J. Math. Soc. Japan
Vol. 35, No. 1, 1983

On the local Hecke series of some classical groups over $\mathfrak{p}$-adic fields

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(Received Sept. 25, 1981)

Introduction.

The purpose of this paper is to prove the rationality for the local Hecke series of some classical groups over $\mathfrak{p}$-adic fields, and to calculate the degrees of its numerator and the denominator.

Let k be a $\mathfrak{p}$-adic field. Let K be either k itself, a quadratic extension of k, or the (unique) central division quaternion algebra over k. We denote by $x \mapsto \bar{x}$ $(x \in K)$ the canonical involution. Let ε be an element of the center of K such that $\varepsilon\bar{\varepsilon}=1$, V be an n-dimensional (right) vector space over K with a non-degenerate ε-hermitian form $\Phi(\ ,\ )$, and L be a maximal lattice in V (cf. § 1-1). Let G be the connected component (in the sense of an algebraic group over k) of

$$\tilde{G}=\{g \in GL(V)\,;\ \Phi(gx, gy)=\mu(g)\,\Phi(x, y) \text{ for all } x, y \in V,\ \mu(g) \in k^{\times}\}.$$

Let U be the subgroup of G consisting of all elements of G which leave L invariant. It is known that U is a maximal compact subgroup of G. For $m \geqq 0$, set

$$X(m)=\{g \in G\,;\ gL \subset L,\ \mathrm{ord}_{\mathfrak{p}}\mu(g)=fm\},$$

where $\mathrm{ord}_{\mathfrak{p}}(x)$ is the $\mathfrak{p}$-order of x for $x \in k$, and the positive integer f is determined by the condition $\mathrm{ord}_{\mathfrak{p}}\mu(G)=f\boldsymbol{Z}$. Let $T(m)$ be the characteristic function of $X(m)$ in G, considered as an element of the Hecke algebra of the group G with respect to U (see § 1-2). Then the (local) Hecke series of the group G with respect to U is by definition

$$Z_{(G,U)}(T)=\sum_{m=0}^{\infty} T(m)\,T^{m},$$

where T is an indeterminate.

Our main result is that the Hecke series $Z_{(G,U)}(T)$ is a rational function in T, and the degree of the numerator is $2^{\nu}-1$ or $2^{\nu}-2$, while that of the denominator is 2^{ν}, where ν is the Witt index of (V, Φ).

When Φ is an alternating form, the Hecke series has been studied in

G. Shimura [12], I. Satake [11], and A. N. Andrianov [1], [2], [3]. The present work owes much to the works of Satake and Andrianov.

The authors sincerely express their hearty thanks to Professor Y. Ihara for drawing their attention to Hecke series.

NOTATIONS AND CONVENTIONS. Throughout this paper, k denotes a $\mathfrak{p}$-adic number field, i. e. a complete discrete valuation field of characteristic 0 with finite residue field. The valuation-ring of k and its maximal ideal are denoted by $\mathfrak{o}$ and $\mathfrak{p}=(\pi)$, respectively. For $x\in k$ we denote by $\mathrm{ord}_{\mathfrak{p}}x$ the $\mathfrak{p}$-order of x. Let p be the number of elements of the residue field $\mathfrak{o}/\mathfrak{p}$. We denote by $|S|$ the cardinality of a finite set S. For a ring R with unit element, we denote by $R^{\times}$ the unit group of R. When $X_1, \cdots, X_n$ are variables, $R[X_1, \cdots, X_n]$ (resp. $R[[X_1, \cdots, X_n]]$) denotes the ring consisting of polynomials in $(X_1, \cdots, X_n)$ (resp. formal power series in $(X_1, \cdots, X_n)$). For any set S, $M_{n,m}(S)$ denotes the set of $n\times m$ matrices with entries in S. Put $M_{n,n}(S)=M_n(S)$. If S is a ring with unit element, $M_n(S)$ forms a ring, and we denote by 1_n the unity of $M_n(S)$. Put $GL_n(S)=M_n(S)^{\times}$. As usual, $\boldsymbol{Z}$, $\boldsymbol{Q}$, and $\boldsymbol{C}$ are the ring of rational integers, the field of rational numbers, and the complex number field, respectively. The δ_{ij} means the usual Kronecker's symbol.

§ 1. Preliminaries.

In this section, we shall recall the definitions and several properties of ε-hermitian forms and their Hecke algebras, following mainly Satake [11] and Hijikata [9].

1-1. Let K be either k itself, a quadratic extension field of k, or the (unique) central division quaternion algebra over k. We denote the maximal order in K and its unique two-sided maximal ideal by $\mathfrak{O}$ and $\mathfrak{P}=(\Pi)$, respectively. Let e be the ramification exponent, i. e., $\pi\mathfrak{O}=\mathfrak{P}^e$. We denote by $x\mapsto \bar{x}$ the canonical involution of K over k. Let ε be an element of the center of K, satisfying $\varepsilon\bar{\varepsilon}=1$. Let V be a right vector space over K of dimension n, and Φ be an ε-hermitian form on V, i. e., a k-bilinear mapping $V\times V\to K$ such that $\Phi(xa, yb)=\bar{a}\Phi(x, y)b$, $\Phi(y, x)=\varepsilon\overline{\Phi(x, y)}$, for any $x, y\in V$, $a, b\in K$.

REMARK 1. There are following five cases.

case(O) $K=k$, $\varepsilon=1$, (Φ is a symmetric form).

case(Sp) $K=k$, $\varepsilon=-1$, (Φ is an alternating form).

case(U) K is a quadratic extension field of k.

case(U^+) K is a division quaternion algebra over k, $\varepsilon=1$, (Φ is a quaternion hermitian form).

case(U^-) K is a division quaternion algebra over k, $\varepsilon=-1$, (Φ is a quaternion anti-hermitian form).

We assume that Φ is non-degenerate, i.e., $\Phi(x, V)=0$ implies $x=0$. Let ν be the Witt index of (V, Φ) (cf. [5]) and put $n=n_0+2\nu$. It is known that $0\leqq n_0\leqq 4$ in case (O), $n_0=0$ in case (Sp), $0\leqq n_0\leqq 2$ in case (U), $0\leqq n_0\leqq 1$ in case (U^+), and $0\leqq n_0\leqq 3$ in case (U^-). An $\mathfrak{O}$-submodule L of V is called a *lattice* if it is finitely generated and $L\otimes K=V$. When $\mathfrak{a}$ is a (fractional) two-sided ideal of $\mathfrak{O}$, a lattice L is called *$\mathfrak{a}$-integral* if it satisfies the following two conditions:

(1) $\Phi(x, y)\in\mathfrak{a}$, for any $(x, y)\in L\times L$,

(2) For any x in L, there exists an element ξ of $\mathfrak{a}$ such that $\Phi(x, x)=\xi+\varepsilon\bar{\xi}$.

REMARK 2. In case (Sp) and (U^+), (2) is a consequence of (1). In case (O) and (U^-), (1) is a consequence of (2) (cf. [9] or [16]).

A lattice L is called *maximal $\mathfrak{a}$-integral* if it is maximal in the class of $\mathfrak{a}$-integral lattices. For any maximal $\mathfrak{P}^l$-integral lattice L in V, there exists, by results of Eichler [7], Bruhat [6], Shimura [13], and Tsukamoto [16], a system of vectors $\{e_i, e_i'(1\leqq i\leqq\nu)\}$ such that

$$L=\sum_{i=1}^{\nu} e_i\mathfrak{O}+\sum_{i=1}^{\nu} e_i'\mathfrak{P}^l+L_0^{(l)}\,, \tag{1.1.1}$$

$$\Phi(e_i, e_j)=\Phi(e_i', e_j')=0,\ \Phi(e_i, e_j')=\delta_{ij} \qquad \text{for all } i, j,$$

where $L_0^{(l)}$ is the unique maximal $\mathfrak{P}^l$-integral lattice in $V_0=(\sum e_iK+\sum e_i'K)^{\perp}$ ($\perp$ denoting the orthogonal complement). For $\xi\in K$, put

$$T_\varepsilon(\xi)=\xi+\varepsilon\bar{\xi}\,. \tag{1.1.2}$$

Then

$$L_0^{(l)}=\{x\in V_0\,;\ \Phi(x, x)\in T_\varepsilon(\mathfrak{P}^l)\}. \tag{1.1.3}$$

We now fix any maximal $\mathfrak{P}^l$-integral lattice L in V, and a system of vectors $\{e_i, e_i'\ (1\leqq i\leqq\nu)\}$ satisfying (1.1.1) once for all. We take furthermore an $\mathfrak{O}$-basis $(f_1, \cdots, f_{n_0})$ of $L_0^{(l)}$, and understand that a K-linear transformation of V is represented by a matrix with respect to the basis $(e_1, \cdots, e_\nu, f_1, \cdots, f_{n_0}, e_1', \cdots, e_\nu')$. For any matrix $g=(g_{ij})$ with entries in K, we denote by g^* a matrix whose (i, j)-entry is $\overline{g_{ji}}$. Then the ε-hermitian form Φ is written as follows:

$$\Phi(x, y)=x^*Ry \qquad \text{for any} \quad x, y\in V=K^n=M_{n,1}(K)\,, \tag{1.1.4}$$

where $R=\begin{pmatrix} & & 1_\nu \\ & R_0 & \\ \varepsilon 1_\nu & & \end{pmatrix}$, $R_0^*=\varepsilon R_0\in M_{n_0}(K)$. Set

$$\tilde{G}=\{g\in GL_n(K)\,;\ g^*Rg=\mu(g)R \quad \text{with} \quad \mu(g)\in k^\times\}, \tag{1.1.5}$$

$$\tilde{G}_0=\{g\in GL_{n_0}(K)\,;\ g^*R_0g=\mu_0(g)R_0 \quad \text{with} \quad \mu_0(g)\in k^\times\}, \tag{1.1.6}$$

$$G\text{(resp. } G_0)=\text{the connected component of } \tilde{G}\text{(resp. } \tilde{G}_0) \text{ as algebraic group over } k. \tag{1.1.7}$$

Define a positive integer f by

$$\mathrm{ord}_{\mathfrak{p}}\mu(G)=f\boldsymbol{Z}. \tag{1.1.8}$$

Consider the following subgroups of G;

$$U=\{u\in G;\ uL=L\} \text{ (an open compact subgroup of } G), \tag{1.1.9}$$

$$H=\left\{\begin{pmatrix} \xi_0(X^*)^{-1} & & \\ & h_0 & \\ & & X \end{pmatrix};\ \begin{array}{l} \xi_0\in k^\times,\ X=\mathrm{diag.}(x_1, \cdots, x_\nu) \\ x_i\in K^\times,\ h_0\in G_0,\ \mu_0(h_0)=\xi_0 \end{array}\right\}, \tag{1.1.10}$$

$$N=\left\{\begin{pmatrix} (X^*)^{-1} & * & * \\ 0 & 1_{n_0} & * \\ 0 & 0 & X \end{pmatrix}\in G;\ X=\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\in M_\nu(K)\right\}. \tag{1.1.11}$$

Then

$$G=UHN=UHU \text{ (cf. [11; § 9-2])}. \tag{1.1.12}$$

REMARK 3. Except for case (O) with n_0=even and case (U^-), $\tilde{G}$ itself is connected, i.e., $\tilde{G}=G$. For case (O) with n_0=even (resp. case (U^-)), G is defined by the additional condition $\det(g)=\mu(g)^{n/2}$ (resp. $\tilde{N}(g)=\mu(g)^n$, $\tilde{N}$ denoting the reduced norm of $M_n(K)/k$).

1-2. Let G be as in § 1-1, and $L(G)$ (resp. $L_{\boldsymbol{Z}}(G)$) be the set of all compactly supported continuous functions on G with values in $\boldsymbol{C}$ (resp. $\boldsymbol{Z}$). Set

$$L(G, U)=\{f\in L(G);\ f(ugu')=f(g) \quad \text{for all} \quad u, u'\in U,\ g\in G\}, \tag{1.2.1}$$
$$L_{\boldsymbol{Z}}(G, U)=L_{\boldsymbol{Z}}(G)\cap L(G, U).$$

For $f_1, f_2\in L(G, U)$, define their product by the convolution

$$(f_1*f_2)(g)=\int_G f_1(gg_1^{-1})f_2(g_1)\,dg_1 \qquad (g\in G), \tag{1.2.2}$$

where dg_1 is the bi-invariant Haar measure on G, normalized by the condition that the total volume of U equals 1. The multiplication (1.2.2) gives the $\boldsymbol{C}$-module $L(G, U)$ (resp. the $\boldsymbol{Z}$-module $L_{\boldsymbol{Z}}(G, U)$) the structure of a $\boldsymbol{C}$-algebra (resp. $\boldsymbol{Z}$-algebra). The algebras $L(G, U)$ and $L_{\boldsymbol{Z}}(G, U)$ are called the *Hecke algebras* of the group G with respect to U. They are commutative, and their structures have been determined by Satake [11]. We quote his results. First, some notation. For $\boldsymbol{r}=(r_1, \cdots, r_\nu; r_0)\in\boldsymbol{Z}^\nu\times\boldsymbol{Z}$, put

$$\boldsymbol{\pi}^{\boldsymbol{r}}=\begin{pmatrix}\pi^{r_0}(D^*)^{-1} & 0\\ 0 & D\end{pmatrix} \quad \text{if} \quad n_0=0,$$

$$=\begin{pmatrix}\mu_0(\varpi)^{r_0}(D^*)^{-1} & 0 & 0\\ 0 & \varpi^{r_0} & 0\\ 0 & 0 & D\end{pmatrix} \quad \text{if} \quad n_0\geqq 1, \tag{1.2.3}$$

where $D=D(\boldsymbol{r})=\mathrm{diag.}(\Pi^{r_1}, \cdots, \Pi^{r_\nu})$, and ϖ denotes an element of G_0 such that $\mathrm{ord}_{\mathfrak{p}}\mu_0(\varpi)=f$. Set

$$\Lambda=\{(r_1, \cdots, r_\nu; r_0)\in \boldsymbol{Z}^\nu\times\boldsymbol{Z}; r_1\leqq\cdots\leqq r_\nu\leqq fer_0/2\}$$
$$\text{unless in case (O) with } n_0=0,$$
$$=\{(r_1, \cdots, r_\nu; r_0)\in \boldsymbol{Z}^\nu\times\boldsymbol{Z}; r_1\leqq\cdots\leqq r_{\nu-1}\leqq \mathrm{Min}(r_\nu, r_0-r_\nu)\}$$
$$\text{in case (O) with } n_0=0. \tag{1.2.4}$$

Define $\boldsymbol{r}^{(i)}$ $(0\leqq i\leqq\nu)$ as follows;

$$\boldsymbol{r}^{(i)}=\begin{cases}(\underbrace{0, \cdots, 0}_{i}, 1, \cdots, 1; 2/fe) & \text{if } fe=1 \text{ or } 2,\\ (\underbrace{1, \cdots, 1}_{i}, 2, \cdots, 2; 1) & \text{if } fe=4,\end{cases} \qquad (0\leqq i\leqq \nu-2)$$

$$\boldsymbol{r}^{(\nu-1)}=\begin{cases}(0, \cdots, 0, 1; 2) & \text{if } fe=1 \text{ and not case (O) with } n_0=0,\\ (0, \cdots, 0, 1; 1) & \text{if } fe=2 \text{ or case (O) with } n_0=0,\\ (1, \cdots, 1, 2; 1) & \text{if } fe=4,\end{cases}$$

$$\boldsymbol{r}^{(\nu)}=\begin{cases}(0, \cdots, 0; 1) & \text{if } fe=1 \text{ or } 2,\\ (1, \cdots, 1; 1) & \text{if } fe=4.\end{cases} \tag{1.2.5}$$

It is clear that in case (O) with $n_0=0$ and $\nu=1$, Λ is generated by $\pm\boldsymbol{r}^{(0)}$, $\pm\boldsymbol{r}^{(1)}$ as semi-group, and that in the other cases Λ is generated by $\pm\boldsymbol{r}^{(0)}$, $\boldsymbol{r}^{(1)}, \cdots, \boldsymbol{r}^{(\nu)}$ as semi-group. Let $c^{(i)}$ $(0\leqq i\leqq\nu)$ be the characteristic function of the double coset $U\boldsymbol{\pi}^{\boldsymbol{r}^{(i)}}U$. Clearly, $c^{(i)}\in L_Z(G, U)\subset L(G, U)$. Then

$$G=\coprod_{\boldsymbol{r}\in\Lambda} U\boldsymbol{\pi}^{\boldsymbol{r}}U \quad \text{(disjoint union)}, \tag{1.2.6}$$

$$L_Z(G, U)=\boldsymbol{Z}[c^{(0)}, (c^{(0)})^{-1}, c^{(1)}, \cdots, c^{(\nu)}], \tag{1.2.7}$$

$$L(G, U)=\boldsymbol{C}[c^{(0)}, (c^{(0)})^{-1}, c^{(1)}, \cdots, c^{(\nu)}], \tag{1.2.8}$$

and $c^{(i)}(0\leqq i\leqq\nu)$ are algebraically independent over $\boldsymbol{C}$ (cf. [**11**; Theorem 1 and §9]). Especially, (1.2.6) implies that U is a maximal compact subgroup of G.

We quote Satake's result on the Fourier transform of $L(G, U)$ in the same language as in Andrianov [4; Proposition 3.1]. Let $X_0, \cdots, X_\nu$ be algebraically independent over $\boldsymbol{C}$ and $\boldsymbol{C}[X_0^{\pm}, \cdots, X_\nu^{\pm}]$ be an algebra generated by $X_0, X_0^{-1}, \cdots, X_\nu, X_\nu^{-1}$. From (1.1.12), for each element $g \in G$, the double coset UgU can be decomposed into left cosets in the form

$$UgU = \coprod_{i \in I} U\boldsymbol{\pi}^{\boldsymbol{m}_i} n_i, \tag{1.2.9}$$

where $\boldsymbol{m}_i = (m_{i1}, \cdots, m_{i\nu}; m_{i0}) \in \boldsymbol{Z}^\nu \times \boldsymbol{Z}$, $n_i \in N$, and I is a finite index set. The set $\{\boldsymbol{m}_i ; i \in I\}$ is uniquely determined by UgU. We denote by $C(g)$ the characteristic function of UgU in G. Put

$$\Psi(C(g)) = \sum_{i \in I} X_0^{m_{i0}} \prod_{j=1}^{\nu} (X_j/q^j)^{m_{ij}}, \tag{1.2.10}$$

where q is the number of elements of $\mathfrak{O}/\mathfrak{P}$. Since $\{C(g) ; g \in G\}$ spans $L(G, U)$ over $\boldsymbol{C}$, we can extend to a $\boldsymbol{C}$-linear mapping from $L(G, U)$ into $\boldsymbol{C}[X_0^{\pm}, \cdots, X_\nu^{\pm}]$. By using the left coset decomposition (1.2.9), one checks that Ψ is a $\boldsymbol{C}$-algebra homomorphism. Let S_ν denote the group of all permutations of the variables $X_1, \cdots, X_\nu$, and $w^{(i)}$ $(1 \leqq i \leqq \nu)$ denote the transformation;

$$\begin{aligned} &X_0 \longmapsto X_0(p^A X_i)^{fe}, \quad X_i \longmapsto p^{-2A} X_i^{-1}, \\ &X_j \longmapsto X_j \quad (j \neq 0, i) \end{aligned} \tag{1.2.11}$$

where $p^A = q^{n_0/2+\tau-1}$ $(\tau = \dim_k(\mathrm{Ker}\, T_\varepsilon)/\dim_k(K))$, i.e.,

$$A = \begin{cases} n_0/2-1 & \text{in case (O)} \\ 0 & \text{in case (Sp)} \\ (n_0-1)/e & \text{in case (U)} \\ n_0-1/2 & \text{in case (U}^+) \\ n_0-3/2 & \text{in case (U}^-). \end{cases} \tag{1.2.12}$$

Let W denote the group of automorphisms of the algebra $\boldsymbol{C}[X_0^{\pm}, \cdots, X_\nu^{\pm}]$ generated by S_ν and $w^{(i)}$ $(1 \leqq i \leqq \nu)$ (resp. S_ν and $w^{(i)} w^{(j)}$ $(1 \leqq i < j \leqq \nu)$) except for case (O) with $n_0 = 0$ (resp. in case (O) with $n_0 = 0$).

THEOREM (S) (Satake [11; Theorem 3]). *The above mapping defines a $\boldsymbol{C}$-algebra isomorphism between $L(G, U)$ and the W-invariant subalgebra $\boldsymbol{C}[X_0^{\pm}, \cdots, X_\nu^{\pm}]^W$.*

§ 2. Main results.

In this section, we shall define the Hecke series (§ 2-1) and state the main results (§ 2-2). We keep all notations in § 1.

2-1. For $m\geqq 0$, set

$$X(m)=\{g\in G\,;\ gL\subset L,\ \mathrm{ord}_{\mathfrak{p}}\mu(g)=fm\}. \tag{2.1.1}$$

(see (1.1.5), (1.1.8))

Let $T(m)(\in L_{\mathbf{Z}}(G, U))$ be the characteristic function of $X(m)$ in G.

DEFINITION 1. The Hecke series (resp. the index function series) of the group G with respect to U is an element of $L_{\mathbf{Z}}(G, U)[[T]]$ (resp. $\boldsymbol{Z}[[T]]$) given by (2.1.2) (resp. (2.1.3)).

$$Z_{(G,U)}(T)=\sum_{m=0}^{\infty} T(m)\, T^{m}\,, \tag{2.1.2}$$

$$z_{(G,U)}(T)=\sum_{m=0}^{\infty} |U\backslash X(m)|\, T^{m}\,. \tag{2.1.3}$$

The series (2.1.2) is of interest in connection with the theory of Hecke operators for automorphic forms on classical groups. We shall show that the series (2.1.2) and (2.1.3) are rational functions of T. Set

$$\zeta_{(G,U)}(T)=\sum_{m=0}^{\infty} \Psi(T(m))\ T^{m}\,. \tag{2.1.4}$$

As in Andrianov [2], [3], in place of $Z_{(G,U)}(T)$, we shall treat chiefly $\zeta_{(G,U)}(T)$, which is an element of $\boldsymbol{C}[X_0^{\pm}, \cdots, X_\nu^{\pm}]^W[[T]]$. When $\nu=0$, we obtain immediately;

$$Z_{(G,U)}(T)=1/(1-C(\varpi)T)\,, \tag{2.1.5}$$

and

$$z_{(G,U)}(T)=1/(1-T)\,. \tag{2.1.6}$$

We assume $\nu\geqq 1$, hereafter.

2-2. Let us state the theorems. Assume $\nu\geqq 1$.

THEOREM 1. *The following formal identity holds:*

$$\zeta_{(G,U)}(T)=\frac{P(X_0, \cdots, X_\nu\,;\ T)}{Q(X_0, \cdots, X_\nu\,;\ T)}\,, \tag{2.2.1}$$

where $Q(X_0, \cdots, X_\nu\,;\ T)=(1-X_0T)\prod_{r=1}^{\nu}\prod_{1\leqq i_1<\cdots<i_r\leqq\nu}(1-(p^{Ar}X_{i_1}\cdots X_{i_r})^{fe}X_0T)\}$ *and* $P(X_0, \cdots, X_\nu\,;\ T)\in\boldsymbol{Q}[X_0, \cdots, X_\nu\,;\ T]\cap\boldsymbol{C}[X_0^{\pm}, \cdots, X_\nu^{\pm}]^W[T]$, *satisfying*

(i) $P(X_0, \cdots, X_\nu\,;\ 0)=1$,

(ii) *the degree in* T *of* $P(X_0, \cdots, X_\nu\,;\ T)$ *is* $2^\nu-2$ *if* $fe=1$, *and* $2^\nu-1$ *if* $fe=2$ *or* 4. *Moreover if* $fe=1$ *or* 2, *the term of highest degree in* T *of* P *is*

$$(-1)^{\nu-1}q^{-\nu(\nu-1)/2}p^{A\nu(2^{\nu-1}-1)}(X_1\cdots X_\nu)^{2^{\nu-1}-1}(X_0T)^{2^\nu-2} \qquad \text{if } fe=1\,,$$

$$(-1)^{\nu-1}q^{-\nu(\nu-1)/2}p^{A\nu(2^{\nu}-1)+B\nu}(X_1\cdots X_\nu)^{2^{\nu}-1}(X_0T)^{2^\nu-1} \qquad \text{if } fe=2\,,$$

where A and B are constants defined at (1.2.12) *and* (3.1.18), *respectively. Finally,* $P(X_0, \cdots, X_\nu; T)$ *and* $Q(X_0, \cdots, X_\nu; T)$ *have no common divisor in* $\boldsymbol{C}[X_0, \cdots, X_\nu; T]$.

REMARK 4. Except for case (O) with $n_0=0$, $Q(X_0, \cdots, X_\nu; T)$ can be expressed as $Q(X_0, \cdots, X_\nu; T)=\prod_{w\in W/S_\nu}(1-w(X_0)T)$. Thus $Q(X_0, \cdots, X_\nu; T)$ is the exact denominator in these cases. In case (O) with $n_0=0$, it is also clear, because

$$\zeta_{(G,U)}(X_0, \cdots, X_\nu; T)=\zeta_{(G',U')}(X_0, p^{-1}X_1, \cdots, p^{-1}X_\nu; T)$$

where $G'=\mathrm{GSp}(\nu, k)$ is the symplectic similitudes group of genus ν (matrix size is 2ν), and $U'=G'\cap GL_{2\nu}(\mathfrak{o})$.

The following corollary is easily obtained by Theorem (S) and Theorem 1.

COROLLARY OF THEOREM 1. *The Hecke series $Z_{(G,U)}(T)$ of the group G with respect to U is a rational function in T with coefficients in $L(G, U)$. The degree of the numerator is $2^\nu-2$ if $fe=1$ and $2^\nu-1$ otherwise. And the degree of the denominator is 2^ν.*

REMARK 5. In case (Sp) the above results have been obtained by Hecke [8], G. Shimura [**12**] ($\nu=2$ with some conjectures on the degrees of the numerator and the denominator for the general n), I. Satake [**11**] (explicit conjecture on the denominator of $\zeta_{(G,U)}(T)$), and A. N. Andrianov [**1**], [**2**], [**3**] ($\nu=3$ and proved Theorem 1). In case (U^+), for $\nu=1$, the result is given in Y. Ihara [**10**] and by Mr. Yoshihiro Furukawa.

By the definition of Ψ((1.2.10)), specializing $X_i=q^i$ $(0\leqq i\leqq\nu)$ we obtain the index function series $z_{(G,U)}(T)$ as a rational function in T. Furthermore we shall prove the following theorem, which gives us more handy expression for $z_{(G,U)}(T)$.

THEOREM 2. *The following identity holds:*

$$z_{(G,U)}T=p(T)/q(T)\,, \tag{2.2.2}$$

where $q(T)=\prod_{r=0}^{\nu}(1-(p^A q^{\nu-(r-1)/2})^{fer}T)$ *and* $p(T)\in\boldsymbol{Z}[T]$ *satisfying* $p(0)=1$ *and the degree of* $p(T)$ *is at most* ν.

REMARK 6. In case (Sp) the above theorem is proved in [**11**; Appendix I-4].

§3. Proofs.

First, we describe an explicit decomposition of the set $X(m)$ into left U-cosets (§3-1). In §3-2 we calculate some partial sum of the multiple Hecke series of $GL_\nu(K)$. By using the results of §3-1 and §3-2, the theorems will be proved.

3-1. Let the notations be as in §1 and §2. For $\nu\geqq1$, set

$$I=\{g\in G\,;\ gL\subset L\},\tag{3.1.1}$$

$$I_\nu=M_\nu(\mathfrak{O})\cap GL_\nu(K),\tag{3.1.2}$$

$$U_\nu=GL_\nu(\mathfrak{O}),\tag{3.1.3}$$

$$I_0=\begin{cases}\mathfrak{o}-\{0\} & \text{if}\quad n_0=0,\\ G_0\cap M_{n_0}(\mathfrak{O}) & \text{if}\quad n_0\geqq 1,\end{cases}\tag{3.1.4}$$

$$U_0=\begin{cases}\mathfrak{o}^\times & \text{if}\quad n_0=0,\\ G_0\cap GL_{n_0}(\mathfrak{O}) & \text{if}\quad n_0\geqq 1.\end{cases}\tag{3.1.5}$$

If $n_0=0$ we put $\mu_0(h)=h$ for any $h\in I_0$, and $\varpi=\mu_0(\varpi)=\pi$. For $C\in I_\nu$ and $h\in I_0$ such that $\mu_0(h)(C^*)^{-1}\in I_\nu$, set

$$F(C,\ h)=\{(Y,\ Z)\in M_\nu(\mathfrak{P}^{-l})\times M_{n_0,\nu}(\mathfrak{P}^{-l}):$$
$$h^*R_0ZC^{-1}\in M_{n_0,\nu}(\mathfrak{O}),\ Y^*C+\varepsilon C^*Y+Z^*R_0Z=0\}.\tag{3.1.6}$$

We define a mapping $\varphi_{(C,h)}$ from $F(C,\ h)$ into I as follows;

$$\varphi_{(C,h)}(Y,\ Z)=\begin{pmatrix}\mu_0(h)(C^*)^{-1} & -(h^*R_0ZC^{-1})^* & Y\\ 0 & h\cdot 1_{n_0} & Z\\ 0 & 0 & C\end{pmatrix}.\tag{3.1.7}$$

We introduce an equivalence relation on the set $F(C,\ h)$ by

$$b_1\sim b_2\quad(b_1,\ b_2\in F(C,\ h)),$$

when there exists some element b_0 in $F(1_\nu,\ 1)$ satisfying

$$\varphi_{(C,h)}(b_2)=\varphi_{(1_\nu,1)}(b_0)\varphi_{(C,h)}(b_1).\tag{3.1.8}$$

Let $\bar{F}(C,\ h)$ be a set of representatives of the equivalence classes of $F(C,\ h)$. For $\boldsymbol{r}=(r_1,\cdots,r_\nu)\in \boldsymbol{Z}^\nu$, $R(\boldsymbol{r})$ denotes a set of representatives of $U_\nu\backslash U_\nu\begin{pmatrix}\Pi^{r_1} & & 0\\ & \ddots & \\ 0 & & \Pi^{r_\nu}\end{pmatrix}U_\nu$ such that each element of $R(\boldsymbol{r})$ is of the form $\begin{pmatrix}\Pi^{m_1} & & *\\ & \ddots & \\ 0 & & \Pi^{m_\nu}\end{pmatrix}$.

LEMMA 1. *For each $m\geqq 0$, let $X(m)\subset G$ be the subset defined at* (2.1.1); *then we have the following disjoint union decomposition of $X(m)$;*

$$X(m)=\coprod_{0\leq r_1\leq\cdots\leq r_\nu\leq fem}\ \coprod_{C\in R(\boldsymbol{r})}\ \coprod_{b\in\bar{F}(C,\varpi^m)} U\varphi_{(C,\varpi^m)}(b),$$

where $\boldsymbol{r}=(r_1,\cdots,r_\nu)$, and ϖ is defined at (1.2.3) (*if $n_0=0$, put $\varpi=\pi$*).

LEMMA 2. *For $u_1,\ u_2\in U_\nu$ and $\varepsilon_1\in U_0$, the mapping*

$$(Y, Z) \longmapsto ((u_1^*)^{-1} Y u_2, Z u_2)$$

defines a bijection between $F(C, h)$ *and* $F(u_1 C u_2, h\varepsilon_1)$, *compatible with the equivalence relation defined at* (3.1.8).

The above two lemmata follow easily from the definitions of $X(m)$ and $\varphi_{(C,h)}$ (see (2.1.1) and (3.1.7)). Especially, $|\bar{F}(C, \varpi^m)|$ depends only on $\boldsymbol{r}$ and m ($C \in R(\boldsymbol{r})$). We shall describe a set of representatives of the equivalence classes of $F(C, h)$ for $C = \mathrm{diag.}(\Pi^{r_1}, \cdots, \Pi^{r_\nu})$ and $h = \varpi^m$.

LEMMA 3. *For* $0 \leqq r_1 \leqq \cdots \leqq r_\nu \leqq fem$, *put* $C = \mathrm{diag.}(\Pi^{r_1}, \cdots, \Pi^{r_\nu})$ *and* $h = \varpi^m$. *Take an element* $Z = (z_1, \cdots, z_\nu)$ *in* $M_{n_0, \nu}(K)$. *Then a necessary and sufficient condition for the existence of* Y *such that* $(Y, Z) \in F(C, h)$ *is* $z_i \in L_0^{(r_i - l)}$ *for all* i. (See (1.1.3) for the definition of $L_0^{(r)}$.)

PROOF. If $(Y, Z) \in F(C, h)$, we have $Y^* C + \varepsilon C^* Y + Z^* R_0 Z = 0$. Denoting by y_{ij} the (i, j)-entry of Y, we have

$$\bar{y}_{ji} \Pi^{r_j} + \varepsilon \bar{\Pi}^{r_i} y_{ij} + z_i^* R_0 z_j = 0 .$$

Especially, $\Phi(z_i, z_i) = T_\varepsilon(-\bar{y}_{ii} \Pi^{r_i})$. This means that $z_i \in L_0^{(r_i - l)}$. Conversely, suppose that $z_i \in L_0^{(r_i - l)}$ for all i. We remark that, for any $t \in \boldsymbol{Z}$ and $\sigma \in G_0$,

$$L_0^{(t)} \Pi = L_0^{(t+2)} , \tag{3.1.9}$$

and

$$\sigma L_0^{(t)} = L_0^{(t')} , \ t' = t + e \cdot \mathrm{ord}_{\mathfrak{p}} \mu_0(\sigma) . \tag{3.1.10}$$

As $r_i \geqq 0$, $z_i \in L_0^{(r_i - l)} \subset L_0^{(l)} \Pi^{-l} = M_{n_0, 1}(\mathfrak{P}^{-l})$. From the definition of $\mathfrak{P}^l$-integrality, we have $R_0 \in M_{n_0}(\mathfrak{P}^l)$. By using (3.1.9) and (3.1.10), we obtain $h^* R_0 z_i \Pi^{-r_i} = \mu_0(h) R_0 h^{-1} z_i \Pi^{-r_i} \in M_{n_0}(\mathfrak{O})$. Define $Y = (y_{ij}) \in M_\nu(K)$ as follows. For each i ($1 \leqq i \leqq \nu$) take $y_{ii} \in \mathfrak{P}^{-l}$ such that $\Phi(z_i, z_i) = T_\varepsilon(-\bar{y}_{ii} \Pi^{r_i})$. Take any element in $\mathfrak{P}^{-l}$ as y_{ij} for $i > j$, and set $y_{ji} = -\bar{\varepsilon} \bar{\Pi}^{-r_j} (\bar{y}_{ij} \Pi^{r_j} + z_j^* R_0 z_i)$. Then it is obvious that (Y, Z) is in $F(C, h)$. q. e. d.

For $Z \in M_{n_0, \nu}(K)$ satisfying the condition in Lemma 3, we choose an element Y_Z in $M_\nu(\mathfrak{P}^{-l})$ such that $(Y_Z, Z) \in F(C, h)$. For $m \in \boldsymbol{Z}$, set

$$L'^{(m)} = \mathfrak{P}^m \cap \mathrm{Ker}\, T_\varepsilon . \tag{3.1.11}$$

$L'^{(m)}$ is a (left) $\mathfrak{o}$-module and has the property

$$\pi L'^{(m)} = L'^{(e+m)} . \tag{3.1.12}$$

By (3.1.9) and (3.1.12), we can regard $L_0^{(m)} / L_0^{(m+1)}$ (resp. $L'^{(m)} / L'^{(m+1)}$) as a vector space over $\mathfrak{O}/\mathfrak{P}$ (resp. $\mathfrak{o}/\mathfrak{p}$). Put

$$\partial = \dim_{\mathfrak{O}/\mathfrak{P}} L_0^{(l)} / L_0^{(l+1)} , \quad \partial' = \dim_{\mathfrak{o}/\mathfrak{p}} L'^{(l)} / L'^{(l+1)} , \tag{3.1.13}$$

where l was determined by the fixed maximal $\mathfrak{P}^l$-integral lattice L. For $\boldsymbol{r} = (r_1, \cdots, r_\nu) \in \boldsymbol{Z}^\nu$ satisfying $0 \leqq r_1 \leqq \cdots \leqq r_\nu$, set

$$Y(\boldsymbol{r})=\left\{Y=(y_{ij})\in M_\nu(K)\,;\ \begin{array}{l}\bar{y}_{ii}\Pi^{r_i}\in L'^{(r_i-l)}/L'^{(2r_i-l)},\ \text{for}\ 1\leqq i\leqq\nu\\ y_{ij}\in\mathfrak{P}^{-l}/\mathfrak{P}^{r_j-l}\ \text{for}\ i>j\\ \bar{y}_{ji}\Pi^{r_j}+\varepsilon\bar{\Pi}^{r_i}y_{ij}=0\end{array}\right\}, \tag{3.1.14}$$

$$Z(\boldsymbol{r})=\{Z=(z_1,\cdots,z_\nu)\in M_{n_0,\nu}(K)\,;\ z_i\in L_0^{(r_i-l)}/L_0^{(2r_i-l)}\ \text{for all}\ \ i\}. \tag{3.1.15}$$

LEMMA 4. *For* $0\leqq r_1\leqq\cdots\leqq r_\nu\leqq fem$, *put* $C=\mathrm{diag.}(\Pi^{r_1},\cdots,\Pi^{r_\nu})$ *and* $h=\varpi^m$. *Then the set* $\{(Y_Z+Y,Z)\,;\ Y\in Y(\boldsymbol{r}),\ Z\in Z(\boldsymbol{r})\}$ *is a system of representatives of the equivalence classes of* $F(C,h)$.

PROOF. By Lemma 3,

$$F(C,h)=\left\{(Y_Z+Y,Z)\,;\ \begin{array}{l}Y=(y_{ij}),\ Z=(z_1,\cdots,z_\nu)\\ z_i\in L_0^{(r_i-l)},\ \bar{y}_{ii}\Pi^{r_i}\in L'^{(r_i-l)}\,(1\leqq i\leqq\nu)\\ y_{ij}\in\mathfrak{P}^{-l}\ \text{for}\ i>j,\ y_{ji}=-\varepsilon\bar{\Pi}^{-r_j}\bar{y}_{ij}\Pi^{r_i}\end{array}\right\}. \tag{3.1.16}$$

If (Y_Z+Y,Z) and $(Y_{Z'}+Y',Z')$ are equivalent, $(Z-Z')C^{-1}$ is in $M_{n_0,\nu}(\mathfrak{P}^{-l})$. Hence in this case we may assume that $Z=Z'\in Z(\boldsymbol{r})$. From the definition of equivalence and (3.1.14) we obtain the required result easily. q. e. d.

For $\boldsymbol{r}=(r_1,\cdots,r_\nu)\in\boldsymbol{Z}^\nu$, put

$$a_{\boldsymbol{r}}(X)=a_{\boldsymbol{r}}(X_1,\cdots,X_\nu)=\sum_{R(\boldsymbol{r})\ni g_i=\begin{pmatrix}\Pi^{m_{i1}} & & *\\ & \ddots & \\ 0 & & \Pi^{m_{i\nu}}\end{pmatrix}}\prod_{j=1}^{\nu}(X_j/q^j)^{m_{ij}}. \tag{3.1.17}$$

PROPOSITION 1.

$$\Psi(T(m))=\sum_{0\leq r_1\leq\cdots\leq r_\nu\leq fem}q^{\rho(\boldsymbol{r})}p^{A\sigma(\boldsymbol{r})+B\varepsilon(\boldsymbol{r})}a_{\boldsymbol{r}}(X)X_0^m,$$

where for $\boldsymbol{r}=(r_1,\cdots,r_\nu)$ *we put* $\rho(\boldsymbol{r})=\sum_{i=1}^{\nu}(\nu-i+1)r_i$, $\sigma(\boldsymbol{r})=\sum_{i=1}^{\nu}r_i$, *and* $\varepsilon(\boldsymbol{r})=|\{1\leqq i\leqq\nu\,;\ r_i=\mathrm{odd}\}|$, *and the half integer* B *is defined by*

$$p^B=q^{-\partial+n_0/2}p^{-\partial'+\kappa/e},\quad \kappa=\dim_k(\mathrm{Ker}\,T_\varepsilon) \tag{3.1.18}$$

(see (3.1.13) for the definitions of ∂ and ∂').

PROOF. By using Lemmas 1-4, we obtain

$$\Psi(T(m))=\sum_{0\leq r_1\leq\cdots\leq r_\nu\leq fem}|Y(\boldsymbol{r})|\cdot|Z(\boldsymbol{r})|\,a_{\boldsymbol{r}}(X)X_0^m.$$

From (3.1.14) and (3.1.15), $|Y(\boldsymbol{r})|$ and $|Z(\boldsymbol{r})|$ are easily calculated. q. e. d.

3-2. In this subsection, denote by N (resp. N') the set $\{1,\cdots,\nu\}$ (resp. $\{1,\cdots,\nu-1\}$). We shall define the multiple Hecke series of $GL_\nu(K)$, which was introduced by Andrianov in [2], [3], as a generalization of the Hecke series of $GL_\nu(K)$ (cf. Tamagawa [14]).

DEFINITION 2. The multiple Hecke series of $GL_\nu(K)$ (resp. the multiple index function series of $GL_\nu(K)$) is an element of $\boldsymbol{C}[X_1, \cdots, X_\nu][[t_1, \cdots, t_\nu]]$ (resp. $\boldsymbol{C}[[t_1, \cdots, t_\nu]]$) given by (3.2.1) (resp. (3.2.2)).

$$F(t; X)=F(t_1, \cdots, t_\nu; X_1, \cdots, X_\nu) \sum_{0\leq r_1\leq\cdots\leq r_\nu} t_1^{r_1}\cdots t_\nu^{r_\nu} a_r(X), \tag{3.2.1}$$

$$f(t)=f(t_1, \cdots, t_\nu)=\sum_{0\leq r_1\leq\cdots\leq r_\nu} t_1^{r_1}\cdots t_\nu^{r_\nu} |R(r)|. \tag{3.2.2}$$

THEOREM (A) (Andrianov [3; Theorem 3]). *For* $1\leqq r\leqq\nu$, *put* $v_r=\prod_{i=1}^{r} t_{\nu-r+i}$. *Then* $F(t; X)$ *is an element of* $\boldsymbol{C}[X][[t]]\cap\boldsymbol{C}[t][[X]]$, *and the following formal identity holds;*

$$F(t_1, \cdots, t_\nu; X_1, \cdots, X_\nu)=\frac{P^{(0)}(t_1, \cdots, t_\nu; X_1, \cdots, X_\nu)}{Q^{(0)}(t_1, \cdots, t_\nu; X_1, \cdots, X_\nu)},$$

where $Q^{(0)}(t; X)=\prod_{r=1}^{\nu}\prod_{1\leq i_1<\cdots<i_r\leq\nu}(1-q^{-r(r+1)/2}X_{i_1}\cdots X_{i_r}v_r)$ *and* $P^{(0)}(t_1, \cdots, t_\nu; X_1, \cdots, X_\nu)\in\boldsymbol{C}[v_1, \cdots, v_{\nu-1}; X_1, \cdots, X_\nu]$ *such that*

(i) $P^{(0)}(t_1, \cdots, t_{\nu-1}, 0; X_1, \cdots, X_\nu)=1$,

(ii) *the term of highest degree in* $(v_1, \cdots, v_{\nu-1})$ *is*

$$M_0(t; X)=(-1)^{\nu-1}q^{-\nu(\nu+3)2^{\nu-3}+\nu}(X_1\cdots X_\nu)^{2^{\nu-1}-1}\prod_{r=1}^{\nu-1} v_r^{\binom{\nu}{r}}.$$

$\left(\binom{\nu}{r}=\frac{\nu!}{r!(\nu-r)!}\right.$, *the binomial coefficient.*$\left.\right)$

EXAMPLES OF $P^{(0)}$.

$$P^{(0)}(t_1; X_1)=1,$$

$$P^{(0)}(t_1, t_2; X_1, X_2)=1-q^{-3}X_1X_2t_2^2,$$

$$\begin{aligned}P^{(0)}(t_1, t_2, t_3; X_1, X_2, X_3)=1&-q^{-3}\{s_2v_1^2+(q^{-1}+q^{-2}+q^{-3})s_3v_1v_2+q^{-4}s_1s_2v_2^2\}\\&+q^{-4}(1+q^{-1})s_3\{v_1^3+q^{-2}s_1v_1^2v_2+q^{-4}s_2v_1v_2^2+q^{-6}s_3v_2^3\}\\&-q^{-8}s_3v_1v_2\{s_2v_1^2+(q^{-1}+q^{-2}+q^{-3})s_3v_1v_2+q^{-4}s_1s_2v_1^2\}+q^{-15}s_3^2v_1^2v_2^3,\end{aligned}$$

where $s_1=X_1+X_2+X_3$, $s_2=X_1X_2+X_2X_3+X_3X_1$, $s_3=X_1X_2X_3$, $v_1=t_3$, $v_2=t_2t_3$, and $v_3=t_1t_2t_3$.

For any subset $I=\{i_1, \cdots, i_r\}\subset N'$ $(1\leqq i_1<\cdots<i_r\leqq\nu-1)$, we define a polynomial $\varphi_{(I)}(v)$ in v by

$$\varphi_{(I)}(v)=\frac{\varphi_\nu(v)}{\varphi_{i_1-i_0}(v)\cdots\varphi_{i_{r+1}-i_r}(v)}, \tag{3.2.3}$$

where $\varphi_i(v)=\prod_{j=1}^{i}(v^j-1)$ for any $i\geqq1$ and put $i_0=0$ and $i_{r+1}=\nu$. (We understand that $\varphi_{(\phi)}(v)=1$.) The following proposition is proved in a quite similar manner to [3; Theorem 3].

PROPOSITION 2. *For* $0 \leqq r \leqq \nu-1$, *put* $z_r = q^{r(\nu-r)} v_{\nu-r}$. *Then*

$$f(t_1, \cdots, t_\nu) = p^{(0)}(t_1, \cdots, t_\nu)/q^{(0)}(t_1, \cdots, t_\nu),$$

where $q^{(0)}(t_1, \cdots, t_\nu) = \prod_{i=0}^{\nu-1}(1-z_i)$ *and*

$$p^{(0)}(t_1, \cdots, t_\nu) = \sum_{I \subset N'} \varphi_{(I)}(q^{-1}) \prod_{i \in I} z_i \prod_{j \in N'-I} (1-z_j).$$

(N'-I *denotes the complement of* I *in* N'.)

We shall compute certain partial sum of $F(t; X)$. For any $I \subset N$, set

$$F_I(t_1, \cdots, t_\nu; X_1, \cdots, X_\nu) = \sum_{\substack{0 \leq r_1 \leq \cdots \leq r_\nu \\ s(r)=I}} t_1^{r_1} \cdots t_\nu^{r_\nu} a_r(X), \tag{3.2.4}$$

and

$$f_I(t_1, \cdots, t_\nu) = \sum_{\substack{0 \leq r_1 \leq \cdots \leq r_\nu \\ s(r)=I}} t_1^{r_1} \cdots t_\nu^{r_\nu} |R(\boldsymbol{r})|, \tag{3.2.5}$$

where, for $\boldsymbol{r} = (r_1, \cdots, r_\nu) \in \boldsymbol{Z}^\nu$,

$$s(\boldsymbol{r}) = \{1 \leqq i \leqq \nu;\ r_i \text{ is odd}\}. \tag{3.2.6}$$

For any subset $I \subset N$, define a function ρ_I^N on $\boldsymbol{Z}^\nu$ as

$$\rho_I^N(\boldsymbol{r}) = \sum_{\substack{I_0 \subset I \\ I_0' \subset N-I}} (-1)^{|I_0|} \prod_{i \in I_0 \cup I_0'} (-1)^{r_i}, \tag{3.2.7}$$

where $\boldsymbol{r} = (r_1, \cdots, r_\nu) \in \boldsymbol{Z}^\nu$.

LEMMA 5.

$$\rho_I^N(\boldsymbol{r}) = \begin{cases} 2^\nu & \text{if } s(\boldsymbol{r}) = I, \\ 0 & \text{otherwise}. \end{cases}$$

In other words, $2^{-\nu}\rho_I^N$ *is the characteristic function of* $\{\boldsymbol{r} \in \boldsymbol{Z}^\nu;\ s(\boldsymbol{r}) = I\}$ *in* $\boldsymbol{Z}^\nu$.

PROOF. If $I = \emptyset$, it is clear. Suppose $I \neq \emptyset$, and take an element a in I. Then

$$\rho_I^N(\boldsymbol{r}) = (1-(-1)^{r_a}) \rho_{I-\{a\}}^{N-\{a\}}(r_1, \cdots, r_{a-1}, r_{a+1}, \cdots, r_\nu).$$

Thus the lemma is proved by induction on $|I|$. q.e.d.

For any subset $I \subset N$, we set $t_I = (t_1', \cdots, t_\nu')$ as follows;

$$t_i' = \begin{cases} -t_i & \text{if } i \in I, \\ t_i & \text{if } i \notin I. \end{cases} \tag{3.2.8}$$

Put

$$P^{(1)}(t; X) = P^{(0)}(t; X) \prod_{r=1}^{\nu} \prod_{1 \leq i_1 < \cdots < i_r \leq \nu} (1 + q^{-r(r+1)/2} X_{i_1} \cdots X_{i_r} v_r), \tag{3.2.9}$$

and

$$p^{(1)}(t) = p^{(0)}(t) \prod_{r=0}^{\nu-1} (1+z_r), \tag{3.2.10}$$

where $z_r=q^{r(\nu-r)}v_{\nu-r}$ for $0\leqq r\leqq\nu-1$.

PROPOSITION 3.

$$F_I(t;X)=P_I^{(1)}(t;X)/Q^{(1)}(t;X),$$
$$f_I(t)=p_I^{(1)}(t)/q^{(1)}(t),$$

where

$$Q^{(1)}(t;X)=\prod_{r=1}^{\nu}\prod_{1\leq i_1<\cdots<i_r\leq\nu}(1-(q^{-r(r+1)/2}X_{i_1}\cdots X_{i_r}v_r)^2),$$

$$q^{(1)}(t)=\prod_{r=1}^{\nu-1}(1-z_r^2)$$

$$P_I^{(1)}(t;X)=2^{-\nu}\sum_{\substack{I_0\subset I\\ I_0'\subset N-I}}(-1)^{|I_0|}P^{(1)}(t_{I_0\cup I_0'};X),$$

and

$$p_I^{(1)}(t)=2^{-\nu}\sum_{\substack{I_0\subset I\\ I_0'\subset N-I}}(-1)^{|I_0|}p^{(1)}(t_{I_0\cup I_0'}).$$

Especially, the total degree of $P_I^{(1)}$ (resp. $p_I^{(1)}$) in $(v_1,\cdots,v_\nu)$ is at most $2^{\nu-1}-3$ (resp. $2\nu-1$), and the term of degree $2^{\nu+1}-3$ (resp. $2\nu-1$) is $\delta_{|I|,\nu}M_1(t;X)$ (resp. $\delta_{|I|,\nu}q^{\nu(\nu-1)(2\nu-1)/6}\prod_{r=1}^{\nu-1}v_r^2\cdot v_\nu$), where

$$M_1(t;X)=(-1)^{\nu-1}q^{-\nu(\nu+3)2^{\nu-2}+\nu}(X_1\cdots X_\nu)^{2^\nu-1}\prod_{r=1}^{\nu-1}v_r^{2\binom{\nu}{r}}\cdot v_\nu. \tag{3.2.11}$$

PROOF. The first part of this proposition is easily verified by Theorem (A), Proposition 2, and Lemma 5. Put $t_I=(t_1',\cdots,t_\nu')$ and $v_r'=\prod_{j=1}^{r}t_{\nu-r+j}'$. Then for $1\leqq r\leqq\nu$, $v_r'=(-1)^{|\bar{r}\cap I|}v_r$, where $\bar{r}=\{\nu-r+1,\cdots,\nu\}$. As the term of highest total degree in $(v_1,\cdots,v_\nu)$ of $P^{(1)}(t;X)$ is $M_1(t;X)$, that of $P_I^{(1)}(t;X)$ is

$$2^{-\nu}\sum_{\substack{I_0\subset I\\ I_0'\subset N-I}}(-1)^{|\bar{\nu}\cap(I_0\cup I_0')|}M_1(t;X)=\delta_{|I|,\nu}M_1(t;X).$$

The statement on $p_I^{(1)}$ is similarly proved. q. e. d.

3-3. Let us prove Theorem 1 and 2. We keep all notations in § 3-2. First, let T be an indeterminate. For any subset $I\subset N$ and $\lambda\geqq 1$, set

$$\zeta(t;X;T)=\sum_{m=0}^{\infty}T^m\sum_{0\leq r_1\leq\cdots\leq r_\nu\leq m}t_1^{r_1}\cdots t_\nu^{r_\nu}a_r(X), \tag{3.3.1}$$

$$\zeta_I^{(\lambda)}(t;X;T)=\sum_{m=0}^{\infty}T^m\sum_{\substack{0\leq r_1\leq\cdots\leq r_\nu\leq 2\lambda m\\ s(r)=I}}t_1^{r_1}\cdots t_\nu^{r_\nu}a_r(X), \tag{3.3.2}$$

$$\bar{\zeta}(t;T)=\sum_{m=0}^{\infty}T^m\sum_{0\leq r_1\leq\cdots\leq r_\nu\leq m}t_1^{r_1}\cdots t_\nu^{r_\nu}|R(\boldsymbol{r})|, \tag{3.3.3}$$

and

$$\bar{\zeta}_I^{(\lambda)}(t\,;\,T)=\sum_{m=0}^{\infty}T^m\sum_{\substack{0\leq r_1\leq\cdots\leq r_\nu\leq 2\lambda m\\ s(r)=I}}t_1^{r_1}\cdots t_\nu^{r_\nu}\,|R(\boldsymbol{r})|\,. \tag{3.3.4}$$

By the definition of Hecke series and Propositions 1 and 2, we have

$$\zeta_{(G,U)}(T)=\begin{cases}\zeta(q^\nu p^A,\,q^{\nu-1}p^A,\,\cdots,\,qp^A\,;\,X\,;\,X_0T) & \text{if } fe=1\,,\\ \sum_{I\subset N}p^{B|I|}\zeta_I^{(fe/2)}(q^\nu p^A,\,q^{\nu-1}p^A,\,\cdots,\,qp^A\,;\,X\,;\,X_0T) & \text{otherwise,}\end{cases}$$

and

$$z_{(G,U)}(T)=\begin{cases}\bar{\zeta}(q^\nu p^A,\,q^{\nu-1}p^A,\,\cdots,\,qp^A\,;\,T) & \text{if } fe=1\,,\\ \sum_{I\subset N}p^{B|I|}\bar{\zeta}_I^{(fe/2)}(q^\nu p^A,\,q^{\nu-1}p^A,\,\cdots,\,qp^A\,;\,T) & \text{otherwise.}\end{cases}$$

As

$$\zeta(t\,;\,X\,;\,T)=F(t_1,\,\cdots,\,t_{\nu-1},\,t_\nu T\,;\,X)/(1-T)\,,$$

and

$$\bar{\zeta}(t\,;\,T)=f(t_1,\,\cdots,\,t_{\nu-1},\,t_\nu T)/(1-T)\,,$$

in the case $fe=1$ Theorems 1 and 2 are proved. (Note that if $fe=1$ then $B=0$). Suppose $fe=2$. Put

$$P_I^{(1)}(t_1,\,\cdots,\,t_{\nu-1},\,t_\nu T\,;\,X)=\sum_{m=0}^{2^{\nu+1}-3}\alpha_{I,m}(t_1,\,\cdots,\,t_\nu\,;\,X)T^m\,.$$

Then we get

$$(1-T)\prod_{r=1}^{\nu}\prod_{1\leq i_1<\cdots<i_r\leq\nu}(1-(q^{-r(r+1)/2}X_{i_1}\cdots X_{i_r}v_r)^2T)\zeta_I^{(1)}(t\,;\,X\,;\,T)$$
$$=\sum_{m=0}^{2^\nu-2}\{\alpha_{I,2m}(t\,;\,X)T^m+\alpha_{I,2m+1}(t\,;\,X)T^{m+1}\}\,.$$

By Proposition 3

$$\alpha_{I,2^{\nu+1}-3}(t\,;\,X)=\delta_{|I|,\nu}M_1(t\,;\,X)\qquad\text{(see (3.2.11))}\,.$$

Therefore in this case Theorem 1 is obtained, and Theorem 2 is similarly proved. Finally, suppose $fe=4$. Put

$$P_I^{(2)}(t\,;\,X)=P_I^{(1)}(t\,;\,X)\prod_{r=1}^{\nu}\prod_{1\leq i_1<\cdots<i_r\leq\nu}(1+(q^{-r(r+1)/2}X_{i_1}\cdots X_{i_r}v_r)^2)\,,$$

and

$$P_I^{(2)}(t_1,\,\cdots,\,t_{\nu-1},\,t_\nu T\,;\,X)=\sum_{m=0}^{2^{\nu+2}-5}\beta_{I,m}(t\,;\,X)T^m\,.$$

Then we have

$$(1-T)\prod_{r=1}^{\nu}\prod_{1\leq i_1<\cdots<i_r\leq\nu}(1-(q^{-r(r+1)/2}X_{i_1}\cdots X_{i_r}v_r)^4T)\zeta_I^{(2)}(t;\ X;\ T)$$

$$=\sum_{m=0}^{2^\nu-2}\{\beta_{I,4m}(t;\ X)T^m+\beta_{I,4m+1}(t;\ X)T^{m+1}+\beta_{I,4m+2}(t;\ X)T^{m+1}+\beta_{I,4m+3}(t;\ X)T^{m+1}\},$$

and similar formula for $\bar{\zeta}_I^{(2)}(t;\ T)$. This proves Theorems 1 and 2 except that the degree of $P(T)$ is $2^\nu-1$. We can easily see that the coefficient of T^ν of $p(T)$ (see Theorem 2) is not 0 as a polynomial in p. Therefore the degree of $P(T)$ is $2^\nu-1$. Now our theorems are proved. q. e. d.

§ 4. Examples.

In this section we shall present some explicit formulae of Hecke series for small ν. Unfortunately we do not know the value f in case (U^-) completely. Thus we omitt case (U^-). Note that fe is 1 or 2 except for case (U^-). Table 1 gives the values f, ∂, and B. For $\nu=1$ the Hecke series are explicitly given in (I). Tables 2 and 3 give the image of the generators of $L_Z(G, U)$ under the isomorphism Ψ for $\nu=2$. (Using these tables we can easily write down Hecke series explicitly.) Finally in (III), we present an example of the Hecke series in the case $fe=4$.

For the classification of anisotropic ε-hermitian forms over $\mathfrak{p}$-adic fields, see [**6**], [**7**], [**13**], [**14**], [**16**]. Note that if L (resp. M) is maximal $\mathfrak{P}^l$-integral (resp. $\mathfrak{P}^{l+fem}$-integral) lattice in V, there exists an element g in G such that $M=gL$. This implies that we may consider l modulo fe. In case (U), we assume $\varepsilon=1$.

[Table 1]

No.		n_0	(R_0, l)	f	∂	B
case (O)	(i)	0	(0, 0)	1	0	0
	(1)	1	$(2s, 0)$ or $(2\pi s, 1)$	2	1	$-1/2$
	(2)	1	$(2\pi s, 0)$ or $(2s, -1)$	2	0	$1/2$
	(ii)	2	$(S_1, 0)$	1	1	0
	(3)	2	$(S_2, 0)$ or $(\pi S_2, 1)$	2	2	-1
	(4)	2	$(S_2, -1)$ or $(\pi S_2, 0)$	2	0	1
	(5)	3	$(2\pi s+S_2, 0)$ or $(2\pi^2 s+\pi S_2, 1)$	2	2	$-1/2$
	(6)	3	$(2s+\pi S_2, 0)$ or $(2\pi s+\pi^2 S_2, 1)$	2	1	$1/2$
	(iii)	4	$(S_2+\pi S_2, 0)$	1	2	0
case (Sp)	(iv)	0	(0, 0)	1	0	0
case (U) with $e=1$	(v)	0	(0, 0)	1	0	0
	(7)	1	$(s, 0)$ or $(\pi s, 1)$	2	1	-1
	(8)	1	$(\pi s, 0)$ or $(s, -1)$	2	0	1

	(vi)	2	$(S_3, 0)$	1	1	0
case (U) with $e=2$	(9)	0	$(0, -\delta)$	1	0	$-1/2$
	(10)	0	$(0, -\delta+1)$	1	0	$1/2$
	(11)	1	$(s, -\delta)$ or $(s, -\delta+1)$	1	0 or 1	0
	(12)	2	$(S_4, -\delta)$	1	0	$1/2$
	(13)	2	$(S_4, -\delta+1)$	1	2	$-1/2$
case (U^+)	(14)	0	$(0, 0)$	1	0	$1/2$
	(15)	0	$(0, 1)$	1	0	$-1/2$
	(16)	1	$(1, 0)$	1	1	$-1/2$
	(17)	1	$(1, -1)$	1	0	$1/2$

where $s\in\mathfrak{o}^\times$, $\mathfrak{P}^\delta$ is the different of K/k in case (U) with $e=2$, and

$$S_1=\begin{pmatrix} 2a_1 & b_1 \\ b_1 & 2c_1 \end{pmatrix};\ a_1\in\mathfrak{o}^\times,\ b_1\in\mathfrak{p},\ c_1\in\mathfrak{p}-\mathfrak{p}^2,$$

$$S_2=\begin{pmatrix} 2a_2 & b_2 \\ b_2 & 2c_2 \end{pmatrix};\ a_2,\ c_2\in\mathfrak{o}^\times,\ b_2\in\mathfrak{o},\ k(\sqrt{-\det S_2})\text{ is unramified over } k,$$

$$S_3=\begin{pmatrix} a_3 & \bar{b}_3 \\ b_3 & c_3 \end{pmatrix};\ a_3\in\mathfrak{o}^\times,\ b_3\in\mathfrak{P},\ c_3\in\mathfrak{p}-\mathfrak{p}^2,$$

$$S_4=\begin{pmatrix} a_4 & \bar{b}_4 \\ b_4 & c_4 \end{pmatrix};\ a_4,\ c_4\in\mathfrak{o}^\times,\ b_4\in\mathfrak{P}^{-\delta+1}.$$

(I) We suppose that $\nu=1$. Then

$$Z_{(G,U)}(T)=\frac{1+A_1T}{1-A_2T+A_3T^2},$$

where

$$A_1=\begin{cases} 0 & \text{if } fe=1, \\ qp^{A+B}c^{(0)} & \text{if } fe=2, \end{cases}$$

$$A_2=\begin{cases} c^{(0)}+c^{(1)} & \text{in case (O) with } n_0=0, \\ c^{(1)} & \text{if } fe=1 \text{ and not case (O) with } n_0=0, \\ c^{(1)}-(qp^{A+B}-1)C^{(0)} & \text{if } fe=2, \end{cases}$$

$$A_3=\begin{cases} c^{(0)}c^{(1)} & \text{in case (O) with } n_0=0, \\ (qp^Ac^{(0)})^{fe} & \text{otherwise}, \end{cases}$$

(see (1.2.5) for the definition of $c^{(i)}$).

(II) We suppose that $\nu=2$. Put $S_1=X_1+X_2$ and $S_2=X_1X_2$. Then

$$Z_{(G,U)}(T)=\frac{1+B_1T-B_2T^2-B_3T^3}{1-B_4T+B_5T^2-B_4B_6T^3+B_6^2T^4},$$

where

$$\Psi(B_1)=\begin{cases}0 & \text{if } fe=1,\\ p^{A+B}X_0\{S_1+p^{2A}S_1S_2+(p^{A+B}+p^{A-B}-q^{-1}p^{A-B})S_2\} & \text{if } fe=2,\end{cases}$$

$$\Psi(B_2)=\begin{cases}q^{-1}p^{2A}X_0^2S_2 & \text{if } fe=1,\\ q^{-1}p^{3A+B}X_0^2S_2\{S_1+p^{2A}S_1S_2+(p^{A+B}+p^{A-B}-qp^{A+B})S_2\} & \text{if } fe=2,\end{cases}$$

$$\Psi(B_3)=\begin{cases}0 & \text{if } fe=1,\\ q^{-1}p^{6A+2B}X_0^3S_2^3 & \text{if } fe=2,\end{cases}$$

$$\Psi(B_4)=\begin{cases}X_0(1+p^AS_1+p^{2A}S_2) & \text{if } fe=1,\\ X_0(1+p^{2A}S_1^2+p^{4A}S_2^2-2p^{2A}S_2) & \text{if } fe=2,\end{cases}$$

$$\Psi(B_5)=\begin{cases}p^AX_0^2(S_1+2p^AS_2+p^{2A}S_1S_2) & \text{if } fe=1,\\ p^{2A}X_0^2\{(S_1+p^{2A}S_1S_2)^2-2S_2(1+p^{2A}S_1^2+p^{4A}S_2^2-p^{2A}S_2)\} & \text{if } fe=2,\end{cases}$$

$$\Psi(B_6)=X_0^2(p^{2A}S_2)^{fe}.$$

(II)-1°; the case $fe=1$.

$$\Psi(c^{(0)})=q^{-3}X_0^2S_2,$$

$$\Psi(c^{(2)})=\begin{cases}X_0(1+p^{-2}S_2) & \text{in case (O) with } n_0=0,\\ X_0(1+p^AS_1+p^{2A}S_2) & \text{otherwise},\end{cases}$$

$\Psi(c^{(1)})$ is given in Table 2.

[Table 2]

No.	$X_0^{-2}\Psi(c^{(1)})$
(i)	$p^{-1}X_0S_1$
(ii)	$p^{-1}S_1+p^{-1}S_1S_2+p^{-3}(p^2-1)S_2$
(iii)	$p^{-1}S_1+pS_1S_2+p^{-3}(p^2-1)(p+1)S_2$
(iv)	$p^{-1}S_1+p^{-1}S_1S_2+p^{-3}(p^2-1)S_2$
(v)	$p^{-2}S_1+p^{-4}S_1S_2+p^{-6}(p^2+1)(p-1)S_2$
(vi)	$p^{-2}S_1+S_1S_2+p^{-6}(p^3-1)(p^2+1)S_2$

(II)-2°; the case $fe=2$.

$$\Psi(c^{(0)})=q^{-3}X_0S_2$$

$$\Psi(c^{(0)})+\Psi(c^{(1)})+\Psi(c^{(2)})=\Psi(T(1)) \qquad \text{(see Proposition 1)}$$

$$\Psi(c^{(1)})=X_0\{q^{-1}S_1+(p)p^{\kappa}q^{n_0-3}S_1S_2+CS_2\}$$

where C is given in Table 3, $\kappa=\dim_k(\mathrm{Ker}\ T_\varepsilon)$, and

$$(p)=\begin{cases} p & \text{(U) with } e=1, \\ 1 & \text{otherwise.}\end{cases}$$

[Table 3]

No.	(1)	(2)	(3)	(4)	(5)
C	0	$p^{-3}(p^2-1)$	0	$p^{-3}(p^2-1)(p+1)$	$p^{-3}(p^2-1)$

No.	(6)	(7)	(8)	(9)
C	$p^{-3}(p^2-1)(p+1)$	$p^{-6}(p^2+1)(p-1)$	$p^{-6}(p^3-1)(p^2+1)$	0

No.	(10)	(11)	(12)	(13)
C	$p^{-3}(p^2-1)$	$p^{-3}(p^2-1)$	$p^{-3}(p^2-1)(p+1)$	$p^{-3}(p^2-1)$

No.	(14)	(15)	(16)	(17)
C	$p^{-6}(p^4-1)$	$p^{-6}(p^2+1)(p-1)$	$p^{-6}(p^4-1)$	$p^{-6}(p^3-1)(p^2+1)$

(III) An example in the case $fe=4$.

Let K_0 be the unique unramified quadratic extension field of k. A division quaternion algebra K is realized as a cyclic algebra (K_0, π), i.e., $K=K_0+K_0\Pi$ where $\Pi^2=\pi$ and $\Pi\beta\Pi^{-1}=\bar{\beta}$ for all $\beta\in K_0$. In case (U$^-$) with $n_0=1$ and $\nu=1$, put $R_0=u-\bar{u}$ ($u\in(K_0\cap\mathfrak{O})^\times$, $u\notin k$) and $l=0$. Then $A=-1/2$, $B=-3/2$, $\partial=1$, and $f=e=2$.

$$Z_{(G,U)}(T)=\frac{1+\{c^{(1)}+p^3c^{(0)}\}T}{1-c^{(0)-1}\{c^{(1)2}-2p^3c^{(0)2}\}T+p^6c^{(0)2}T^2}.$$

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