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On analytic families of submanifolds and holomorphical convexity of neighbourhoods of P^1

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§ 1. Introduction.

Let X be a complex manifold and A a compact complex submanifold of X. We say that A is exceptionally embedded in X or exceptional in X if there exist a neighbourhood U of A and a proper holomorphic map $\mathfrak{p}$ from U to a domain of C^N that $\mathfrak{p}|_{U-A}$ is a biholomorphic map from $U-A$ to $\mathfrak{p}(U-A)$. In 1962, H. Grauert proved in [1] the following criterion for exceptional sets.

THEOREM I. (Grauert, Satz 8 in [1]).

Under the above notations, if the normal bundle of A in X has its zero section as an exceptional set, then A is exceptional in X.

In view of Theorem I, it is natural to ask whether the converse of the above statement holds or not, and it is also shown by Grauert that the converse is false. So the more detailed description of exceptional sets are desirable. In the present article, in particular, we examine the case where $A \cong P^1$.

The main result is

THEOREM 2. *Let X be a complex manifold containing a complex analytic submanifold A. Suppose that $A \cong P^1$ and $N_{A/X}$, the normal bundle of A in X, is seminegative in the sense that every rank 1 holomorphic subbundle of $N_{A/X}$ has nonpositive degree. Then A has a holomorphically convex neighbourhood in X.*

As a corollary, we have

COROLLARY. *Under the above assumptions, if moreover there exists a neighbourhood of A in X which contains no compact complex analytic subvariety of dimension $\geqq 1$ except for A, then A is exceptional in X.*

In order to prove Theorem 2, we prove the following theorem by using Kodaira's technique in [3].

THEOREM 1. *Let X be a complex manifold of dimension r and A a complex analytic submanifold of X. Suppose that $A \cong P^1$ and*

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$$N_{A/X}\cong\bigoplus_{\alpha=1}^{m+n}H^{a_\alpha}$$

where $r=m+n$, $m>0$, $n\geqq 0$, *H denotes the holomorphic line bundle of degree* 1 *over* $\boldsymbol{P}^1$, *and*

$$a_1\leqq\cdots\leqq a_m<0\leqq a_{m+1}\leqq\cdots\leqq a_{m+n}.$$

Then there exists a complex analytic family $\pi:\mathcal{V}\to\Delta^l$ *of locally closed complex analytic submanifolds of X which satisfies the following conditions:*

0) $$A\subset\pi^{-1}(0)$$

1) $$N_{A/\pi^{-1}(0)}\cong\bigoplus_{\alpha=1}^{m}H^{a_\alpha}$$

2) *The infinitesimal displacement*

$$\sigma:\ T_0(\Delta^l)\longrightarrow H^0(A,\ \underline{N_{\pi^{-1}(0)/X}|_A})$$

is bijective.

Here $\Delta^l=\{(t_1,\cdots,t_l)\in\boldsymbol{C}^l;\ |t_\nu|<1,\ \nu=1,2,\cdots,l\}$ *and* $T_0(\Delta^l)$ *denotes the holomorphic tangent space of* Δ^l *at* 0.

Theorem 1 follows from the theorem of Knorr-Schneider (cf. [2], Satz 3.4) and Kodaira's theorem (cf. [3], Main Theorem).

We note that Theorem 2 does not describe the whole aspect of the exceptional embeddings of $\boldsymbol{P}^1$ in the sense that there exists an exceptional embedding of $\boldsymbol{P}^1$ whose normal bundle is not seminegative, which we construct in the appendix.

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§ 2. Construction of $\pi^{-1}(0)$.

Let X be a complex manifold of dimension r containing a complex analytic submanifold A which is biholomorphic to $\boldsymbol{P}^1$. It is well known (cf. [5]) that A is covered by two coordinate neighbourhoods U_0 and U_1 with local coordinates $(t_i,u_i)=(t_i,u_i^1,\cdots,u_i^r)$, $i=0,1$, such that $U_i\cap A$ coincides with the subspace of U_i determined by $u_i^1=\cdots=u_i^r=0$. Let the coordinate transformation be $(t_0,u_0)=g(t_1,u_1)=(g^0(t_1,u_1),g^1(t_1,u_1),\cdots,g^r(t_1,u_1))$.

Since by the Grothendieck-Birkhoff's theorem $N_{A/X}$ is a direct sum of holomorphic line bundles over A, we may assume

(1) $$g^0(t_0,0,\cdots,0)=\frac{1}{t_0}$$

$$\frac{\partial(g^0, \cdots, g^r)}{\partial(t_0^0, u_0^1, \cdots, u_0^r)}\bigg|_{u_0^1=\cdots=u_0^r=0} = \begin{pmatrix} t_0^{-2} & \theta_1(t_0) \cdots & \cdots \theta_r(t_0) \\ & t_0^{-a_1} & & 0 \\ & & \ddots & \\ 0 & & & t_0^{-a_r} \end{pmatrix}. \tag{2}$$

Here $\theta_1(t_0), \cdots, \theta_r(t_0)$ are holomorphic functions on $U_0 \cap U_1 \cap A$ and a_α ($\alpha=1, 2, \cdots, r$) are integers satisfying $N_{A/X} \cong \bigoplus_{\alpha=1}^{r} H^{a_\alpha}$. In what follows we restrict ourselves to the case where $r=m+n$, $m>0$, $n \geqq 0$, and

$$a_1 \leqq \cdots \leqq a_m < 0 \leqq a_{m+1} \leqq \cdots \leqq a_{m+n}. \tag{3}$$

We denote by $\mathcal{J}$ the ideal sheaf of A in X.

LEMMA 1. *Under the above conditions, we can choose the coordinates* (t_i, u_i), $i=0, 1$, *so that they satisfy* (1), (2) *and moreover*

(I) $$g^0(t_0, u_0^1, \cdots, u_0^m, 0, \cdots, 0) \equiv \frac{1}{t_0} \quad \mathrm{mod}\, \mathcal{J}^{-a_1+1}$$

(II) *There exist polynomials f^α in (t_1, u_1), for $1 \leqq \alpha \leqq m$, such that the degrees of f^α in u_1 are $\leqq -a_1$, and*

$$g^\alpha(t_1, u_1^1, \cdots, u_1^m, 0, \cdots, 0) \equiv f^\alpha(t_1, u_1^1, \cdots, u_1^m) \quad \mathrm{mod}\, \mathcal{J}^{-a_1+1} \tag{4}$$

$$f^\alpha(0, u_1^1, \cdots, u_1^m)=0, \tag{5}$$

(III) $$g^\beta(t_1, u_1^1, \cdots, u_1^m, 0, \cdots, 0) \equiv 0 \quad \mathrm{mod}\, \mathcal{J}^{-a_1+1}, \qquad \textit{for } m+1 \leqq \beta \leqq m+n.$$

PROOF. We prove, for every positive integer k, there exist coordinates (t_{0k}, u_{0k}) and (t_{1k}, u_{1k}) in the neighbourhoods of $U_0 \cap A$ and $U_1 \cap A$, respectively, such that, letting $(t_{0k}, u_{0k}) = g_k(t_{1k}, u_{1k}) = (g_k^0(t_{1k}, u_{1k}), \cdots, g_k^r(t_{1k}, u_{1k}))$ be the coordinate transformation, (1), (2) and the following conditions hold.

(I_k) $$g_k^0(t_{1k}, u_{1k}^1, \cdots, u_{1k}^m, 0, \cdots, 0) \equiv 1/t_{1k} \quad \mathrm{mod}\, \mathcal{J}^k$$

(II_k) There exist polynomials f_k^α ($\alpha=1, \cdots, m$) in (t_{1k}, u_{1k}) such that the degrees of f_k^α in u_{1k} are $\leqq -a_1$ and

$$g_k^\alpha(t_{1k}, u_{1k}^1, \cdots, u_{1k}^m, 0, \cdots, 0) \equiv f_k^\alpha(t_{1k}, u_{1k}^1, \cdots, u_{1k}^m) \quad \mathrm{mod}\, \mathcal{J}^k,$$

$$f_k^\alpha(0, u_{1k})=0.$$

(III_k) $$g_k^\beta(t_{1k}, u_{1k}^1, \cdots, u_{1k}^m, 0, \cdots, 0) \equiv 0 \quad \mathrm{mod}\, \mathcal{J}^k \qquad \text{for } m+1 \leqq \beta \leqq m+n.$$

We prove this by induction on k.

If $k=1$, then (1) and (2) already implies (I_1), (II_1), (III_1), since $-a_\alpha>0$ for $1 \leqq \alpha \leqq m$.

Assume that there exist coordinates (t_{0k}, u_{0k}), (t_{1k}, u_{1k}) in the neighbourhoods of $U_i \cap A$, satisfying (I_k), (II_k), (III_k). Let $\{g_k^0\}_k'$ denote the k-th homogeneous term of $g_k^0(t_{1k}, u_{1k}^1, \cdots, u_{1k}^m, 0, \cdots, 0)$. By the Laurent expansion theorem there exist homogeneous polynomials $g_k^{0+}(t_{1k}, u_{1k}^1, \cdots, u_{1k}^m)$ and $g_k^{0-}(t_{1k}, u_{1k}^1, \cdots, u_{1k}^m)$ in variables $u_{1k}^1, \cdots, u_{1k}^m$ with coefficients in the power series rings $\boldsymbol{C}\{t_{1k}\}$ and $\boldsymbol{C}\{1/t_{1k}\}$, respectively, such that

$$\{g_k^0\}_k' = g_k^{0+} + g_k^{0-}\,. \tag{6}$$

We set

$$t_{1k+1} = t_{1k} - t_{1k}^2 g_k^{0+} \tag{7}$$

$$t_{0k+1} = t_{0k} - g_k^{0-}(1/t_{0k}, t_{0k}^{-a_1}u_{0k}^1, \cdots, t_{0k}^{-a_m}u_{0k}^m)\,. \tag{8}$$

Then we have

$$\begin{aligned} t_{0k+1} - 1/t_{1k+1} &= t_{0k} - g_k^{0-}(1/t_{0k}, t_{0k}^{-a_1}u_{0k}^1, \cdots, t_{0k}^{-a_m}u_{0k}^m) \\ &\qquad -(t_{1k}(1-t_{1k}g_k^{0+}(t_{1k}, u_{1k}^1, \cdots, u_{1k}^m)))^{-1} \\ &\equiv t_{0k} - g_k^{0+}(t_{1k}, u_{1k}^1, \cdots, u_{1k}^m) \\ &\qquad -1/t_{1k} - g^{0+}(t_{1k}, u_{1k}^1, \cdots, u_{1k}^m) \qquad \bmod \mathcal{J}^{k+1} \\ &\equiv \{g_k^0\}_k' - (g_k^{0+} + g_k^{0-}) = 0 \qquad \bmod \mathcal{J}^{k+1}\,. \end{aligned}$$

Thus, if we take (t_{0k+1}, u_{0k}) and (t_{1k+1}, u_{1k}) as coordinates, then (I_{k+1}), (II_k), (III_k) are satisfied.

Let $(t_{0k+1}, u_{0k}) = g_{*k}(t_{1k+1}, u_{1k}) = (g_{*k}^0, \cdots, g_{*k}^r)$ be the coordinate transformation. Letting $\{g_{*k}^\alpha\}_k'$ be the k-th homogeneous term of $g_{*k}^\alpha(t_{1k+1}, u_{1k}^1, \cdots, u_{1k}^m, 0, \cdots, 0)$ for $1 \leqq \alpha \leqq r$, by the Laurent expansion theorem we have homogeneous polynomials $g_k^{\alpha+}$ and $g_k^{\alpha-}$ satisfying

$$\{g_{*k}^\alpha\}_k' = g_k^{\alpha+} + g_k^{\alpha-}\,, \tag{9}$$

where the coefficients of $g_k^{\alpha+}$ (or $g_k^{\alpha-}$) are contained in $\boldsymbol{C}\{t_{1k+1}\} - \boldsymbol{C}$ (or $\boldsymbol{C}\{1/t_{1k+1}\}$). We set

$$g_k^{\alpha+} = g_{k\square}^\alpha + t_{1k+1}{}^{-a_1+1} g_{k\Delta}^\alpha\,, \qquad \text{for } 1 \leqq \alpha \leqq m\,. \tag{10}$$

Here the coefficients of $g_{k\Delta}^\alpha$ are contained in $\boldsymbol{C}\{t_{1k+1}\}$ and the coefficients of $g_{k\square}^\alpha$ are polynomials of degree $\leqq -a_1$ in t_{1k+1}. We put

$$\begin{aligned} u_{1k+1}^\alpha &= u_{1k}^\alpha + t_{1k+1}{}^{-a_1+a_\alpha+1} g_{k\Delta}^\alpha \\ u_{0k+1}^\alpha &= u_{0k}^\alpha - g_k^{\alpha-}(1/t_{0k+1}, t_{0k+1}{}^{-a_1}u_{0k}^1, \cdots, t_{0k+1}{}^{-a_m}u_{0k}^m)\,, \\ &\text{for } 1 \leqq \alpha \leqq m\,, \end{aligned} \tag{11}$$

(12) $$u^{\beta}_{1k+1}=u^{\beta}_{1k}+t_{1k+1}{}^{a_\beta}g^{\beta+}_{k}$$

$$u^{\beta}_{0k+1}=u^{\beta}_{0k}-g^{\beta-}_{k}(1/t_{0k+1}, t_{0k+1}{}^{-a_1}u^{1}_{0k}, \cdots, t_{0k+1}{}^{-a_m}u^{m}_{0k}),$$

for $m+1\leqq\beta\leqq m+n$.

Then we have

(13) $$u^{\alpha}_{0k+1}\underset{\text{by (2)}}{\equiv}u^{\alpha}_{0k}-g^{\alpha-}_{k}(t_{1k+1}, u^{1}_{1k}, \cdots, u^{m}_{1k}) \quad \mathrm{mod}\, \mathcal{I}^{k+1}$$

$$\underset{\text{by (9)}}{\equiv}f^{\alpha}_{k}(t_{1k+1}, u^{1}_{1k}, \cdots, u^{m}_{1k})+\{g^{\alpha}_{*k}\}'_{k}-g^{\alpha-}_{k} \quad \mathrm{mod}\, \mathcal{I}^{k+1}$$

$$\underset{\text{by (2)}}{\equiv}f^{\alpha}_{k}(t_{1k+1}, u^{1}_{1k+1}, \cdots, u^{m}_{1k+1})+g^{\alpha}_{k\square} \quad \mathrm{mod}\, \mathcal{I}^{k+1},$$

for $1\leqq\alpha\leqq m$.

Since $g^{\alpha}_{k\square}$ is a polynomial in t_{1k+1} and $u^{1}_{1k+1}, \cdots, u^{m}_{1k+1}$, of degree $\leqq -a_1$ satisfying $g^{\alpha}_{k\square}(0, u^{1}_{1k}, \cdots, u^{m}_{1k})=0$, (II_{k+1}) is satisfied if we put

(14) $$f^{\alpha}_{k+1}(t_{1k+1}, u^{1}_{1k+1}, \cdots, u^{m}_{1k+1})$$

$$=f^{\alpha}_{k}(t_{1k+1}, u^{1}_{1k+1}, \cdots, u^{m}_{1k+1})+g^{\alpha}_{k\square}(t_{1k+1}, u^{1}_{1k+1}, \cdots, u^{m}_{1k+1}).$$

In view of (2), (III_k) and (12), it follows immediately that (III_{k+1}) is satisfied by (t_{0k+1}, u_{0k+1}) and (t_{1k+1}, u_{1k+1}). q. e. d.

Let (t_i, u_i), $i=0, 1$, be the coordinates satisfying the conditions (2), (I), (II), (III). We may assume that the coordinate neighbourhoods U_i are polydiscs defined by $U_i=\{(t_i, u_i); |t_i|<3, |u_i|<1\}$, and (t_i, u_i) are defined on the closures of U_i.

For sufficiently small positive number ε there exists a complex manifold V^{ε} of dimension $1+m$ obtained by patching the polydiscs $V^{\varepsilon}_0=\{(s_0, z^1_0, \cdots, z^m_0); |s_0|<2, |z^{\alpha}_0|<\varepsilon, 1\leqq\alpha\leqq m\}$ and $V^{\varepsilon}_1=\{(s_1, z^1_1, \cdots, z^m_1); |s_1|<2, |z^{\alpha}_1|<\varepsilon, 1\leqq\alpha\leqq m\}$ by the following coordinate transformation:

(15) $$s_0=1/s_1$$

(16) $$z_0=f^{\alpha}(s_1, z^1_1, \cdots, z^m_1), \qquad 1\leqq\alpha\leqq m.$$

Note that V^{ε} contains $\boldsymbol{P}^1$ as an analytic submanifold and z^{α}_0, $s_0z^{\alpha}_0$, $(1\leqq\alpha\leqq m)$ can be extended as holomorphic functions on V^{ε}, which we identify with z^{α}_0, $s_0z^{\alpha}_0$. We identify s_0 with a meromorphic function which extends s_0 by (15). Set

(17) $$z_i=(z^1_i, \cdots, z^m_i) \qquad i=0, 1,$$

(18) $$z_0=f(s_1, z_1)$$

(19) $$z_1=f^*(s_0, z_0).$$

We denote by $\mathcal{J}$ the ideal sheaf of $\boldsymbol{P}^1$ in V^ε.

PROPOSITION 1. *For sufficiently small ε, there exists a holomorphic embedding of V^ε into X which is compatible with the embeddings of $\boldsymbol{P}^1$ into V^ε and X.*

PROOF. Proposition 1 is equivalent to the assertion that for sufficiently small ε we can find holomorphic embeddings of V_i^ε into U_i satisfying the compatibility condition to make them into a well defined holomorphic map from V^ε to X which is compatible with the embedding of $\boldsymbol{P}^1$. To prove this, as usual, it suffices to show the existence of vectors of convergent power series $\tilde{\varphi}_0(s_0, z_0)$ (in z_0) and $\tilde{\varphi}_1(s_1, z_1)$ (in z_1) whose coefficients are holomorphic functions on $\{s_0; |s_0|\leqq 2\}$ and $\{s_1; |s_1|\leqq 2\}$, respectively, such that

$$\tilde{\varphi}_1(s_1, z_1)=g(\tilde{\varphi}_0(s_0, z_0)) \tag{20}$$

for any $(s_0, z_0)\in V_0^\varepsilon$ and $(s_1, z_1)\in V_1^\varepsilon$ satisfying (15) and (16).

We shall construct such $\tilde{\varphi}_0$ and $\tilde{\varphi}_1$ satisfying the following additional conditions:

(A′) The orders of $\tilde{\varphi}_0(s_0, z_0)-(s_0, z_0, 0)$ and $\tilde{\varphi}_1(s_1, z_1)-(s_1, z_1, 0)$ are at least $-a_1$.

(B′) Regarding $\tilde{\varphi}_1(1/s_0, f^*(s_0, z_0))-(1/s_0, f^*(s_0, z_0), 0)$ as a power series in z_0, its coefficients are holomorphic functions on $\{s_0; |s_0|\geqq 1/2\}$ whose orders at ∞ are at most $-a_1$.

We set $\varphi_0(s_0, z_0)=\tilde{\varphi}_0(s_0, z_0)-(s_0, z_0, 0)$ and $\varphi_1(s_1, z_0)=\tilde{\varphi}_1(s_1, f^*(1/s_1, z_0))-(s_1, f^*(1/s_1, z_0), 0)$. Then (A′) is equivalent to

(A) The orders of $\varphi_0(s_0, z_0)$ and $\varphi_1(s_1, z_0)$ are at least $-a_1$,

and (B′) is equivalent to

(B) The coefficients of $\varphi_1(s_1, z_0)$ are holomorphic on $\{s_1; 0\leqq|s_1|\leqq 2\}$ and have poles of orders at most $-a_1$ at $\{s_1=0\}$.

Note that (A) and (B) implies that $\varphi_1(s_1, f(s_1, z_1))$ is a vector of power series whose coefficients are holomorphic functions on $|s_1|\leqq 2$, since $f(0, z_1)=0$ by (5). Therefore we have only to find holomorphic functions φ_0 and φ_1 on V_0^ε and V_1^ε respectively, satisfying (A), (B) and

$$\text{(C)}\qquad \begin{aligned}&(1/s_0, f^*(s_0, z_0), 0)+\varphi_1(1/s_0, z_0)\\ &\quad =g((s_0, z_0, 0)+\varphi_0(s_0, z_0)), \qquad \text{for } (s_0, z_0)\in V_0^\varepsilon\cap V_1^\varepsilon.\end{aligned}$$

To show the existence of such φ_0 and φ_1, it suffices to find sequences of vectors of polynomials $\{\varphi_0^\nu(s_1, z_0)\}$ and $\{\varphi_1^\nu(s_0, z_0)\}$ ($\nu=0, 1, \cdots$) in z_0, satisfying the following conditions:

(A_ν) The degrees and the orders of φ_0^ν and φ_1^ν are $\leqq -a_1+\nu$ and $\geqq -a_1$, respectively.

(B_ν) The coefficients of φ_1^ν are holomorphic functions on $\{s_1; 0<|s_1|\leqq 2\}$ and have poles of orders at most $-a_1$ at $\{s_1=0\}$.

(C_ν) $$(1/s_0, f((s_0, z_0), 0)+\varphi_1^\nu(1/s_0, z_0)$$
$$\equiv g((s_0, z_0, 0)+\varphi_0^\nu(s_0, z_0)) \qquad \bmod \mathcal{J}^{-a_1+\nu}$$

(D_ν) $$\varphi_i^{\nu-1}\equiv\varphi_i^\nu \qquad \bmod \mathcal{J}^{-a_1+\nu} \quad i=0, 1.$$

(E_ν) There exists a convergent power series $A(z_0)$ in z_0 with coefficients in $\boldsymbol{C}$ which is independent of ν, such that

$$\varphi_0^\nu(s_0, z_0)\ll A(z_0) \qquad \text{for } |s_0|\leqq 2 \tag{21}$$

and

$$\varphi_1^\nu(s_1, z_0)\ll A(z_0) \qquad \text{for } 1/2\leqq|s_1|\leqq 2. \tag{22}$$

Here $\ll$ means that the coefficients of the right hand side are greater than the absolute values of the corresponding coefficients of the left hand side. We shall prove (E_ν) for the convergent power series

$$A(z_0)=\frac{c}{16b}\sum_{k=1}^{\infty}\frac{b^k(z_0^1+\cdots+z_0^m)^k}{k^2}$$

for suitable b and c.

The existence of the sequences satisfying (A_ν)$\sim$(E_ν) is proved by the induction on ν. If $\nu=0$ we set $\varphi_1^0(s_1, z_0)=0$ and $\varphi_0^0(s_0, z_0)=0$. Let us assume that for $\nu\leqq r$ there exist $\varphi_1^\nu(s_1, z_0)$ and $\varphi_0^\nu(s_0, z_0)$ satisfying (A_ν)$\sim$(E_ν). We let

$$\psi(s_0, z_0)=\Big\{\Big(\frac{1}{s_0}, f^*(s_0, z_0), 0\Big)+\varphi_1^r\Big(\frac{1}{s_0}, z_0\Big)$$
$$-g((s_0, z_0, 0)+\varphi_0^r(s_0, z_0))\Big\}_{-a_1+1+r},$$

where $\{\ \}_{-a_1+1+r}$ denotes the $(-a_1+1+r)$-th homogeneous term. Obviously

$$\psi(s_0, z_0)=\{(0, f^*(s_0, z_0), 0)-g((s_0, z_0, 0)+\varphi_0^r(s_0, z_0))\}_{-a_1+1+r}. \tag{23}$$

First we estimate $\{f^*(s_0, z_0)\}_{-a_1+1+r}$. There exists b_0 such that for some constant K_0,

$$f^*(s_0, z_0)\ll\frac{K_0}{c}A(z_0), \tag{24}$$

for $b>b_0$, $c>16$, and $1/2\leqq|s_0|\leqq 2$. Thus,

$$\{f^*(s_0, z_0)\}_{-a_1+1+r} \ll \frac{K_0}{c} A(z_0) \tag{25}$$

for $b>b_0$, $c>16$, and $1/2 \leqq |s_0| \leqq 2$. Next we estimate

$$\{g((s_0, z_0, 0)+\varphi_0^r(s_0, z_0))\}_{-a_1+1+r}. \tag{26}$$

We expand $g(s_0+y_0, y_1, \cdots, y_{m+n})$ into a power series in $y_0, \cdots, y_{m+n}$. Since g is a vector of holomorphic functions of $s_0, y_0, \cdots, y_{m+n}$, $1/2 \leqq |s_0| \leqq 2$, $|y_0| \leqq 1/6$, $\cdots$, $|y_{m+n}| \leqq 1/2$, there exist constants K_1 and L such that

$$g\Big((s_0+y_0, y_1, \cdots, y_{m+n})-\frac{1}{s_0}, 0, \cdots, 0\Big) \tag{27}$$

$$\ll K_1 \sum_{k=1}^{\infty} L^k (y_0+\cdots+y_{m+n})^k,$$

for $1/2 \leqq |s_0| \leqq 2$, $|y_0| \leqq 1/6, \cdots, |y_{m+n}| \leqq 1/2$. It is well known*) that for a natural number k,

$$A(z_0)^k \ll \left(\frac{c}{b}\right)^{k-1} A(z_0). \tag{28}$$

Hence, if

$$b>2L(m+n+1)c \tag{29}$$

we obtain

$$\{g((s_0, z_0, 0)+\varphi_0^r(s_0, z_0))\}_{-a_1+1+r} \ll K_1 \sum_{k=2}^{\infty} L^k (m+n+1)^k A(z_0)^k \tag{30}$$

$$\ll K_1 A(z_0) \sum_{k=2}^{\infty} L^k (m+n+1)^k \left(\frac{c}{b}\right)^{k-1}$$

$$\ll \frac{K_1 L(m+n+1)c}{b} A(z_0).$$

Combining this with (25) we obtain

$$\phi(s_0, z_0) \ll \left\{\frac{KK_0}{c} + \frac{K_1 L(m+n+1)c}{b}\right\} A(z_0), \tag{31}$$

for $1/2 \leqq |s_0| \leqq 2$. We define $\eta_I(s_0)$ by

$$\phi(s_0, z_0) = \sum_{|I|=-a_1+1+r} \eta_I(s_0) z_0^I, \tag{32}$$

where I is a multi-index of length m. Expanding $\eta_I(s_0)$ into a Laurent series

*) See [4], p. 50, Corollary.

(33) $$\eta_I(s_0)=\sum_{-\infty<\mu<\infty}\eta_{I\mu}s_0^{\mu},$$

where $\eta_{I\mu}=(\eta_{I\mu}^0, \cdots, \eta_{I\mu}^{m+n})\in \boldsymbol{C}^{m+n+1}$, we let

(34) $$\eta_I^{\alpha 1}(s_0)=\sum_{\mu\leqq -a_\alpha}\eta_{I\mu}s_0^{\mu},$$

and

(35) $$\eta_I^{\alpha 0}(s_0)=\sum_{\mu>-a_\alpha}\eta_{I\mu}s_0^{\mu+a_\alpha},$$

for $1\leqq\alpha\leqq m+n$. Further we define $\hat{\eta}_I^0(s_0)$ by

(36) $$\hat{\eta}_I^0(s_0)=\eta_I^0(s_0)-\sum_{1\leqq\alpha\leqq m+n}\theta_\alpha(s_0)\eta_I^{\alpha 0}(s_0),$$

where $\eta_I(s_0)=(\eta_I^0(s_0), \cdots, \eta_I^{m+n}(s_0))$. We expand

(37) $$\hat{\eta}_I^0(s_0)=\sum_{-\infty<\mu<\infty}\hat{\eta}_{I\mu}^0 s_0^{\mu}, \qquad \hat{\eta}_{I\mu}^0\in\boldsymbol{C}.$$

We put

(40) $$\psi^1(s_0, z_0)=\Big(\sum_{\mu<-2,\,|I|=-a_1+1+r}\hat{\eta}_{I\mu}^0 s_0^{\mu}z_0^I, \sum_{\mu<-a_1,\,|I|=-a_1+1+r}\eta_{I\mu}^1 s_0^{\mu}z_0^I, \cdots, \sum_{\mu<-a_{m+n},\,|I|=-a_1+1+r}\eta_{I\mu}^{m+n}s_0^{\mu}z_0^I\Big),$$

and

(41) $$\psi^0(s_0, z_0)=\Big(\sum_{\mu\geqq-2,\,|I|=-a_1+1+r}\hat{\eta}_{I\mu}^0 s_0^{\mu+2}z_0^I, \sum_{\mu\geqq-a_1\,|I|=-a_1+1+r}\eta_{I\mu}^1 s_0^{\mu+a_1}z_0^I, \cdots, \sum_{\mu\geqq-a_{m+n},\,|I|=-a_1+1+r}\eta_{I\mu}^{m+n}s_0^{\mu+a_{m+n}}z_0^I\Big).$$

It is easily checked that

(42) $$\psi(s_0, z_0)=\psi^1(s_0, z_0)+\psi^0(s_0, z_0)\begin{pmatrix} s_0^{-2} & & 0 \\ \theta_1(s_0) & \ddots & \\ \vdots & 0 & \ddots \\ \theta_{m+n}(s_0) & & s_0^{-a_{m+n}}\end{pmatrix}$$
$$\text{for } \frac{1}{2}\leqq|s_0|\leqq 2.$$

The following lemma is proved easily by Cauchy's integral formula.

LEMMA. *There exists a constant K_2 such that for any holomorphic function*

(43) $$\eta(s_0)=\sum_{-\infty<\mu<\infty}\eta_\mu s_0^{\mu} \qquad \textit{on } \frac{1}{2}\leqq|s_0|\leqq 2,$$

we have

(44) $$\sup_{|s_0|\leqq 2}\Big|\sum_{\mu\geqq 0}\eta_\mu s_0^\mu\Big|\leqq K_2\sup_{1/2\leqq|s_0|<2}|\eta(s_0)|,$$

and

(45) $$\sup_{1/2\leqq|s_0|}\Big|\sum_{\mu<0}\eta_\mu s_0^\mu\Big|\leqq K_2\sup_{1/2\leqq|s_0|\leqq 2}|\eta(s_0)|.$$

Thus, letting

(46) $$K_3=(m+n+1)2^{-a_1+a_{m+n+1}}\times K_2\Big(\sum_{1\leqq\alpha\leqq m+n}\sum_{1/2\leqq|s_0|\leqq 2}|\theta_\alpha(s_0)|+1\Big)$$

we obtain

(47) $$\psi^1(s_0, z_0)\ll\Big\{\frac{K_0}{c}+\frac{K_1L(m+n+1)c}{b}\Big\}K_3A(z_0),$$

for $1/2\leqq|s_0|\leqq 2$, and

(48) $$\psi^0(s_0, z_0)\ll\Big\{\frac{K_0}{c}+\frac{K_1L(m+n+1)c}{b}\Big\}K_3A(z_0),$$

for $|s_0|\leqq 2$.

Therefore, if we set

(49) $$\varphi_1^{r+1}(s_1, z_0)=\varphi_1^r(s_1, z_0)-\psi^1\Big(\frac{1}{s_1}, z_0\Big)$$

and

(50) $$\varphi_0^{r+1}(s_0, z_0)=\varphi_0^r(s_0, z_0)+\psi^0(s_0, z_0),$$

then $\varphi_1^{r+1}(s_1, z_0)$ and $\varphi_0^{r+1}(s_0, z_0)$ satisfy (E_{r+1}) if

(51) $$c>\max\{16, 2K_0K_3\},$$

and

(52) $$b>\max\{b_0, 2K_1K_3L(m+n+1)c\},$$

which are conditions independent of r. The remaining conditions $(A_{r+1})\sim(D_{r+1})$ are easily checked. q. e. d.

§ 3. Analytic families.

Let the notations be as in § 1 and § 2. In virtue of Proposition 1, we can identify V_0^ε, V_1^ε and V^ε with locally closed analytic submanifolds of X. By (2), we may assume

(53) $$|f(s_1, z_1^1, \cdots, z_1^m)|>|z_1| \quad \text{for } |s_1|=2 \text{ and } |z_1|<\varepsilon,$$

(54) $$|f(s_1, z_1^1, \cdots, z_1^m)|<|z_1| \quad \text{for } |s_1|\leqq 1/2 \text{ and } |z_1|<\varepsilon,$$

(55) $$|f^*(s_0, z_0^1, \cdots, z_0^m)|>|z_0| \quad \text{for } |s_0|=2 \text{ and } |z_0|<\varepsilon.$$

We set

$$V_i=\{x\in V^{\varepsilon};\ |s_i|\leqq 1/2\} \qquad i=0,\ 1\ ,$$

$$V^i=V_i\cup(V_0^{\varepsilon}\cap V_1^{\varepsilon}) \qquad i=0,\ 1$$

and

$$V=V^0\cup V^1\ .$$

Since $N_{A/V}$ is a direct sum of line bundles of negative degree, A is exceptional in V and since V is locally an intersection of polydiscs, for sufficiently small ε, V is a holomorphically convex manifold with maximal compact analytic subset A.

Let p represent a point in V whose coordinate in V^i is $(s_i,\ z_i)$. We define a function $\tilde{\delta}$ on V by

$$\begin{aligned}\tilde{\delta}(p)&=\min(\varepsilon-|z_0|,\ \varepsilon-|z_1|) \qquad \text{for } p\in V^0\cap V^1\ ,\\ &=\varepsilon-|z_0|, \qquad \text{for } p\in V^0-V^1\\ &=\varepsilon-|z_1|, \qquad \text{for } p\in V^1-V^0\ .\end{aligned}$$

By (53) and (55), $\tilde{\delta}$ is continuous on V. Note that

$$\tilde{\delta}<\varepsilon-|z_0| \qquad \text{on } V\ , \tag{56}$$

by (54).

LEMMA 2. *There exists a constant $\tilde{K}_0$ such that for any positive integer κ and holomorphic function f on $V^0\cap V^1$ satisfying $|f(p)|<\tilde{\delta}^{-\kappa}$, there exist holomorphic functions f_0 and f_1, defined on V^0 and V^1 respectively, satisfying*

$$f(p)=f_0(p)-f_1(p) \qquad \textit{on } V^0\cap V^1\ , \tag{57}$$

$$|f_0(p)|<\tilde{K}_0\tilde{\delta}^{-\kappa} \qquad \textit{on } V^0 \tag{58}$$

and

$$|f_1(p)|<\tilde{K}_0\tilde{\delta}^{-\kappa} \qquad \textit{on } V^1\ . \tag{59}$$

PROOF. Let f be a holomorphic function on $V^0\cap V^1$. We expand

$$f=\sum_{-\infty<\mu<\infty,\ I} f_{\mu I}s_0^{\mu}z_0^I\ .$$

We put

$$f_0=\sum_{\substack{\mu\geqq 0\\ I}} f_{\mu I}s_0^{\mu}z_0^I$$

$$f_1=\sum_{\substack{\mu<0\\ I}} f_{\mu I}s_0^{\mu}z_0^I\ .$$

Then f_i are holomorphic on V^i, $i=0,\ 1$, and satisfy (57).

By (53) there exists a positive number e such that if $|s_0|\leqq 1/2+e$, then $\tilde{\delta}(p)=\varepsilon-|z_0|$. We note that there exists a constant K_0' such that for every holomorphic function $g(s_0)=\sum_{-\infty<\mu<\infty} g_\mu s_0^\mu$ on $\{s_0; 1/2\leqq|s_0|\leqq 1/2+e\}$, we have

$$\sup_{|s_0|\leqq 1/2+e} \sum_{0\leqq\mu} g_\mu s_0^\mu \leqq K_0' \sup_{1/2\leqq|s_0|\leqq 1/2+e} |g(s_0)| \tag{60}$$

$$\sup_{|s_0|\geqq 1/2} \sum_{\mu<0} g_\mu s_0^\mu \leqq K_0' \sup_{1/2\leqq|s_0|\leqq 1/2+e} |g(s_0)|. \tag{61}$$

Let $(s_0, z_0)\in V^0$. If $|s_0|\leqq 1/2+e$, then

$$\begin{aligned} |f_0(s_0, z_0)| &\leqq K_0' \sup_{1/2\leqq|s_0|\leqq 1/2+e} |f(s_0, z_0)| \qquad \text{(by (60))} \\ &\leqq K_0' \sup_{1/2\leqq|s_0|\leqq 1/2+e} \tilde{\delta}(s_0, z_0)^{-\kappa} \\ &(=K_0'(\varepsilon-|z_0|)^{-\kappa})=K_0'\tilde{\delta}^{-\kappa}. \end{aligned} \tag{62}$$

If $|s_0|\geqq 1/2+e$, then similarly as above,

$$\begin{aligned} |f_0(s_0, z_0)| &\leqq |f(s_0, z_0)|+|f_1(s_0, z_0)| \\ &\leqq \tilde{\delta}^{-\kappa}+K_0'(\varepsilon-|z_0|)^{-\kappa} \\ &\leqq (1+K_0')\tilde{\delta}^{-\kappa} \qquad \text{(by (56))}. \end{aligned} \tag{63}$$

Hence (58) holds. On the other hand, if $p\in V^1-V^0$, then $|z_0(p)|<|z_1(p)|$ and $|s_0(p)|\geqq 1/2$ ($s_i(p)$, $z_i(p)$ denote the values of s_j, z_j at p). Hence by (61),

$$|f_1(p)|\leqq K_0'(\varepsilon-|z_0|)^{-\kappa}\leqq K_0'\tilde{\delta}(p)^{-\kappa}. \tag{64}$$

Therefore, for any $p\in V^1$ we have

$$|f_1(p)|\leqq K_0'(\varepsilon-|z_0|)^{-\kappa}\leqq K_0'\tilde{\delta}(p)^{-\kappa}. \tag{65}$$

Hence we obtain (58) and (59) for $\tilde{K}_0=K_0'+1$.

Let $N_{V/X}|_A$ denote the restriction of $N_{V/X}$ to A. We denote by $\underline{N_{V/X}}$ ($\underline{N_{V/X}|_A}$) the sheaf of germs of holomorphic sections of $N_{V/X}$ ($N_{V/X}|_A$).

LEMMA 3. *The restriction map* $\rho_0: H^0(V, \underline{N_{V/X}})\to H^0(A, \underline{N_{V/X}|_A})$ *is surjective.*

PROOF. We have the following exact sequence.

$$0 \longrightarrow N_{V/X}|_A \longrightarrow N_{A/X} \longrightarrow N_{A/V} \longrightarrow 0.$$

Since $N_{A/X}\cong \bigoplus_{\gamma=1}^{m+n} H^{a_\gamma}$, $N_{A/V}\cong \bigoplus_{\alpha=1}^{m} H^{a_\alpha}$, and

$$\operatorname{Hom}\Big(\bigoplus_{\beta=m+1}^{m+n} H^{a_\beta}, \bigoplus_{\alpha=1}^{m} H^{a_\alpha}\Big)=0, \tag{66}$$

it follows that

$$N_{V/X}|_A\cong \bigoplus_{\beta=m+1}^{m+n} H^{a_\beta}. \tag{67}$$

We have the following exact sequence.

$$(68)\qquad 0 \longrightarrow \mathcal{J}^{\nu+1}\otimes \underline{N_{V/X}} \longrightarrow \mathcal{J}^{\nu}\otimes \underline{N_{V/X}} \longrightarrow \mathcal{J}^{\nu}/\mathcal{J}^{\nu+1}\otimes \underline{N_{V/X}|_A} \longrightarrow 0\,.$$

Since $\mathcal{J}^{\nu}/\mathcal{J}^{\nu+1}\cong \underline{S^{\nu}(N_{A/V}^*)}$ (S^{ν} denotes the ν-th symmetric tensor product), by (67) we have

$$(69)\qquad H^1(A,\ \mathcal{J}^{\nu}/\mathcal{J}^{\nu+1}\otimes \underline{N_{V/X}|_A})=0 \qquad \text{for } \nu\geqq 1\,.$$

By Remmert's reduction theorem there exists a Stein space $\hat{V}$ and a proper holomorphic map $\hat{p}: V\to\hat{V}$ such that $\hat{p}|_{V-A}$ is a biholomorphic map from $V-A$ to $\hat{V}-\hat{p}(A)$. By Grauert's direct image theorem, $p_*^1\underline{N_{V/X}}$ is a coherent analytic sheaf on $\hat{V}$ whose support is $\hat{p}(A)$. Therefore for sufficiently large μ, $\boldsymbol{m}^{\mu}p_*^1\underline{N_{V/X}}=0$, where $\boldsymbol{m}$ denotes the maximal ideal sheaf of $\hat{p}(A)$ in $\hat{V}$. Thus, for sufficiently large μ, $p_*^0\mathcal{J}^{\mu}p_*^1\underline{N_{V/X}}=0$. Hence by Grauert's comparison theorem, we have

$$(70)\qquad H^1(V,\ \mathcal{J}^{\mu}\otimes \underline{N_{V/X}})=0\,, \qquad \text{for sufficiently large } \mu\,.$$

Combining (68) and (69) with (70), we obtain $H^1(V,\ \mathcal{J}\otimes \underline{N_{V/X}})=0$. Therefore the restriction map ρ_0 is surjective.

Let M be a complex manifold of dimension r and B a complex manifold of dimension 1. A complex manifold $\mathcal{V}$ over B with a smooth holomorphic map $\pi: \mathcal{V}\to B$ is called an analytic family of (locally closed) analytic submanifolds of dimension q of M if $\mathcal{V}$ is a locally closed analytic submanifold of $M\times B$ of codimension $r-q$ such that π is compatible with the projection $p_2: M\times B\to B$.

From now on we assume B is a polydisc containing 0, $M=X$, and $\pi^{-1}(0)=V$. We may assume that there are two coordinate neighbourhoods U_0 and U_1 in X such that

$$U_i\cap V=V^i\,, \qquad i=0,\ 1\,.$$

Further, we may choose the coordinates

$$(\tilde{s}_i,\ \tilde{z}_i,\ w_i)=(\tilde{s}_i,\ \tilde{z}_i^1,\ \cdots,\ \tilde{z}_i^m,\ w_i^1,\ \cdots,\ w_i^n)$$

of U_i such that they are holomorphic on a neighbourhood of the closure of U_i, V is defined by $w_i^1=\cdots=w_i^n=0$, and

$$\tilde{s}_i|_{V\cap U_i}=s_i$$
$$\tilde{z}_i^{\alpha}|_{V\cap U_i}=z_i\,, \qquad 1\leqq\alpha\leqq m\,.$$

Let the coordinate transformations be

$$(\tilde{s}_1,\ \tilde{z}_1)=\tilde{g}^*(\tilde{s}_0,\ \tilde{z}_0,\ w_0)$$
$$w_1=\tilde{h}^*(\tilde{s}_0,\ \tilde{z}_0,\ w_0)\,,$$

and

$$(\tilde{s}_0,\ \tilde{z}_0)=\tilde{g}(\tilde{s}_1,\ \tilde{z}_1,\ w_1)$$

$$w_0=\tilde{h}(\tilde{s}_1,\ \tilde{z}_1,\ w_1),$$

on $U_0 \cap U_1$.

From now on we identify (s_i, z_i) with $(\tilde{s}_i, \tilde{z}_i)$ and we denote a point of V^i by its coordinates (s_i, z_i).

Let $\pi: \mathcal{V}\to\Delta^l$ be an analytic family of locally closed submanifolds of X having

$$\Delta^l=\{t=(t_1, \cdots, t_l)\in \boldsymbol{C}^l\,;\ |t_j|<1,\ 1\leqq j\leqq l\}$$

as the parameter space. It is clear that the submanifolds $\pi^{-1}(t)$ are defined in a neighbourhood of V^i by simultaneous equations of the form

$$w_i^\beta=\varphi_i^\beta(s_i, z_i, t), \qquad \beta=1, \cdots, n,$$

where the $\varphi_i^\beta(s_i, z_i, t)$ are holomorphic functions of (s_i, z_i, t) defined in a neighbourhood of $V^i\times 0$ in $V^i\times\Delta^l$. We set

$$\varphi_i(s_i, z_i, t)=(\varphi_i^1(s_i, z_i, t), \cdots, \varphi_i^n(s_i, z_i, t)).$$

Take an arbitrary tangent vector

$$\frac{\partial}{\partial t}=\sum_{i=1}^{l}\gamma_i\frac{\partial}{\partial t_i}$$

of Δ^l at 0, and set

$$\psi_i(s_i, z_i, t)=\left.\frac{\partial\varphi_i(s_i, z_i, t)}{\partial t}\right|_{t=0}.$$

$\{\psi_i(s_i, z_i, t)\}$ $(i=0, 1)$ represents a holomorphic section of $N_{V/X}$.

We call $\{\psi_i(s_i, z_i)\}$ the infinitesimal displacement of V along the tangent vector $\partial/\partial t\in T_0(\Delta^l)$ and denote it by $\sigma(\partial/\partial t)$.

THEOREM 1. *There exists an analytic family $\pi: \mathcal{V}\to\Delta^l$ of locally closed submanifolds of X which satisfies the following conditions.*

0) $A\subset\pi^{-1}(0)$ *(We identify A with $A\times 0$.)*

1) $N_{A/\pi^{-1}(0)}\cong\bigoplus_{\alpha=1}^{m}H^{a_\alpha}$

2) *The infinitesimal displacement*

$$\sigma: T_0(\Delta^l)\longrightarrow H^0(A, \underline{N_{\pi^{-1}(0)/X}|_A})$$

is bijective.

PROOF. By Lemma 3, there exist $\beta_k\in H^0(V, \underline{N_{V/X}})$, $k=1, \cdots, l$, such that

$\rho_0(\beta_k)$ $k=1, \cdots, l$ are linearly independent and span $H^0(A, \underline{N_{V/X}|_A})$. To prove the theorem it suffices to show the existence of vectors of holomorphic functions

(71) $$\varphi_i(s_i, z_i, t)=(\varphi_i^1(s_i, z_i, t), \cdots, \varphi_i^n(s_i, z_i, t)),$$

defined on a neighbourhood of $V_i\times 0$ in $V_i\times \Delta^l$, such that

(i) $$\varphi_i(s_i, z_i, 0)=0$$

(ii) $$\left.\frac{\partial\varphi_i(s_i, z_i, t)}{\partial t^\rho}\right|_{t=0}=\beta_\rho^i(s_i, z_i), \qquad 1\leqq\rho\leqq l,$$

where $\{\beta_\rho^i(s_i, z_i)\}=\beta_\rho$, and

(iii) $$\varphi_1(\tilde{g}^*(s_0, z_0, \varphi_0(s_0, z_0, t)), t)=\tilde{h}^*(s_0, z_0, \varphi_0(s_0, z_0, t))$$

for $(s_0, z_0, \varphi_0(s_0, z_0, t))\in U_0\cap U_1$.

To prove this, it suffices to show the existence of the sequence $\{\varphi_i^\mu(s_i, z_i, t)\}$ satisfying the following properties.

(a) $\varphi_i^\mu(s_i, z_i, t)$ is a (vector valued) polynomial in t of degree μ whose coefficients are vectors of holomorphic functions on V^i, such that

(b) $$\varphi_i^1(s_i, z_i, t)=\sum_{\rho=1}^{l}\beta_\rho^i(s_i, z_i)t_\rho$$

(c) $$\varphi_i^\mu(s_i, z_i, 0)=0$$

(d) $$\varphi_i^\mu(s_i, z_i, t)\equiv\varphi_i^{\mu-1}(s_i, z_i, t) \qquad \mathrm{mod}\, t^\mu$$

(e) $$\varphi_1^\mu(\tilde{g}^*(s_0, z_0, \varphi_0^\mu(s_0, z_0, t)), t)\equiv\tilde{h}^*(s_0, z_0, \varphi_0^\mu(s_0, z_0, t)) \qquad \mathrm{mod}\, t^\mu$$

(f) $$\varphi_i^\mu(s_i, z_i, t)\ll\tilde{A}(t), \qquad \text{for } (s_i, z_i)\in V^i.$$

Here,

(72) $$\tilde{A}(t)=\frac{c}{16\tilde{b}}\sum_{k=1}\frac{\tilde{b}^k(t_1+\cdots+t_l)^k}{k^2}$$

c is a positive constant, and $\tilde{b}$ is a function on V with positive values which are determined later.

We prove this by a similar method as in the proof of Theorem 1. First, we set

(73) $$\varphi_i^1(s_i, z_i, t)=\sum_{\rho=1}\beta_\rho^i(s_i, z_i)t_\rho.$$

We may assume that $\beta_\rho^i(s_i, z_i)$ are holomorphic on a neighbourhood of the closure of V^i, hence if we choose c large enough (f) holds for $\varphi_i^1(s_i, z_i, t)$ and (a)$\sim$(e) are obviously satisfied. Assume that we have $\varphi_i^\mu(s_i, z_i, t)$ satisfying (a)$\sim$(f) for $\mu\leqq r$. We set

$$\tilde{\phi}(t)=\{\varphi_1^r(\tilde{g}^*(s_0, z_0, \varphi_0^r(s_0, z_0, t)), t)-\tilde{h}^*(s_0, z_0, \varphi_0^r(s_0, z_0, t))\}_{r+1}, \tag{74}$$

and

$$\tilde{\phi}^*(t)=\{\varphi_0^r(\tilde{g}(s_1, z_1, \varphi_1^r(s_1, z_1, t)), t)-\tilde{h}(s_1, z_1, \varphi_1^r(s_1, z_1, t))\}_{r+1}. \tag{75}$$

It is easy to check that

$$\tilde{\phi}(t)=-\tilde{\phi}^*(t)^t\left(\frac{\partial w_1}{\partial w_0}\right)_{w_0=0}, \quad \text{on } V^0\cap V^1. \tag{76}$$

We may assume that

$$\left(\frac{\partial w_1}{\partial w_0}\right)\Big|_{w_0=0}=\begin{pmatrix} s_0^{-a_{m+1}} & & 0 \\ & \ddots & \\ 0 & & s_0^{-a_{m+n}} \end{pmatrix}. \tag{77}$$

We are going to estimate $\tilde{\phi}(t)$. We expand $\tilde{g}^*(s_0, z_0, w_0)$ and $\tilde{h}^*(s_0, z_0, w_0)$ into power series of $w_0=(w_0^1, \cdots, w_0^n)$:

$$\tilde{g}^*(s_0, z_0, w_0)=\sum_{j_1,\cdots,j_n\geq 0} g^*_{j_1\cdots j_n}(s_0, z_0)(w_0^1)^{j_1}\cdots(w_0^n)^{j_n}, \tag{78}$$

$$\tilde{h}^*(s_0, z_0, w_0)=\sum_{j_1,\cdots,j_n\geq 0} h^*_{j_1\cdots j_n}(s_0, z_0)(w_0^1)^{j_1}\cdots(w_0^n)^{j_n}. \tag{79}$$

Since $\tilde{g}^*(s_0, z_0, w_0)$ and $\tilde{h}^*(s_0, z_0, w_0)$ are holomorphic on a neighbourhood of the closure of $U_0\cap U_1$, there exists a constant M such that

$$\sum_{j_1,\cdots,j_n\geq 0} g^*_{j_1\cdots j_n}(s_0, z_0)(w_0^1)^{j_1}\cdots(w_0^n)^{j_n}\ll\sum_{k=1}^{\infty}M^k(w_0^1+\cdots+w_0^n)^k, \tag{80}$$

and

$$\sum_{j_1,\cdots,j_n\geq 0} h^*_{j_1\cdots j_n}(s_0, z_0)(w_0^1)^{j_1}(w_0^2)^{j_2}\cdots(w_0^n)^{j_n}\ll\sum_{k=1}^{\infty}M^k(w_0^1+\cdots+w_0^n)^k. \tag{81}$$

First, we estimate

$$\{\tilde{h}^*(s_0, z_0, \varphi_0^r(s_0, z_0, t))\}_{r+1}. \tag{82}$$

Since

$$\{\tilde{A}(t)\}^k\ll\frac{c}{b}^{k-1}\tilde{A}(t) \tag{83}$$

for a natural number k, we have as in the proof of Theorem 1,

$$\{\tilde{h}^*(s_0, z_0, \varphi_0^r(s_0, z_0, t))\}_{r+1}\ll\tilde{A}(t)\sum_{k=1}^{\infty}\left(\frac{Mnc}{b}\right)^k Mn. \tag{84}$$

Hence, if

(85)
$$\frac{Mnc}{\tilde{b}}<\frac{1}{2}$$

we obtain

(86)
$$\{\tilde{h}^*(s_0, z_0, \varphi_0^r(s_0, z_0, t))\}_{r+1} \ll \frac{2M^2n^2c}{\tilde{b}}\tilde{A}(t).$$

Now we estimate

(87)
$$\{\varphi_1^r(\tilde{g}^*(s_0, z_0, \varphi_0^r(s_0, z_0, t)), t)\}_{r+1}.$$

We set

(88)
$$\tilde{g}^*(s_0, z_0, \varphi_0^r(s_0, z_0, t))=(s_1, z_1)+\zeta(s_0, z_0, t).$$

We have

(89)
$$\zeta(s_0, z_0, t) \ll \tilde{K}_1\tilde{A}(t)$$

for some constant $\tilde{K}_1$. We fix δ so that $0<\delta<1$, and we set

(90)
$$V^{i,\delta}=\{(s_i, z_i)\in V^i;\ |s_i|<2-\delta\}.$$

There exists a constant $\tilde{K}_2$ (>0) such that if $p=(s_1, z_1)\in V^0\cap V^{1,\delta}$, then the coefficients of

(92)
$$\varphi_1^r(s_1+y_0, z_1^1+y_1, \cdots, z_1^m+y_m, t)$$

are holomorphic functions of $y_0, \cdots, y_m$, $|y_0|\leqq K_2\tilde{\delta}(p), \cdots, |y_m|\leqq K_2\tilde{\delta}(p)$. From now on we denote $\tilde{\delta}(p)$ simply by $\tilde{\delta}$. Expanding the coefficients of $\varphi_1^r(s_1+y_0, z_1^1+y_1, \cdots, z_1^m+y_m, t)$ into power series in $y_0, \cdots, y_m$, we obtain the inequality,

(93)
$$\varphi_1^r(s_1+y_0, z_1^1+y_1, \cdots, z_1^m+y_m, t)$$
$$\ll \tilde{A}(t)\prod_{\alpha=0}^{\infty}\Big(1-\frac{y_\alpha}{\tilde{K}_2\tilde{\delta}}\Big)^{-1}, \qquad \text{on } V^0\cap V^{1,\delta}.$$

Combining this with (88) and (89), we obtain

(94)
$$\{\varphi_1^r(\tilde{g}^*(s_0, z_0, \varphi_0^r(s_0, z_0, t)), t)\}_{r+1}$$
$$=\{\varphi_1^r((s_1, z_1)+\zeta(s_0, z_0, t), t)-\varphi_1^r(s_1, z_1, t)\}_{r+1}$$
$$\ll \tilde{A}(t)\Big\{\Big(1-\frac{\tilde{K}_1A(t)}{\tilde{K}_2\tilde{\delta}}\Big)^{-(m+1)}-1\Big\}$$
$$\ll \sum_{k=1}^{\infty}\binom{m+k}{k}\Big(\frac{\tilde{K}_1}{\tilde{K}_2\tilde{\delta}}\Big)^k\tilde{A}(t)^{k+1}.$$

Hence, if we assume

(95) $$\frac{\tilde{K}_1 c}{\tilde{b}\tilde{K}_2\tilde{\delta}} < \frac{1}{2}$$

we have

(96) $$\{\varphi_1^r(\tilde{g}^*(s_0, z_0, \varphi_0^r(s_0, z_0, t)), t)\}_{r+1} \ll 2^{m+2}\frac{\tilde{K}_1 c}{\tilde{b}\tilde{K}_2\tilde{\delta}}\tilde{A}(t), \quad \text{on } V^0 \cap V^{1,\delta}.$$

Consequently, we have

(97) $$\tilde{\psi}(t) \ll \frac{c}{\tilde{b}}\Big(\tilde{K}_3 + 2^{m+2}\frac{\tilde{K}_1}{\tilde{K}_2\tilde{\delta}}\Big)\tilde{A}(t), \quad \text{on } V^0 \cap V^{1,\delta}$$

where $\tilde{K}_3 = 2M^2n^2$. Similarly, there are constants M', $\tilde{K}_1'$, $\tilde{K}_2'$, and $\tilde{K}_3'$, such that, if

(98) $$\frac{M'nc}{\tilde{b}} < \frac{1}{2}$$

and

(99) $$\frac{\tilde{K}_1' c}{\tilde{b}\tilde{K}_2'\tilde{\delta}} < \frac{1}{2},$$

we have

(100) $$\tilde{\psi}^*(t) \ll \frac{c}{\tilde{b}}\Big(\tilde{K}_3' + 2^{m+2}\frac{\tilde{K}_1'}{\tilde{K}_2'\tilde{\delta}}\Big)\tilde{A}(t), \quad \text{on } V^1 \cap V^{0,\delta}.$$

Combining this with (76), (77) and (97) we obtain

(101) $$\tilde{\psi}(t) \ll K_4 \frac{c}{\tilde{b}\tilde{\delta}}\tilde{A}(t), \quad \text{on } V^0 \cap V^1,$$

for some constant K_4.

We let

(108) $$\tilde{b} = b' c K_4^{-1} \tilde{\delta}^{-1},$$

where b' is a positive constant determined later. Then we have

(109) $$\tilde{\psi}(t) \ll \frac{1}{b'}\tilde{A}(t) \quad \text{on } V^0 \cap V^1.$$

By Lemma 2, there exist vectors of polynomials $\psi_0(s_0, z_0, t)$ and $\psi_1(s_1, z_1, t)$ which are homogeneous of degree $r+1$*) and satisfy

(110) $$\tilde{\psi}(t) = \psi_0(s_0, z_0, t)\begin{pmatrix} s_0^{-a_{m+1}} & & 0 \\ & \ddots & \\ 0 & & s_0^{-a_{m+n}} \end{pmatrix} - \psi_1(s_1, z_1, t),$$

*) The coefficients of $\psi_0(\psi_1)$ are holomorphic functions on V^0 (resp. V^1).

on $V^0 \cap V^1$,

(111) $$\psi_0(s_0, z_0, t) \ll \frac{2^{a\,m+n} K_5}{b'} \tilde{A}(t), \qquad \text{on } V^0,$$

and

(112) $$\psi_1(s_1, z_1, t) \ll \frac{2^{a\,m+n} K_5}{b'} \tilde{A}(t), \qquad \text{on } V^1.$$

Therefore, if we choose b' so that

(113) $$\frac{MnK_4\tilde{\delta}}{b'} < \frac{1}{2},$$

(114) $$\frac{M'nK_4\tilde{\delta}}{b'} < \frac{1}{2},$$

(115) $$\frac{\tilde{K}_1 K_4}{b\tilde{K}_2} < \frac{1}{2},$$

(116) $$\frac{\tilde{K}'_1 K_4}{b'\tilde{K}'_2} < \frac{1}{2},$$

and

(117) $$\frac{2^{a\,m+n} K_5}{b'} < 1,$$

we obtain

(118) $$\psi_0(s_0, z_0, t) \ll \tilde{A}(t), \qquad \text{on } V^0,$$

and

(119) $$\psi_1(s_1, z_1, t) \ll \tilde{A}(t), \qquad \text{on } V^1.$$

We set

(120) $$\varphi_0^{r+1}(s_0, z_0, t) = \varphi_0^r(s_0, z_0, t) + \psi_0(s_0, z_0, t),$$

and

(121) $$\varphi_0^{r+1}(s_1, z_1, t) = \varphi_1^r(s_1, z_1, t) + \psi_1(s_1, z_1, t).$$

It is clear that φ_0^{r+1}, and φ_1^{r+1} satisfy the conditions. q. e. d.

Let $\iota : \mathcal{V} \to X \times \Delta^l$ be the inclusion map and $p_1 : X \times \Delta^l \to X$ the projection onto the first factor. In Theorem 1, if in particular we assume $a_\beta = 0$ for $m+1 \leqq \beta \leqq m+n$, then by 2), $p_1 \circ \iota$ is a biholomorphic map on a neighbourhood of A. Since π is a 1-convex map on a neighbourhood of A in the sense of Knorr-Schneider (cf. [2]) there exists a holomorphically convex neighbourhood of A in X (cf. Satz 3.4 in [2]). Therefore combining this with Kodaira's theorem on analytic families of compact complex manifolds (cf. Main Theorem in [3]), we have

THEOREM 2. *Let X be a complex manifold containing a complex analytic submanifold A. Suppose that $A\cong \boldsymbol{P}^1$ and $N_{A/X}$ is seminegative in the sense that every rank* 1 *holomorphic subbundle of $N_{A/X}$ has nonpositive degree. Then A has a holomorphically convex neighbourhood in X.*

As a corollary we have

COROLLARY. *Under the above assumptions, if moreover there exists a neighbourhood of A in X which contains no compact complex subvariety of dimension $\geqq 1$ except for A, then A is exceptional in X.*

PROBLEM. Find an algebraic proof of Theorem 2. Namely, let X be a nonsingular projective algebraic variety over an algebraically closed field k. Assume that X contains a nonsingular rational curve A and $N_{A/X}$ is seminegative. Then, is there an algebraic space Y (in the sense of M. Artin) and a proper morphism $\varpi: X\to Y$ such that $\varpi(A)$ is a point?

Appendix.

We show an example of exceptional embedding of $\boldsymbol{P}^1$ whose normal bundle is not seminegative.

Let ζ be an inhomogeneous coordinate of $\boldsymbol{P}^1$. We set $U_0=\{\zeta;\ |\zeta|<2\}$ and $U_1=\{\zeta;\ 1/2<|\zeta|\leqq\infty\}$. We define an analytic space U by patching $U_0\times \boldsymbol{C}^2$ and $U_1\times \boldsymbol{C}^2$ along $(U_0\cap U_1)\times \boldsymbol{C}^2$ as follows,

$$u_1=\zeta^4u_0+\zeta v_0^2\,, \tag{1}$$

$$v_1=\zeta^{-1}v_0\,. \tag{2}$$

Here (ζ, u_0, v_0) and (ζ^{-1}, u_1, v_1) denote the coordinates of $U_0\times \boldsymbol{C}^2$ and $U_1\times \boldsymbol{C}^2$, respectively. Note that U is the total space of an analytic fiber bundle over $\boldsymbol{P}^1$ whose fiber is biholomorphic to $\boldsymbol{C}^2$ and structure group is the complex analytic diffeomorphism group of $\boldsymbol{C}^2$. U contains $\boldsymbol{P}^1$ as an analytic submanifold of codimension 2 which is defined by the local equations $u_1=v_1=0$ and $u_0=v_0=0$. Clearly $N_{\boldsymbol{P}^1/U}=H\oplus H^{-4}$.

In the similar way as above we define an analytic space V by the following equations:

$$u_1'=\zeta^4u_0'+\zeta v_0'\,, \tag{3}$$

$$v_1'=\zeta^{-2}v_0'\,. \tag{4}$$

V is the total space of an analytic vector bundle over $\boldsymbol{P}^1$. Since

$$\begin{pmatrix}1 & 0\\ -\zeta^{-3} & 1\end{pmatrix}\begin{pmatrix}\zeta^4 & \zeta\\ 0 & \zeta^{-2}\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & \zeta^3\end{pmatrix}=\begin{pmatrix}\zeta & 0\\ 0 & \zeta\end{pmatrix}, \tag{5}$$

$V \cong H^{-1} \oplus H^{-1}$. Therefore V contains $\boldsymbol{P}^1$ as an exceptional subset.

On the other hand, there exists a double covering from U to V, defined by $u_i' = u_i$ and $v_i' = v_i^2$ $(i=0, 1)$. Hence $\boldsymbol{P}^1$ has a holomorphically convex neighbourhood system in U whose member contains no compact analytic subset of positive dimension other than $\boldsymbol{P}^1$ itself. Therefore $\boldsymbol{P}^1$ is exceptional in U.

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Added in proof: During the submission of our paper H. Laufer has published the following two articles:

[6] Versal deformations for two-dimensional pseudoconvex manifolds, Ann. Scuola Norm. Sup. Pisa (4), **7** (1980), 511-521.
[7] On $\boldsymbol{CP}^1$ as an exceptional set, Ann. of Math. Studies, Princeton Univ. Press., 1981, 261-275.

In [7] he studies the exceptional embedding of $\boldsymbol{P}^1$ in detail and gives interesting examples similar to that we gave here in the appendix. Bearing Kodaira's theory in mind, the connection between [6] and ours will be clear.