mathcal{J}_Y\longrightarrow 0.$$

Thus $\mathcal{E}^{\vee}=\mathcal{E}(D-(p+1)F)$ contains $\mathcal{O}_X$.

(3.12) Taking dual of the exact sequence (3.10.1) and tensoring it with $\mathcal{O}_X(F)$, we have

$$0 \longrightarrow \mathcal{O}_X(D-(p+2)F) \longrightarrow \mathcal{E}^\vee \xrightarrow{t} \mathcal{O}_X(F) \longrightarrow 0.$$

If $H^0(X, \mathcal{E}^\vee) \neq (0)$, let s be a nonzero section of $\mathcal{E}^\vee$, then $t(s) \neq 0$ because $H^0(X, \mathcal{O}_X(D-(p+2)F))=(0)$. Now consider the following diagram

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \mathcal{O}_F(1) \arrow[r] \arrow[u] & \mathcal{E}^\vee \otimes \mathcal{O}_F \arrow[r] \arrow[u] & \mathcal{O}_F \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{O}_X(D-(p+2)F) \arrow[r] \arrow[u] & \mathcal{E}^\vee \arrow[r] \arrow[u] & \mathcal{O}_X(F) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{O}_X(D-(p+3)F) \arrow[r] \arrow[u] & \mathcal{E}^\vee(-F) \arrow[r] \arrow[u] & \mathcal{O}_X \arrow[r] \arrow[u, "u"'] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \tag{3.12.1}$$

where the morphism u is defined by the nonzero section $t(s)$ of $\mathcal{O}_X(F)$. Take cohomology groups of the diagram (3.12.1) and then we get

$$\begin{tikzcd}
0 \arrow[r] & H^0(X, \mathcal{E}^\vee) \arrow[r, "t"] & H^0(X, \mathcal{O}_X(F)) \arrow[r, "v"] & H^1(X, \mathcal{O}_X(D-(p+2)F)) \\
 & 0 \arrow[r] \arrow[u] & H^0(X, \mathcal{O}_X) \arrow[r, "\phi"] \arrow[u, "u"'] & H^1(X, \mathcal{O}_X(D-(p+3)F)) \arrow[u, "h"'] \\
 & & 0 \arrow[u] &
\end{tikzcd}$$

PROPOSITION (3.13). *ξ is contained in* $\ker h$.

PROOF. $h(\xi)=h\phi(1)=vu(1)=vt(0)=0$.

(3.14) Let x be a closed point of $\boldsymbol{P}^1$ and $F_x=\pi^{-1}(x)$. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X(D-(p+3)F) \longrightarrow \mathcal{O}_X(D-(p+2)F) \longrightarrow \mathcal{O}_{F_x}(1) \longrightarrow 0$$

and take cohomology groups. Then we get the exact sequence

$$\begin{aligned} 0 \longrightarrow & H^0(F_x, \mathcal{O}_{F_x}(1)) \\ & \longrightarrow H^1(X, \mathcal{O}_X(D-(p+3)F)) \xrightarrow{h_x} H^1(X, \mathcal{O}_X(D-(p+2)F)) \longrightarrow 0. \end{aligned}$$

Let ξ be a nonzero element of $H^1(X, \mathcal{O}_X(D-(p+3)F))$ and $\mathcal{E}$ be the vector bundle defined by ξ as in (3.10). The converse of Proposition (3.13) holds.

PROPOSITION (3.15). *If ξ is contained in* $\ker h_x$ *for some closed point x of*

$\boldsymbol{P}^1$ then $\mathcal{E}$ is not stable.

PROOF. Standard diagram chasing shows that $H^0(X, \mathcal{E}^\vee)\neq(0)$. By Lemma (3.11), $\mathcal{E}$ is not stable.

Denote by S the union of $\ker h_x$'s where x runs through all closed points of $\boldsymbol{P}^1$.

LEMMA (3.16). *The set S is not contained in any proper linear subspace of $H^1(X, \mathcal{O}_X(D-(p+3)F))$.*

PROOF. Let x be a closed point of $\boldsymbol{P}^1$ and u_x be a morphism

$$\mathcal{V}^\vee(p)\longrightarrow\mathcal{V}^\vee(p+1)$$

which vanishes at x. Then the morphisms

$$h_x : H^1(X, \mathcal{O}_X(D-(p+3)F))\longrightarrow H^1(X, \mathcal{O}_X(D-(p+2)F))$$

and

$$u_x : H^0(\boldsymbol{P}^1, \mathcal{V}^\vee(p))\longrightarrow H^0(\boldsymbol{P}^1, \mathcal{V}^\vee(p+1))$$

are dual to each other. Thus it suffices to prove that the intersection of $\mathrm{im}\,u_x$'s is (0). Let $x_1, \cdots, x_l$ $(l=p+2-a)$ be distinct closed points of $\boldsymbol{P}^1$. Then $\mathrm{im}\,u_{x_1}\cap\cdots\cap\mathrm{im}\,u_{x_l}=(0)$ because

$$\mathcal{V}^\vee(p+1)=\mathcal{O}_{\boldsymbol{P}^1}(p+1-a)\oplus\mathcal{O}_{\boldsymbol{P}^1}(p+1-b)\oplus\mathcal{O}_{\boldsymbol{P}^1}(p+1).$$

The projective version of Lemma (3.16) is available.

PROPOSITION (3.17). *$\Psi(\tilde{X})$ is not contained in any hyperplane of $\boldsymbol{P}$.*

(3.18) Proposition (3.17) shows that the morphism η in (3.8) induces an injective morphism of global sections. By Remark (3.9), η induces an isomorphism of global sections so that Ψ is defined by the complete linear system $|\mathcal{O}_{\tilde{X}}(\tilde{D}+(p+1)\tilde{F})|$. Putting all these above together, an argument similar to the proof of Theorem 2.2 shows the following theorem.

THEOREM (3.19). *The moduli on $\boldsymbol{M}(1;0,1)$ is the complement of the dual 3-dimensional rational scroll $(\tilde{X}, \mathcal{O}_{\tilde{X}}(\tilde{D}+(p+1)\tilde{F}))$ and has a universal family.*

References

[1] A. Grothendieck and J. Dieudonné, Éléments de géométrie algébrique, Publ. Math. I.H.E.S., Nos. 4, 8, 11, …, Paris, (1960-).

[2] R. Hartshorne, Stable vector bundles of rank 2 on $\boldsymbol{P}^3$, Math. Ann., **238** (1978), 229-280.

[3] M. Maruyama, On a family of algebraic vector bundles, Number Theory, Algebraic Geometry and Commutative Algebra, in honor of Y. Akizuki, Kinokuniya, Tokyo, 1973, 95-146.

[4] C. Okonek et al., Vector bundles on complex projective spaces, Birkhäuser, Boston,

1980.
[5] F. Takemoto, Stable vector bundles on algebraic surfaces, Nagoya Math. J., **47** (1972), 29-48.

Toshio HOSOH
Department of Mathematics
Faculty of Science and Technology
Science University of Tokyo
Noda, Chiba 278
Japan

Sadao ISHIMURA
Faculty of General Education
Toho University
Miyama, Funabashi, Chiba 274
Japan