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On the spaces of self homotopy equivalences of certain CW complexes

Dedicated to Professor Nobuo Shimada on his 60th birthday

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§ 0. Introduction.

Let X be a connected locally finite CW complex with non-degenerate base point and let $G(X)$ and $G_0(X)$ be the spaces of self homotopy equivalences of X and self homotopy equivalences of X preserving the base point respectively.

It seems that little is known about the homotopy type of $G(X)$ except in the following two cases. When X is an Eilenberg-MacLane complex $K(\pi, n)$, the weak homotopy type of $G(X)$ is determined completely. That is, Thom noted that if π is an abelian group $G(K(\pi, n))$ has the same weak homotopy type as $\mathrm{Aut}(\pi)\times K(\pi, n)$, where $\mathrm{Aut}(\pi)$ denotes the group of automorphisms of π [**7**]. Gottlieb proved that $G(K(\pi, 1))$ has the same weak homotopy type as $\mathrm{Out}(\pi)\times K(Z(\pi), 1)$, where $\mathrm{Out}(\pi)$ denotes the group of automorphisms of π modulo the inner automorphisms and $Z(\pi)$ denotes the center of π [**1**]. When X is the n-sphere $S^n\,(n\geqq 1)$, it is known that $\pi_i(G_0(S^n))\cong\pi_{n+i}(S^n)$ $(i\geqq 1)$.

In this paper, we shall show the following two theorems and their applications.

THEOREM A. *Let X and Y be connected locally finite CW complexes with base points. For a given $n>0$, assume that $\pi_i(X)=0$ for every $i>n$ and $\pi_i(Y)=0$ for every $i\leqq n$. Then we have*

$$G(X\times Y)=G(X)^Y\times G(Y)^X,$$

$$G_0(X\times Y)=(G(X),\, G_0(X))^{(Y,\, y_0)}\times(G(Y),\, G_0(Y))^{(X,\, x_0)},$$

where $(Z, Z')^{(K,L)}$ denotes the space of maps of (K, L) into (Z, Z').

THEOREM B. *Let X be a connected locally finite CW complex with base point whose dimension is not greater than n and let Y be an n-connected locally finite CW complex with base point. Then the same formulas as in Theorem A hold for $G(X\times Y)$ and $G_0(X\times Y)$.*

§1. $G(X)$ and $G_0(X)$.

Let X and Y be Hausdorff spaces with non-degenerate base points. Then Y^X and Y_0^X will denote the space of maps of X to Y with the compact open topology and the space of maps of X to Y preserving the base points respectively. Also $(Y, Y')^{(X, A)}$ will denote the space of maps of (X, A) to (Y, Y'). This work concerns the space of self homotopy equivalences of connected locally finite CW complex X. In what follows, by a CW complex with base point we mean a connected locally finite CW complex with a chosen vertex.

Let X be a CW complex. Then every arcwise connected component of $G(X)$ has the same homotopy type. The same thing holds for $G_0(X)$. More generally, we have the following

PROPOSITION 1. *Let X be a homotopy associative H-space with unit e. Suppose for each element x of X there exists an element x' of X such that $x \cdot x'$ and $x' \cdot x$ are both contained in the arcwise connected component of e. Then, every arcwise connected component of X has the same homotopy type.*

The proof is easy, so it is omitted.

It should be noted that the hypotheses of this proposition are satisfied in the following three cases:

(1) X=the space Z^Y of maps of a locally compact Hausdorff space Y to a connected H-group Z,

(2) X=the space Z_0^Y of maps of a CW complex Y with base point to a connected homotopy associative H-space Z [**2**],

(3) X=the space Z_0^{SY} of maps $(SY, *) \to (Z, z_0)$, where SY is the suspension of a CW complex Y with base point.

We now consider the relation between $G(X)$ and $G_0(X)$ of a CW complex X with base point. There is the following well-known fibration

$$G_0(X) \longrightarrow G(X) \xrightarrow{\omega} X,$$

where ω is the evaluation map on the base point of X. This fibration is not always weakly splittable, that is, $G(X)$ not always has the same weak homotopy type as $X \times G_0(X)$. However the following holds.

PROPOSITION 2. *Let X be a CW complex with an H-structure. Then $G(X)$ and $X \times G_0(X)$ have the same weak homotopy type.*

PROOF. Let f be the map of X to X^X defined as follows:

$$f(x)(x') = \mu(x, x') = x \cdot x',$$

where μ denotes the multiplication in X with unit e. Then $f(e)$ is contained in the arcwise connected component of id_X in $G(X)$. Note that, since X is connected, for each x of X $f(x)$ can be joined by an arc to id_X in $G(X)$. Thus f can be regarded as a map of X to $G(X)$. Furthermore, we can see easily that $\omega\circ f$ is homotopic to id_X relative to e. By using the composition in $G(X)$, define a map $\varphi: X\times G_0(X)\to G(X)$ by

$$\varphi(x, g)=f(x)\cdot g.$$

Then, it can be proved easily that φ induces isomorphisms of the homotopy groups of the arcwise connected components of $X\times G_0(X)$. In other words, $G(X)$ has the same weak homotopy type as $X\times G_0(X)$.

By Proposition 2, we see that $G(K(\pi, n))$ has the same weak homotopy type as $K(\pi, n)\times G_0(K(\pi, n))$ if π is abelian. Furthermore, we can observe that $G_0(K(\pi, n))$ is weakly homotopy equivalent to $\mathrm{Aut}(\pi)$ if π is abelian.

§2. $G(X\times Y)$ and $G_0(X\times Y)$.

Let X and Y be CW complexes with base points. Then there exist the following homeomorphisms [**6**]

$$(X\times Y)^{X\times Y}\cong X^{X\times Y}\times Y^{X\times Y}\cong (X^X)^Y\times (Y^Y)^X,$$

$$(X\times Y)_0^{X\times Y}\cong X_0^{X\times Y}\times Y_0^{X\times Y}\cong (X^X, X_0^X)^{(Y, y_0)}\times (Y^Y, Y_0^Y)^{(X, x_0)}.$$

Using these correspondences we have

THEOREM A. *Let X and Y be CW complexes with base points. For a given $n>0$, assume that $\pi_i(X)=0$ for every $i>n$ and $\pi_i(Y)=0$ for every $i\leqq n$. Then we have*

$$G(X\times Y)=G(X)^Y\times G(Y)^X,$$

$$G_0(X\times Y)=(G(X), G_0(X))^{(Y, y_0)}\times (G(Y), G_0(Y))^{(X, x_0)}.$$

PROOF. First we shall show the second equality. Let f be a self homotopy equivalence of $X\times Y$ preserving the base point (x_0, y_0). Then, using the second correspondence above, f determines an element (f_1, f_2) of $(X^X, X_0^X)^{(Y, y_0)}\times (Y^Y, Y_0^Y)^{(X, x_0)}$. Since f induces automorphisms of the homotopy groups of $X\times Y$, by using the hypotheses on X and Y $f_1(y_0)$ and $f_2(x_0)$ induce automorphisms of the homotopy groups of X and Y respectively. Thus $f_1(y_0)$ is a self homotopy equivalence of the based complex X. Because Y is connected, this implies that f_1 is a map of (Y, y_0) to $(G(X), G_0(X))$. Similarly we see that f_2 is a map of (X, x_0) to $(G(Y), G_0(Y))$. Therefore we have

$$G_0(X\times Y)\subset (G(X), G_0(X))^{(Y, y_0)}\times (G(Y), G_0(Y))^{(X, x_0)}.$$

Conversely, it is easily verified that each element (f_1, f_2) of $(G(X), G_0(X))^{(Y, y_0)}\times (G(Y), G_0(Y))^{(X, x_0)}$ is contained in $G_0(X\times Y)$ by considering its induced homomorphisms of the homotopy groups of $X\times Y$. This proves the second equality.

For a proof of the first equality, let f be an element of $G(X\times Y)$ which corresponds to an element (f_1, f_2) of $(X^X)^Y\times(Y^Y)^X$. Then there exists a self homotopy equivalence f' of $X\times Y$ with $f'(x_0, y_0)=(x_0, y_0)$ and homotopic to f. Let (f_1', f_2') be the corresponding element of $(X^X)^Y\times(Y^Y)^X$ to f' then f_1' and f_2' are homotopic to f_1 and f_2, respectively. By the second equality we have

$$(f_1', f_2')\in G(X)^Y\times G(Y)^X.$$

Consequently (f_1, f_2) is an element of $G(X)^Y\times G(Y)^X$, that is, $G(X\times Y)\subset G(X)^Y \times G(Y)^X$.

Conversely it can be proved easily that each element of $G(X)^Y\times G(Y)^X$ can be joined by an arc with an element of

$$(G(X), G_0(X))^{(Y, y_0)}\times(G(Y), G_0(Y))^{(X, x_0)}=G_0(X\times Y).$$

This implies that $G(X)^Y\times G(Y)^X\subset G(X\times Y)$. Our proof is completed.

Let us introduce a proposition which will be used later on the weak homotopy type of space of maps. We write $X\underset{w}{\simeq}Y$ if X and Y have the same weak homotopy type.

PROPOSITION 3. *Suppose* $X\underset{w}{\simeq}Y$, *then for every CW complex* Z, *we have* $X^Z\underset{w}{\simeq}Y^Z$.

PROOF. We may assume without loss of generality that X and Y are arcwise connected and there is a map f of X to Y which induces an isomorphism of $\pi_n(X, x_0)$ onto $\pi_n(Y, y_0)$ for each n. Let $\tilde{f}$ be the map of X^Z to Y^Z induced by the map f. Then we shall show that the homomorphisms $\tilde{f}_*$ from $\pi_n(X^Z, \alpha)$ to $\pi_n(Y^Z, \tilde{f}(\alpha))$ induced by $\tilde{f}$ is an isomorphism for each n and for every $\alpha\in X^Z$.

To see this, let h be a map of $(S^n, *)$ to $(Y^Z, \tilde{f}(\alpha))$ which represents an element $[h]$ of $\pi_n(Y^Z, \tilde{f}(\alpha))$ and let $\bar{h}$ be the map of $S^n\times Z$ to Y associated with h. Define a map $\bar{g}'$ of $*\times Z$ to X by $\bar{g}'(*, z)=\alpha(z)$. Then we have $f\circ\bar{g}' =\bar{h}|*\times Z$. Since f is a weak homotopy equivalence, there exists a map $\bar{g}$ of $S^n\times Z$ to X such that $f\circ\bar{g}$ is homotopic to $\bar{h}$ relative to $*\times Z$ and $\bar{g}$ is an extention of $\bar{g}'$. Let g be a map of $(S^n, *)$ to (X^Z, α) defined by $\bar{g}$. Immediately we see $\tilde{f}_*([g])=[h]$. This proves that $\tilde{f}_*$ is epimorphic.

To see that $\tilde{f}_*$ is monomorphic, let g be a map of $(S^n, *)$ to (X^Z, α) such that $\tilde{f}_*([g])=0$, and let $\bar{g}$ be the map of $S^n\times Z$ to X associated with g. Then we have a homotopy $\bar{H}$: $S^n\times Z\times I\to Y$ of $f\circ\bar{g}$ to $f\circ\alpha$ satisfying $\bar{H}(*, z, t)=f\circ\alpha(z)$ for $z\in Z$. Since f is a weak homotopy equivalence, we can prove easily that there exists a map $\bar{G}$ of $S^n\times Z\times I$ to X satisfying

$$\bar{G}(\lambda, z, 0)=\bar{g}(\lambda, z) \qquad (\lambda\in S^n,\ z\in Z,\ t\in I)$$
$$\bar{G}(\lambda, z, 1)=\alpha(z)$$
$$\bar{G}(*, z, t)=\alpha(z)$$

and furthermore $f\circ\bar{G}$ is homotopic to $\bar{H}$ relative to $S^n\times Z\times 0\cup S^n\times Z\times 1\cup *\times Z\times I$. Let G be a map of $(S^n, *)\times I$ to (X^Z, α) defined by $\bar{G}$. Then we see that G is a homotopy of g to the constant map. This implies $[g]=0$. Thus our proof is completed.

REMARK. In Proposition 3, let X and Y be spaces with base points which have the same weak homotopy type and let Z be a CW complex with base point. Then, in a manner similar to our proof of Proposition 3, we can show $X^Z_0\underset{w}{\simeq}Y^Z_0$.

Putting $X=K(\pi, n)$ in Theorem A, we can prove the following.

THEOREM 4. *Let X be $K(\pi, n)$ with a chosen base point and let Y be an n-connected CW complex with base point. Then we have*

$$G_0(X\times Y)\underset{w}{\simeq}\mathrm{Aut}(\pi)\times G_0(Y)\times G(Y)^X_0.$$

PROOF. Put $Z=(G(X), G_0(X))^{(Y,y_0)}$, then we obtain the following two fibrations

$$\begin{array}{ccccc} G(X)^Y_0 & \longrightarrow & G(X)^Y & \xrightarrow{\omega} & G(X) \\ \| & & \cup & & \cup \\ G(X)^Y_0 & \longrightarrow & Z & \xrightarrow{\omega} & G_0(X) \end{array}$$

where ω is the evaluation map on the base point y_0 of Y. Here $G(X)^Y$ has the same weak homotopy type as $G(X)\times G(X)^Y_0$ because $G(X)$ is an H-space. This splitting induces that

$$Z\underset{w}{\simeq}G_0(X)\times G(X)^Y_0$$
$$\underset{w}{\simeq}\mathrm{Aut}(\pi)\times G(X)^Y_0.$$

For $n>1$, by Remark of Proposition 3 we have

$$G(X)^Y_0\underset{w}{\simeq}(K(\pi, n)\times\mathrm{Aut}(\pi))^Y_0$$
$$\underset{w}{\simeq}K(\pi, n)^Y_0.$$

Since Y is n-connected, $K(\pi, n)^Y_0$ is weakly homotopy equivalent to one point. Thus we obtain

$$Z\underset{w}{\simeq}\mathrm{Aut}(\pi).$$

For $n=1$, we have

$$G(X)_0^Y \underset{w}{\simeq} (K(Z(\pi),\ 1)\times \mathrm{Out}(\pi))_0^Y$$
$$\underset{w}{\simeq} K(Z(\pi),\ 1)_0^Y$$
$$\underset{w}{\simeq} 0.$$

In this case, we have also $Z \underset{w}{\simeq} \mathrm{Aut}(\pi)$.

Similarly we see that

$$(G(Y),\ G_0(Y))^{(X,\, x_0)} \underset{w}{\simeq} G_0(Y)\times G(Y)_0^X .$$

Consequently, by Theorem A we have

$$G_0(X\times Y) \underset{w}{\simeq} \mathrm{Aut}(\pi)\times G_0(Y)\times G(Y)_0^X .$$

As a special case of this theorem, we have

COROLLARY. *Let X and Y be $K(\pi,\ m)$ and $K(\pi',\ n)$ $(1\leqq m<n)$ respectively. Then we have*

$$G_0(X\times Y) \underset{w}{\simeq} \mathrm{Aut}(\pi)\times \mathrm{Aut}(\pi')\times Y_0^X .$$

Note that in the corollary, Y_0^X has the same weak homotopy type as

$H^n(K(\pi,\ m),\ \pi')\times$ {certain product space of Eilenberg-Maclane complexes}

by the theorem of J. C. Moore [4].

As mentioned in the introduction, our second main result is as follows.

THEOREM B. *Let X be a CW complex with base point whose dimension is not greater than n and let Y be an n-connected CW complex with base point. Then we have the same formulas as in Theorem A for $G(X\times Y)$ and $G_0(X\times Y)$.*

PROOF. We shall show first that

$$G_0(X\times Y)=(G(X),\ G_0(X))^{(Y,\, y_0)}\times (G(Y),\ G_0(Y))^{(X,\, x_0)}$$

under the identification

$$(X\times Y)_0^{X\times Y}=X_0^{X\times Y}\times Y_0^{X\times Y}=(X^X,\ X_0^X)^{(Y,\, y_0)}\times (Y^Y,\ Y_0^Y)^{(X,\, x_0)} .$$

Let f be an element of $G_0(X\times Y)$ which corresponds to the element $(f_1,\ f_2)$ of $(X^X,\ X_0^X)^{(Y,\, y_0)}\times (Y^Y,\ Y_0^Y)^{(X,\, x_0)}$. Since f induces automorphisms of the homotopy groups of $X\times Y$, by assumption on X and Y the automorphism $f_*: \pi_k(X\times Y)\to \pi_k(X\times Y)$ may be regarded as the induced homomorphism $f_1(y_0)_*: \pi_k(X)\to\pi_k(X)$ for each $k\leqq n$. This shows that $f_1(y_0)_*: \pi_k(X)\to\pi_k(X)$ is an automorphism for each $k\leqq n$. By the theorem of J. H. C. Whitehead [8] $f_1(y_0)$ is a self homotopy equivalence of X, that is, $f_1(y_0)$ is an element of $G_0(X)$. Since Y is connected, $f_1(y)$ is an element of $G(X)$ for each element y of Y. This implies $f_1\in(G(X),$

$G_0(X))^{(Y, y_0)}$.

Let i_1 and i_2 be the inclusion maps of X and Y into $X \times Y$ respectively:

$$i_1(x)=(x, y_0) \qquad (x \in X),$$

$$i_2(y)=(x_0, y) \qquad (y \in Y).$$

Let p_1 and p_2 be the projections of $X \times Y$ onto X and Y respectively. Define an isomorphism $\lambda: \pi_k(X) \oplus \pi_k(Y) \to \pi_k(X \times Y)$ by $\lambda(\alpha, \beta)=i_{1*}(\alpha)+i_{2*}(\beta)$. Then we have the following sequence of isomorphisms for each k

$$\pi_k(X) \oplus \pi_k(Y) \xrightarrow{\lambda} \pi_k(X \times Y) \xrightarrow{f_*} \pi_k(X \times Y)$$

$$\xrightarrow{(p_{1*}, p_{2*})} \pi_k(X) \oplus \pi_k(Y).$$

Here we have

$$\begin{aligned} p_{1*} \circ f_* \circ \lambda(\alpha, \beta) &= p_{1*} \circ f_* \circ i_{1*}(\alpha)+p_{1*} \circ f_* \circ i_{2*}(\beta) \\ &= f_1(y_0)_*(\alpha)+h_{1*}(\beta), \\ p_{2*} \circ f_* \circ \lambda(\alpha, \beta) &= p_{2*} \circ f_* \circ i_{1*}(\alpha)+p_{2*} \circ f_* \circ i_{2*}(\beta) \\ &= h_{2*}(\alpha)+f_2(x_0)_*(\beta), \end{aligned}$$

where $h_1:(Y, y_0) \to (X, x_0)$ is the map defined by $h_1(y)=f_1(x_0, y)$ and $h_2:(X, x_0) \to (Y, y_0)$ is the map defined by $h_2(x)=f_2(x, y_0)$. Since Y is n-connected and $\dim X \leqq n$, h_2 is homotopic to the constant map. Thus we obtain an automorphism of $\pi_k(X) \oplus \pi_k(Y)$:

$$(1) \qquad (p_{1*}, p_{2*}) \circ f_* \circ \lambda(\alpha, \beta)=(f_1(y_0)_*(\alpha)+h_{1*}(\beta), f_2(x_0)_*(\beta)).$$

Therefore $f_2(x_0)_*$ is an automorphism of $\pi_k(Y)$ for each k. Hence $f_2(x_0):(Y, y_0) \to (Y, y_0)$ is a homotopy equivalence. Because X is arcwise connected, this implies that $f_2(x)$ is a self homotopy equivalence of Y for each x. That is, $f_2 \in (G(Y), G_0(Y))^{(X, x_0)}$. Finally, we have

$$G_0(X \times Y) \subset (G(X), G_0(X))^{(Y, y_0)} \times (G(Y), G_0(Y))^{(X, x_0)}.$$

Conversely, let f be an element of $(X \times Y)^{X \times Y}$ which corresponds to an element (f_1, f_2) of $(G(X), G_0(X))^{(Y, y_0)} \times (G(Y), G_0(Y))^{(X, x_0)}$. Then we see that $f_1(y_0)_*$ and $f_2(x_0)_*$ are automorphisms of $\pi_k(X)$ and $\pi_k(Y)$ for each k respectively. Note that the formula (1) holds in this situation. Thus f_* is an automorphism of $\pi_k(X \times Y)$ for each k. Consequently f is a self homotopy equivalence of $X \times Y$ preserving the base point (x_0, y_0), that is, $f \in G_0(X \times Y)$. Hence our proof of the assertion on $G_0(X \times Y)$ is completed.

The assertion about $G(X \times Y)$ can be proved in a similar way to our proof of Theorem A.

§ 3. Applications.

Note that the following theorem can be deduced from Theorem A and Theorem B.

THEOREM C. *For a given $n>0$, let X be a CW complex with base point and let Y be an n-connected CW complex with base point. Assume that* $\dim X \leqq n$ *or* $\pi_i(X)=0$ *for every $i>n$. Then the following hold:*

$$G(X\times Y)\underset{w}{\simeq} G(X)\times G(Y)\times G(X)_0^Y\times G(Y)_0^X,$$

$$G_0(X\times Y)\underset{w}{\simeq} G_0(X)\times G_0(Y)\times G(X)_0^Y\times G(Y)_0^X.$$

PROOF. With the help of Theorems A and B, we can give a proof in a similar manner to the proof of Theorem 4 by using the fact that $G(X)$ and $G(Y)$ are H-spaces. We omit the details.

If X is a finite CW complex with base point, by the theorem of J. Milnor [**3**] $G(X)$ and $G_0(X)$ have the same homotopy types as CW complexes. Thus, as a special case of Theorem C, we have the following

COROLLARY. *Let X be a simply connected finite CW complex with base point. Then it holds that*

$$G(S^1\times X)\simeq O(2)\times G(X)\times \Omega G(X),$$

$$G_0(S^1\times X)\simeq \boldsymbol{Z}_2\times G_0(X)\times \Omega G(X),$$

where $O(2)$ is the orthogonal group of degree 2 and $\Omega G(X)$ is the space of loops on $G(X)$ based at id_X.

Now, let $\varepsilon(X)$ be the group of based homotopy classes of self homotopy equivalences of CW complex X with base point, and let Y be a CW complex with base point. Then we shall define an action of the direct product group $\varepsilon(X)\times\varepsilon(Y)$ on the group $[X, G(Y)]_0$ whose multiplication is induced by the H-structure in $G(Y)$ [**2**].

Let k be an element of $G_0(Y)$ and let $G_i(Y)$ is the arcwise connected component of $G(Y)$ containing the identity map id_Y. We define a self map $\tilde{k}$ of $G_i(Y)$ by using the multiplication in $G(Y)$ as follows:

$$\tilde{k}(\alpha)=k^{-1}\cdot\alpha\cdot k \qquad (\alpha\in G_i(Y))$$

where k^{-1} is a fixed element representing $[k]^{-1}$. Obviously the homotopy class $[\tilde{k}]$ is independent of the choice of k^{-1} and it depends only on $[k]$.

Let $[\bar{f}]$ be an element of $[X, G(Y)]_0=[X, G_i(Y)]_0$, then we define a multiplication of $\varepsilon(X)\times\varepsilon(Y)$ on $[X, G_i(Y)]_0$ as follows:

$$([h], [k])^*[\bar{f}]=[\tilde{k}\circ\bar{f}\circ h].$$

With this multiplication we have

LEMMA 5. *The direct product group* $\varepsilon(X)\times\varepsilon(Y)$ *acts on the group* $[X, G_i(Y)]_0$.

PROOF. We have

$$([h], [k])^*([\bar{f}]\cdot[\bar{g}])=([h], [k])^*[\bar{f}\cdot\bar{g}]$$
$$=[\tilde{k}\circ(\bar{f}\cdot\bar{g})\circ h]$$
$$=[\tilde{k}\circ((\bar{f}\circ h)\cdot(\bar{g}\circ h))],$$

because

$$(\bar{f}\cdot\bar{g})\circ h(x)=\bar{f}(h(x))\cdot\bar{g}(h(x))=(\bar{f}\circ h)\cdot(\bar{g}\circ h)(x) \qquad (x\in X).$$

Furthermore, since $\tilde{k}$ is an H-map, we have

$$[\tilde{k}\circ((\bar{f}\circ h)\cdot(\bar{g}\circ h))]=[(\tilde{k}\circ\bar{f}\circ h)\cdot(\tilde{k}\circ\bar{g}\circ h)]$$
$$=(([h], [k])^*[\bar{f}])\cdot(([h], [k])^*[\bar{g}]).$$

Next,

$$(([h], [k])([h'], [k']))^*[\bar{f}]=([hh'], [kk'])^*[\bar{f}]$$
$$=[\widetilde{kk'}\circ\bar{f}\circ(hh')]$$
$$=([h'], [k'])^*[\tilde{k}\circ\bar{f}\circ h]$$
$$=([h'], [k'])^*(([h], [k])^*[\bar{f}]).$$

Obviously we have

$$([\mathrm{id}_X], [\mathrm{id}_Y])^*[\bar{f}]=[\bar{f}].$$

Thus our proof is completed.

Suppose $\pi_j(X)=0$ for every $j>n$ and Y is n-connected. Let us define the correspondence λ from $\varepsilon(X\times Y)$ to the semi-direct product group $(\varepsilon(X)\times\varepsilon(Y))\otimes[X, G_i(Y)]_0$ by using the action introduced above. Let $[f]$ be an element of $\varepsilon(X\times Y)$. Note that, as we already observed in the proof of Theorem A, $p_1\circ f\circ i_1=f_1\circ i_1$ and $p_2\circ f\circ i_2=f_2\circ i_2$ are self homotopy equivalences of (X, x_0) and (Y, y_0) respectively, where $i_1\colon X\to X\times Y$, $i_2\colon Y\to X\times Y$ are the inclusion maps and $p_1\colon X\times Y\to X$, $p_2\colon X\times Y\to Y$ are the projection maps. Putting $h=f_1\circ i_1$, $k=f_2\circ i_2$, $\bar{f}_2(x)(y)=f_2(x, y)$ $((x, y)\in X\times Y)$ and

$$(k^{-1}\cdot\bar{f}_2)(x)=k^{-1}\cdot\bar{f}_2(x) \qquad (x\in X),$$

then $k^{-1}\cdot\bar{f}_2(x_0)$ and id_Y are joined by an arc in $G_0(Y)$. Thus by the homotopy extension theorem there exists a map $\bar{f}_2'$ of (X, x_0) to $(G(Y), \mathrm{id}_Y)$ such that $\bar{f}_2'$ is homotopic to $k^{-1}\cdot\bar{f}_2$ under a homotopy keeping x_0 in $G_0(Y)$. Since $(G(Y), G_0(Y))$ is an H-space pair, the homotopy class $[\bar{f}_2']$ in $[X, G_i(Y)]_0$ is independent of the choice of f_2, k^{-1} and $\bar{f}_2'$. Define a correspondence λ of $\varepsilon(X\times Y)$ to $(\varepsilon(X)\times\varepsilon(Y))\otimes[X, G_i(Y)]_0$ as follows:

$$\lambda([f])=([f_1\circ i_1], [f_2\circ i_2], [\bar{f}_2']).$$

Then we have the following result.

THEOREM 6. *For a given $n>0$, let X be a CW complex with base point such that $\pi_i(X)=0$ for every $i>n$ and let Y be an n-connected CW complex with base point. Then we have an isomorphism λ:*

$$\varepsilon(X\times Y) \longrightarrow (\varepsilon(X)\times\varepsilon(Y))\otimes[X, G(Y)]_0,$$

where $\otimes$ denotes a semi-direct product defined by the above action.

PROOF. We shall show that λ is a homomorphism. Let f and g be self homotopy equivalences of $(X\times Y, (x_0, y_0))$. First we see that

$$f_1\circ g\circ i_1\simeq(f_1\circ i_1)\circ(g_1\circ i_1) \qquad \mathrm{rel}\, x_0.$$

To see this, let f_0 be a map of $X\times Y$ to X defined by $f_0(x, y)=f_1(x, y_0)$ $((x, y)\in X\times Y)$. Then, by obstruction theory under our assumptions $\pi_k(X)=0$ for every $k>n$ and $\pi_k(Y)=0$ for every $k\leqq n$ we can see that f_0 is homotopic to f_1 relative to $X\times y_0$. Thus we have $f_1\circ g\circ i_1\simeq f_0\circ g\circ i_1$ rel x_0. On the other hand,

$$\begin{aligned} f_0\circ g\circ i_1(x)&=f_0(g_1(x, y_0), g_2(x, y_0))\\ &=f_1(g_1(x, y_0), y_0)\\ &=(f_1\circ i_1)\circ(g_1\circ i_1)(x). \end{aligned}$$

Combining these two, we have $[f_1\circ g\circ i_1]=[f_1\circ i_1]\cdot[g_1\circ i_1]$.

Next we shall show that $f_2\circ g\circ i_2\simeq(f_2\circ i_2)\circ(g_2\circ i_2)$ rel y_0. Let h be a map of Y to X defined by $h(y)=g_1(x_0, y)$. Then, since Y is n-connected and $\pi_k(X)=0$ for every $k>n$, h is homotopic to the constant map relative to y_0. By the homotopy extension theorem there exists a map g_0 of $X\times Y$ to X which is homotopic to g_1 relative to (x_0, y_0) and satisfies $g_0(x_0\times Y)=x_0$. Let us define a self map g' of $X\times Y$ by $g'(x, y)=(g_0(x, y), g_2(x, y))$. Obviously, we have $g\simeq g'$ rel (x_0, y_0). This implies $f_2\circ g\circ i_2\simeq f_2\circ g'\circ i_2$ rel y_0. Furthermore we have

$$\begin{aligned} f_2\circ g'\circ i_2(y)&=f_2(g_0(x_0, y), g_2(x_0, y))\\ &=f_2(x_0, g_2(x_0, y)) \qquad (y\in Y). \end{aligned}$$

That is, $f_2\circ g'\circ i_2=(f_2\circ i_2)\circ(g_2\circ i_2)$. These imply

$$[f_2\circ g\circ i_2]=[f_2\circ i_2]\cdot[g_2\circ i_2].$$

Putting $h'=g_1\circ i_1$ and $k'=g_2\circ i_2$, we will compute $(kk')^{-1}\cdot\overline{f_2\circ g}$. Since we have $f_2\circ g\simeq f_2\circ(g_0, g_2)$ rel(x_0, y_0) as the argument above, we see

$$\overline{f_2\circ g}\simeq\overline{f_2\circ(g_0, g_2)}.$$

Furthermore, it holds that

$$\begin{aligned}\overline{f_2\circ(g_0, g_2)}(x)(y)&=f_2(g_1(x, y_0), g_2(x, y))\\&=\bar f_2(g_1(x, y_0))(g_2(x, y))\\&=\bar f_2(h'(x))(g_2(x, y))\\&=(\bar f_2(h'(x))\cdot\bar g_2(x))(y).\end{aligned}$$

Hence we have

$$\begin{aligned}&(kk')^{-1}\cdot\bar f_2(h'(x))\cdot\bar g_2(x)\\&\qquad=k'^{-1}\cdot k^{-1}\cdot\bar f_2(h'(x))\cdot\bar g_2(x).\end{aligned}$$

Let $\bar g_2'$ be a map of (X, x_0) to $(G(Y), \mathrm{id}_Y)$ which is homotopic to $k'^{-1}\cdot\bar g_2$. Since $k^{-1}\cdot\bar f_2$ is homotopic to $\bar f_2'$, we have

$$\begin{aligned}k'^{-1}\cdot k^{-1}\cdot(\bar f_2\circ h')\cdot\bar g_2&\simeq k'^{-1}\cdot(\bar f_2'\circ h')\cdot\bar g_2\\&\simeq k'^{-1}\cdot(\bar f_2'\circ h')\cdot k'\cdot k'^{-1}\cdot\bar g_2\\&\simeq k'^{-1}\cdot(\bar f_2'\circ h')\cdot k'\cdot\bar g_2'\\&=(\tilde k'\circ\bar f_2'\circ h')\cdot\bar g_2'.\end{aligned}$$

Hence we have

$$[(\tilde k'\circ\bar f_2'\circ h')\circ\bar g_2']=(([h'], [k'])^*[\bar f_2'])\cdot[\bar g_2'].$$

Finally it holds that

$$\begin{aligned}\lambda([f][g])&=\lambda([f\circ g])\\&=([h][h'], [k][k'], (([h'], [k'])^*[\bar f_2'])[\bar g_2'])\\&=([h], [k], [\bar f_2'])([h'], [k'], [\bar g_2'])\\&=\lambda([f])\lambda([g]),\end{aligned}$$

that is, λ is a homomorphism

We now show that λ is epimorphic. Let $([h], [k], [\bar l])$ be an element of $(\varepsilon(X)\times\varepsilon(Y))\otimes[X, G(Y)]_0$. We define a self map f of $(X\times Y, (x_0, y_0))$ as follows:

$$f(x, y)=(h(x), (k\cdot\bar{l}(x))(y))$$
$$=(h(x), k(l(x, y))),$$

where l is the map of $(X\times Y, (x_0, y_0))$ to (Y, y_0) associated with $\bar{l}$. Then we see easily

$$\lambda([f])=([h], [k], [\bar{l}]).$$

Furthermore we can see easily that λ is monomorphic. Hence our proof is completed.

As a special case of Theorem 6, we have a generalization of the theorem of S. Sasao and Y. Ando [5] as follows.

COROLLARY. *Let X be an n-connected CW complex with base point. Then we have the following isomorphism* λ:

$$\varepsilon(K(\pi, n)\times X) \longrightarrow (\mathrm{Aut}(\pi)\times\varepsilon(X))\otimes[K(\pi, n), G(X)]_0,$$

where the right group is the semi-direct product group defined by the action given in Lemma 5.

PROOF. $[X, G(K(\pi, n))]_0=[X, G_i(K(\pi, n))]_0$ is trivial, because X is n-connected and $G_i(K(\pi, n))$ has the same weak homotopy type as $K(\pi, n)$ or $K(Z(\pi), 1)$ according to $n>1$ or $n=1$. Furthermore we have $\varepsilon(K(\pi, n))=\mathrm{Aut}(\pi)$. Therefore by Theorem 6, we see that λ is an isomorphism.

By Lemma 5, we have the action of the direct product $\varepsilon(X)\times\varepsilon(Y)$ of the groups $\varepsilon(X)$ and $\varepsilon(Y)$ on the group $[X, G(Y)]_0=[X, G_i(Y)]_0$. In other words, we can say that the direct product $\varepsilon(X)\times\varepsilon(Y)$ of the groups $\varepsilon(X)$ and $\varepsilon(Y)$ acts on the group $[Y, G(X)]_0=[Y, G_i(X)]_0$. Consequently, we have the semi-direct product group $(\varepsilon(X)\times\varepsilon(Y))\otimes[Y, G(X)]_0$ defined by this action.

If X is a CW complex of dimension less than or equal to n with base point and Y is an n-connected CW complex with base point, then we shall define a correspondence λ of $\varepsilon(X\times Y)$ to the semi-direct product $(\varepsilon(X)\times\varepsilon(Y))\otimes[Y, G_i(X)]_0$ of the groups $\varepsilon(X)\times\varepsilon(Y)$ and $[Y, G_i(X)]_0$ in the following way.

Let $[f]$ be an element of $\varepsilon(X\times Y)$. As we already observed in the proof of Theorem B, $p_1\circ f\circ i_1=f_1\circ i_1$ and $p_2\circ f\circ i_2=f_2\circ i_2$ are self homotopy equivalences of (X, x_0) and (Y, y_0) respectively. Putting $h=f_1\circ i_1$, $k=f_2\circ i_2$, $\bar{f}_1(y)(x)=f_1(x, y)$ $((x, y)\in X\times Y)$ and

$$(h^{-1}\cdot\bar{f}_1)(y)=h^{-1}\cdot\bar{f}_1(y) \qquad (y\in Y),$$

then we see $h^{-1}\cdot\bar{f}_1(y)$ and id_X can be joined by an arc in $G_0(X)$. By the homotopy extension theorem there exists a map $\bar{f}_1'$ of (Y, y_0) to $(G(X), \mathrm{id}_X)$ such that $\bar{f}_1'$ is homotopic to $h^{-1}\cdot\bar{f}_1$ under a homotopy keeping y_0 in $G_0(X)$. Here we should note that $[\bar{f}_1']$ is uniquely determined as before. Define a correspondence

λ of $\varepsilon(X\times Y)$ to $(\varepsilon(X)\times\varepsilon(Y))\otimes[Y, G_i(X)]_0$ as follows:

$$\lambda([f])=([f_1\circ i_1], [f_2\circ i_2], [\bar{f}_1']).$$

Then we have

THEOREM 7. *For a given $n>0$, let X be a CW complex of* $\dim X\leqq n$ *with base point and let Y be an n-connected CW complex with base point. Suppose that $[X, G(Y)]_0$ is trivial, then λ is an isomorphism of $\varepsilon(X\times Y)$ onto the semi-direct product group $(\varepsilon(X)\times\varepsilon(Y))\otimes[Y, G(X)]_0$ defined by the action introduced previously.*

PROOF. We shall show that λ is a homomorphism. Let f and g be self homotopy equivalences of $(X\times Y, (x_0, y_0))$. Then the map $g_2|X\times y_0: (X\times y_0, (x_0, y_0))\to(Y, y_0)$ is homotopic to the constant map relative to (x_0, y_0) because $\dim X\leqq n$ and Y is n-connected. By the homotopy extension theorem, there exists a map g_0 of $X\times Y$ to Y which is homotopic to g_2 relative to (x_0, y_0) and satisfies $g_0(X\times y_0)=y_0$. Put $g'(x, y)=(g_1(x, y), g_0(x, y))$ $((x, y)\in X\times Y)$, then g' is homotopic to g relative to (x_0, y_0). Thus it holds that

$$f_1\circ g\circ i_1\simeq f_1\circ g'\circ i_1=f_1\circ i_1\circ g_1\circ i_1.$$

Therefore, we have

$$[f_1\circ g\circ i_1]=[f_1\circ i_1\circ g_1\circ i_1]=[f_1\circ i_1][g_1\circ i_1].$$

Next we shall show

$$f_2\circ g\circ i_2\simeq(f_2\circ i_2)\circ(g_2\circ i_2) \qquad \mathrm{rel}\, y_0.$$

Let $\bar{f}_2$ be the map of (X, x_0) to (Y^Y, Y_0^Y) associated with the map f_2 of $(X\times Y, (x_0, y_0))$ to (Y, y_0). Then, as we already observed in the proof of Theorem B, $\bar{f}_2$ is the map of (X, x_0) to $(G(Y), G_0(Y))$. Consequently, $k^{-1}\cdot\bar{f}_2$ is a map of (X, x_0) to $(G(Y), G_0(Y))$ such that $k^{-1}\cdot\bar{f}_2(x_0)$ and id_Y can be joined by an arc in $G_0(Y)$. Thus, by the homotopy extension theorem there exists a map $\bar{f}_2'$ of (X, x_0) to $(G(Y), \mathrm{id}_Y)$ which is homotopic to $k^{-1}\cdot\bar{f}_2$. By our assumption $[X, G(Y)]_0=1$, $\bar{f}_2'$ is homotopic to the constant map. These imply that $\bar{f}_2$ is homotopic to the constant map c_k which maps (X, x_0) to (k, k). That is, f_2 is homotopic to $f_2\circ i_2\circ p_2$ relative to (x_0, y_0). Therefore we have

$$f_2\circ g\circ i_2\simeq f_2\circ i_2\circ p_2\circ g\circ i_2=f_2\circ i_2\circ g_2\circ i_2.$$

Hence it holds that

$$[f_2\circ g\circ i_2]=[f_2\circ i_2\circ g_2\circ i_2]=[f_2\circ i_2][g_2\circ i_2].$$

Putting $h'=g_1\circ i_1$ and $k'=g_2\circ i_2$, in the following we shall show that $(hh')^{-1}\cdot\overline{f_1\circ g}=h'^{-1}\cdot h^{-1}\cdot\overline{f_1\circ g}$ is homotopic to $h'^{-1}\cdot(\bar{f}_1'\circ k')\cdot h'\cdot\bar{g}_1'$, where $\bar{g}_1'$ is a map of (Y, y_0) to $(G(X), \mathrm{id}_X)$ which is homotopic to $h'^{-1}\cdot\bar{g}_1$. Since $[X, G(Y)]_0$ is trivial,

g_2 is homotopic to $k' \circ p_2$ relative to (x_0, y_0). Thus we have

$$\overline{f_1 \circ g} = \overline{f_1 \circ (g_1, g_2)} \simeq \overline{f_1 \circ (g_1, k' \circ p_2)}.$$

Furthermore, if we put $k'^* \bar{f}_1 = \bar{f}_1 \circ k'$ it holds that

$$\begin{aligned}(k'^* \bar{f}_1 \cdot \bar{g}_1)(y)(x) &= \{k'^* \bar{f}_1(y) \cdot \bar{g}_1(y)\}(x)\\ &= \bar{f}_1(k'(y))(\bar{g}_1(y)(x))\\ &= f_1(g_1(x, y), k'(y))\\ &= f_1(g_1(x, y), k' \circ p_2(x, y))\\ &= f_1 \circ (g_1, k' \circ p_2)(y)(x).\end{aligned}$$

Hence we have

$$\begin{aligned}h'^{-1} \cdot h^{-1} \cdot k'^* \bar{f}_1 \cdot \bar{g}_1 &= h'^{-1} \cdot k'^*(h^{-1} \cdot \bar{f}_1) \cdot \bar{g}_1\\ &\simeq h'^{-1} \cdot (k'^* \bar{f}_1') \cdot \bar{g}_1\\ &\simeq h'^{-1} \cdot (k'^* \bar{f}_1') \cdot h' \cdot h'^{-1} \cdot \bar{g}_1\\ &\simeq h'^{-1} \cdot (k'^* \bar{f}_1') \cdot h' \cdot \bar{g}_1',\end{aligned}$$

$$[h'^{-1} \cdot (k'^* \bar{f}_1') \cdot h' \cdot \bar{g}_1'] = (([h'], [k'])^* [\bar{f}_1']) \cdot [\bar{g}_1'].$$

We now see λ is a homomorphism,

$$\begin{aligned}\lambda([f][g]) &= \lambda([f \circ g])\\ &= ([h][h'], [k][k'], (([h'], [k'])^* [\bar{f}_1'])[\bar{g}_1'])\\ &= ([h], [k], [\bar{f}_1'])([h'], [k'], [\bar{g}_1'])\\ &= \lambda([f])\lambda([g])\end{aligned}$$

Next, we shall show that λ is epimorphic. Let $([a], [b], [\bar{c}])$ be an element of $(\varepsilon(X) \times \varepsilon(Y)) \otimes [Y, G(X)]_0$ where a is a self homotopy equivalence of (X, x_0), b is a self homotopy equivalence of (Y, y_0) and $\bar{c}$ is a map of (Y, y_0) to $(G(X), \mathrm{id}_X)$. Then we define a self map $f_{(a,b)}$ of $(X \times Y, (x_0, y_0))$ by

$$f_{(a,b)}(x, y) = (a(x), b(y)) \qquad ((x, y) \in X \times Y).$$

We easily see that

$$\lambda([f_{(a,b)}]) = ([a], [b], 1).$$

Also define a self map $f_{\bar{c}}$ of $(X \times Y, (x_0, y_0))$ by

$$f_{\bar{c}}(x,\ y)=(\bar{c}(y)(x),\ y) \qquad ((x,\ y)\in X\times Y).$$

We can easily see that

$$\lambda([f_{\bar{c}}])=([\mathrm{id}_X],\ [\mathrm{id}_Y],\ [\bar{c}]).$$

Consequently we have

$$\begin{aligned}\lambda([f_{(a,b)}\circ f_{\bar{c}}])&=([a],\ [b],\ 1)([\mathrm{id}_X],\ [\mathrm{id}_Y],\ [\bar{c}])\\&=([a],\ [b],\ [\bar{c}]).\end{aligned}$$

Furthermore we can see easily that $\mathrm{Ker}\lambda$ is just $[\mathrm{id}_{X\times Y}]$. Hence our proof is completed.

As a special case of Theorem 7, we have

COROLLARY. *For a given $n>0$, let X be a CW complex of* $\dim X\leqq n$ *with base point. Then we have the following isomorphism λ:*

$$\varepsilon(X\times K(\pi,\ n+1)) \longrightarrow (\varepsilon(X)\times \mathrm{Aut}(\pi))\otimes[K(\pi,\ n+1),\ G(X)]_0.$$

PROOF. Since $G_i(K(\pi,\ n+1))$ is weakly homotopy equivalent to $K(\pi,\ n+1)$ and $\dim X\leqq n$, we see that $[X,\ G_i(K(\pi,\ n+1))]_0$ is isomorphic to the group $[X,\ K(\pi,\ n+1)]_0$ which is trivial. By Theorem 7, this corollary follows immediately.

REMARK. In this paper we studied $G(X)$ for a connected locally finite CW complex X. However one can use arguments similar to ours within the category of compactly generated spaces and maps. Consequently our assumption of X being a locally finite CW complex can be relaxed, namely Propositions 2, 3, Theorems C, 4, 6 and 7 hold for connected CW complexes instead of connected locally finite CW complexes.

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