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On the remainder estimates of asymptotic formulas for eigenvalues of operators associated with strongly elliptic sesquilinear forms

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1. Introduction.

The purpose of this paper is to improve the results of K. Maruo and H. Tanabe [4], K. Maruo [5] on the eigenvalue distribution.

Let Ω be a bounded domain in real space R^n with generic point $x=(x_1, \cdots, x_n)$. We denote by $\alpha=(\alpha_1, \cdots, \alpha_n)$ a multi-index of length $|\alpha|=\alpha_1+\cdots+\alpha_n$ and use the notations

$$D^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}, \qquad D_k=-\sqrt{-1}\,\partial/\partial x_k .$$

For an integer $m\geqq 0$, $H_m(\Omega)$ is to be the set of all functions whose distribution derivatives of order up to m belong to $L^2(\Omega)$ and we introduce in it the usual norm

$$\|u\|_m=\|n\|_{m,\Omega}=\left(\int_\Omega \sum_{|\alpha|\leq m}|D^\alpha u|^2 dx\right)^{1/2}.$$

$\mathring{H}_m(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H_m(\Omega)$.

Let B be a symmetric integro-differential sesquilinear form of order m with bounded coefficients

$$B[u, v]=\int_\Omega \sum_{|\alpha|,|\beta|\leq m} a_{\alpha\beta}(x)D^\alpha u\overline{D^\beta v}dx$$

satisfying

$$B[u, u]\geqq\delta\|u\|_m^2 \qquad \text{for any } u\in V$$

where δ is some positive constant and V is some closed subspace of $H_m(\Omega)$ containing $\mathring{H}_m(\Omega)$. Let A be the operator associated with this sesquilinear form: an element u of V belongs to $D(A)$ and $Au=f\in L^2(\Omega)$ if $B[u, v]=(f, v)$ is valid for any $v\in V$. It is well known that A is a positive definite self-adjoint operator in $L^2(\Omega)$. In this paper we assume that Ω has restricted cone property and that $2m>n$ as in [4], [5]. For $t>0$ let $N(t)$ be the number of the eigenvalues of A which do not exceed t.

Maruo and Tanabe [4] and Maruo [5] investigated the asymptotic distribution of eigenvalues of the operator A, and under various smoothness asumptions on the coefficients of B deduce formulas with remainder estimates. In particular, Maruo [5] proved that

$$N(t)=c_0t^{n/2m}+O(t^{(n-\theta)/2m}) \qquad \text{as } t\to\infty \tag{1.1}$$

for any $\theta<(h+1)/(h+3)$ if the coefficients $a_{\alpha\beta}$ ($|\alpha|=|\beta|=m$) belong to the class C^{1+h} ($0<h\leqq 1$) in some domain containing Ω. But it is impossible to prove (1.1) for $1/2\leqq\theta<1$ by the method used in [4], [5], even if all the coefficients $a_{\alpha\beta}$ belong to the class $C^\infty(\bar{\Omega})$. Without assuming $2m>n$, Metivier [6] proved that (1.1) holds for $\theta=h/(h+1)$ if the coefficients $a_{\alpha\beta}$ belong to the class C^h. But in [6] no information about asymptotic behavior of the spectral function of A was obtained. Seeley [8] proved that for Laplace operator under the Dirichlet boundary condition (1.1) holds for $\theta=1$, which is the best estimate. Let $\mathscr{B}^\tau(\Omega)$ ($0<\tau<\infty$) be the space of functions u in Ω such that $\partial_x^\alpha u$ are bounded and continuous for $|\alpha|\leqq[\tau]$ and $|\partial_x^\alpha u(x)-\partial_x^\alpha u(y)|/|x-y|^{\dot{\tau}}$ ($|x-y|\leqq 1$, $x, y\in\Omega$) bounded for $|\alpha|=[\tau]$, when $\dot{\tau}=\tau-[\tau]>0$. Here and in what follows, for a real number τ we denote by $[\tau]$ the largest integer which is not larger than τ, set $\dot{\tau}=\tau-[\tau]$ and use the notation $\partial_x^\alpha=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$. The conclusion of this paper is that (1.1) holds for any $\theta<\tau/(\tau+2)$ if the coefficients of B satisfy the following conditions. For $|\alpha|=|\beta|=m$ $a_{\alpha\beta}$ belongs to $\mathscr{B}^\tau(\Omega)$, and when $\tau>2$, for $|\alpha|+|\beta|=2m-1$ $a_{\alpha\beta}$ belongs to $\mathscr{B}^{\tau'}(\Omega)$ ($\tau'=(\tau-2)/2$). In the proof of our theorem the result of Tsujimoto [11] plays an important role.

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2. Main theorem.

As was stated in the introduction let Ω be a bounded domain in R^n having the restricted cone property and it is assumed that $2m>n$. For $x\in\Omega$ we write $\delta(x)=\min\{1, \operatorname{dist}(x, \partial\Omega)\}$. Suppose that

$$\int_\Omega \delta(x)^{-p}dx<\infty \tag{2.1}$$

for some positive number $p<1$ which will be specified later.

We state smoothness assumption on the coefficients of B:

$$\textit{For} \quad |\alpha|=|\beta|=m \qquad a_{\alpha\beta} \textit{ belongs to } \mathscr{B}^\tau(\Omega) \tag{2.2}$$

and when $\tau>2$,

$$\textit{for} \quad |\alpha|+|\beta|=2m-1 \qquad a_{\alpha\beta} \textit{ belongs to } \mathscr{B}^{\tau'}(\Omega) \tag{2.3}$$

where $\tau'=(\tau-2)/2$.

MAIN THEOREM. *In the situation stated above the following asymptotic formula for* $N(t)$ *holds as* $t\rightarrow\infty$:

$$N(t)=c_0 t^{n/2m}+O(t^{(n-\theta)/2m})$$

for any number θ *satisfying* $0<\theta<\tau/(\tau+2)$ *where*

$$c_0=\int_{\Omega} c(x)dx\ ,$$

$$c(x)=(2\pi)^{-n}\int_{\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^{\alpha+\beta}<1} d\xi\ .$$

3. Some lemmas.

Following Maruo and Tanabe [4], we extend the operator A to a mapping on V to V^* where V^* is the antidual of V. This extended operator which is again denoted by A is definded by

$$B[u, v]=(Au, v) \qquad \text{for any } v\in V$$

where the bracket on the right stands for the duality between V^* and V in this case. Identifying $L^2(\Omega)$ with its antidual we may consider $V\subset L^2(\Omega)\subset V^*$ algebraically and topologically, and as is easily seen V is a dense subspace of V^* under this convention. The resolvent of A thus extended is a bounded linear operator on V^* to V. We use the same notation as those of [4] to denote various norms. Let λ be a complex number which is not on the positive real axis and $d(\lambda)$ be the distance from the point λ to the positive real axis.

LEMMA 3.1. *There exists a constant* C *such that*

(i) $\|(A-\lambda)^{-1}\|_{L^2\rightarrow L^2}\leqq 1/d(\lambda)$,

(ii) $\|(A-\lambda)^{-1}\|_{L^2\rightarrow V}\leqq C|\lambda|^{1/2}/d(\lambda)$,

(iii) $\|(A-\lambda)^{-1}\|_{V^*\rightarrow V}\leqq C|\lambda|/d(\lambda)$,

(iv) $\|(A-\lambda)^{-1}\|_{V^*\rightarrow L^2}\leqq C|\lambda|^{1/2}/d(\lambda)$.

PROOF. See [4].

REMARK. Since all coefficients of B are bounded, it follows that for any $u, v\in V$ $|B[u, v]|\leqq K\|u\|_m\|v\|_m$ for some constant K. We note that the constant C of Lemma 3.1 is independent of Ω and depends on only δ, K.

LEMMA 3.2. *Let* T *be a bounded operator on* V^* *to* V. *Then* T *has a kernel* M *in the following sense:*

$$(Tf)(x)=\int_{\Omega}M(x, y)f(y)dy \qquad for\ f\in L^2(\Omega).$$

$M(x, y)$ is continuous in $\bar{\Omega}\times\bar{\Omega}$ and there exists a constant C such that for any $x, y\in\Omega$

$$|M(x, y)|\leqq C\|T\|_{V^*\to V}^{n^2/4m^2}\|T\|_{V^*\to L^2}^{n/2m-n^2/4m^2}\|T\|_{L^2\to V}^{n/2m-n^2/4m^2}\|T\|_{L^2\to L^2}^{(1-n/2m)^2}.$$

PROOF. See [4].

REMARK. For $x^0\in\mathrm{R}^n$, $0<r\leqq 1$ we set $S_r(x^0)=\{x:|x-x^0|<r\}$. When $V=\mathring{H}_m(S_r(x^0))$, the constant C of Lemma 3.2 is independent of x^0, r.

4. Estimates of resolvent kernels ——1.

Let A_0 be the operator associated with B under the Dirichlet boundary conditions. That is, an element u of $\mathring{H}_m(\Omega)$ belongs to $D(A_0)$ and $A_0u=f\in L^2(\Omega)$ if $B[u, v]=(f, v)$ is valid for any $v\in\mathring{H}_m(\Omega)$. Let K_λ, K_λ^0 be the kernels of $(A-\lambda)^{-1}$ and $(A_0-\lambda)^{-1}$ respectively.

LEMMA 4.1. *For any $p>0$ the following inequality holds:*

$$|K_\lambda(x^0, x^0)-K_\lambda^0(x^0, x^0)|\leqq C_p\frac{|\lambda|^{n/2m}}{d(\lambda)}(|\lambda|^{1-1/2m}/\delta(x^0)d(\lambda))^p$$

$$for\ any\ x^0\in\Omega,\ |\lambda|\geqq 1, \qquad (4.1)$$

where C_p is a constant depending on p but not on x^0 and λ.

PROOF. See [4].

Next, we consider B on $S_r(x^0)$ $(0<r\leqq\delta(x^0))$. Let $A_{1,r}$ be the operator associated with B under the Dirichlet boundary conditions. By definition for any $u, v\in\mathring{H}_m(S_r(x^0))$ we have

$$B[u, v]=(A_{1,r}u, v)$$

where the bracket on the right denotes the pairing between the antidual $H_{-m}(S_r(x^0))$ of $\mathring{H}_m(S_r(x^0))$ and $\mathring{H}_m(S_r(x^0))$ in this case. Let $K_\lambda^{1,r}$ be the kernel of $(A_{1,r}-\lambda)^{-1}$.

LEMMA 4.2. *For any $q>0$ the following inequality holds:*

$$|K_\lambda^0(x^0, x^0)-K_\lambda^{1,r}(x^0, x^0)|\leqq C_q\frac{|\lambda|^{n/2m}}{d(\lambda)}(|\lambda|^{1-1/2m}/r\,d(\lambda))^q$$

$$|\lambda|\geqq 1, \qquad (4.2)$$

where C_q is a constant depending on q but not x^0, r and λ.

PROOF. We extend the operator $(A_{1,r}-\lambda)^{-1}$ to a mapping on $H_{-m}(\Omega)$ to $\mathring{H}_m(\Omega)$. This extended operator which is denoted by $\overline{(A_{1,r}-\lambda)^{-1}}$ is defined by

$$\overline{(A_{1,r}-\lambda)^{-1}}f=\begin{cases}(A_{1,r}-\lambda)^{-1}\hat{f} & \text{on } S_r(x^0)\quad\text{for } f\in L^2(\Omega)\\ 0 & \text{on } \Omega-S_r(x^0)\end{cases}$$

where $\hat{f}=f|_{S_r(x^0)}$. For an operator S on $H_{-m}(\Omega)$ to $\mathring{H}_m(\Omega)$ we denote by $\|S\|_{(-m,m)}$, $\|S\|_{(-m,0)}$, $\|S\|_{(0,m)}$, $\|S\|_{(0,0)}$ the norms of S considered as an operator on $H_{-m}(\Omega)$ to $\mathring{H}_m(\Omega)$, on $H_{-m}(\Omega)$ to $L^2(\Omega)$, on $L^2(\Omega)$ to $\mathring{H}_m(\Omega)$, on $L^2(\Omega)$ to $L^2(\Omega)$ respectively. Moreover in the case of $\Omega=S_r(x^0)$ we denote by $\|S\|_{(-m,m)_0}$, $\|S\|_{(-m,0)_0}$, $\|S\|_{(0,m)_0}$, $\|S\|_{(0,0)_0}$ them respectively. We note that for $j=-m$, 0, $k=0$, m

$$\|\overline{(A_{1,r}-\lambda)^{-1}}\|_{(j,k)}\leqq\|(A_{1,r}-\lambda)^{-1}\|_{(j,k)_0}\,. \tag{4.3}$$

We take a real valued function $\xi\in C_0^\infty(\Omega)$ such that $\xi(x)\equiv1$ for $|x|<1/2$, $\xi(x)\equiv0$ for $|x|>3/4$ and put $\xi_r(x)=\xi(x-x^0/r)$. Let $u=(A_0-\lambda)^{-1}f-\overline{(A_{1,r}-\lambda)^{-1}}f$ and $v=\xi_r u$. Now,

$$\begin{aligned}B[u,\xi_r v]-\lambda(u,\xi_r v)&=B[(A_0-\lambda)^{-1}f,\xi_r v]-\lambda((A_0-\lambda)^{-1}f,\xi_r v)\\&\quad-B[(A_{1,r}-\lambda)^{-1}\hat{f},\xi_r v]+\lambda((A_{1,r}-\lambda)^{-1}\hat{f},\xi_r v)=0\,.\end{aligned}$$

So that

$$B[v,v]-\lambda(v,v)=B[v,v]-B[u,\xi_r v]\,.$$

Hence, noting (4.3) the present lemma can be proved just as Lemma 4.2 of [4].

5. Approximation of coefficients by smooth functions.

We set for $|\alpha|=|\beta|=m$

$$\tilde{a}_{\alpha\beta}=\sum_{|\gamma|\leqq[\tau]}\frac{1}{\gamma\,!}(\partial_x^\gamma a_{\alpha\beta})(x^0)(x-x^0)^\gamma\,, \tag{5.1}$$

and for $|\alpha|+|\beta|=2m-1$

$$\tilde{a}_{\alpha\beta}=\sum_{|\gamma|\leqq[\tau']}\frac{1}{\gamma\,!}(\partial_x^\gamma a_{\alpha\beta})(x^0)(x-x^0)^\gamma\,. \tag{5.2}$$

Moreover for $|\alpha|+|\beta|\leqq2m-2$ we put $\tilde{a}_{\alpha\beta}=0$. If $\tau<2$, for $|\alpha|+|\beta|\leqq2m-1$ we put $\tilde{a}_{\alpha\beta}=0$. We shall consider the following symmetric sesquilinear form:

$$B_2'[u,v]=\sum_{|\alpha|,|\beta|\leqq m}\int_{S_r(x^0)}\tilde{a}_{\alpha\beta}(x)D^\alpha u\overline{D^\beta v}dx\,.$$

We note that there exist constants r^0, C^0 independent of x^0 such that for $0<r\leqq r^0<1$, $x\in S_r(x^0)$ and $\xi\in\mathrm{R}^n$

$$\sum_{|\alpha|=|\beta|=m}\tilde{a}_{\alpha\beta}(x)\xi^{\alpha+\beta}\geqq C^0|\xi|^{2m}\,. \tag{5.3}$$

Hence we have for any $u\in\mathring{H}_m(S_r(x^0))$

$$B_2'[u,u]\geqq C^0E^0\|u\|_m^2-\lambda^0\|u\|_0^2$$

where E^0, λ^0 are constants independent of x^0, r. We put

$$B_2[u, v]=B_2'[u, v]+\lambda^0(u, v).$$

Let $A_{2,r}$ be the operator associated with B_2 under the Dirichlet boundary conditions. By definition for any $u, v\in \mathring{H}_m(S_r(x^0))$ we have $B_2[u, v]=(A_{2,r}u, v)$ where the bracket on the right denotes the pairing between the antidual $H_{-m}(S_r(x^0))$ of $\mathring{H}_m(S_r(x^0))$ and $\mathring{H}_m(S_r(x^0))$ this case. We denote by $K_\lambda^{2,r}$ the resolvent kernel of $A_{2,r}$. For $\theta\in(0, \pi/2)$ we set $\Lambda=\{\lambda: \theta\leqq\arg\lambda\leqq 2\pi-\theta, |\lambda|>0\}$.

Lemma 5.1. *There exist constants C_1, C_2 independent of x^0, r such that*

$$|K_\lambda^{2,r}(x, y)|\leqq C_1|\lambda|^{n/2m-1}e^{-C_2|x-y||\lambda|^{1/2m}} \tag{5.4}$$

for $x, y\in S_r(x^0)$, $\lambda\in\Lambda$.

Proof. We set for $\eta\in\mathrm{R}^n$

$$\begin{aligned}B_\eta[u, v]&=B_2[e^{-\langle x,\eta\rangle}u, e^{\langle x,\eta\rangle}v]\\&=\sum_{|\alpha|,|\beta|\leq m}\int_{S_r(x^0)}\tilde{a}_{\alpha\beta}(x)(D+i\eta)^\alpha u\overline{(D-i\eta)^\beta v}\,dx\end{aligned}$$

where $\langle x, \eta\rangle=\sum_{j=1}^n x_j\eta_j$. Let A_η be the operator associated with B_η under the Dirichlet boundary conditions. That is, an element u of $\mathring{H}_m(S_r(x^0))$ belongs to $D(A_\eta)$ and $A_\eta u=f\in L^2(S_r(x^0))$ if $B_\eta[u, v]=(f, v)$ is valid for any $v\in\mathring{H}_m(S_r(x^0))$. Let

$$R_\eta[u, v]=B_\eta[u, v]-B_2[u, v].$$

We note that for $0\leqq k\leqq m$, $v\in\mathring{H}_m(S_r(x^0))$

$$\|v\|_{m-k}\leqq C|\lambda|^{-k/2m}(\|v\|_m+|\lambda|^{1/2}\|v\|_0)$$

where C is a constant independent of r, x^0. Hence we get

$$|R_\eta[v, v]|\leqq C\sum_{k=1}^m|\lambda|^{-k/2m}|\eta|^k(\|v\|_m+|\lambda|^{1/2}\|v\|_0)^2.$$

On the other hand, we have for $v\in\mathring{H}_m(S_r(x^0))$

$$\begin{aligned}|B_2[v, v]-\lambda(v, v)|&\geqq C\frac{d(\lambda)}{|\lambda|}(\|v\|_m+|\lambda|^{1/2}\|v\|_0)^2\\&\geqq C(\|v\|_m+|\lambda|^{1/2}\|v\|_0)^2\end{aligned}$$

for $\lambda\in\Lambda$. Hence, for sufficiently small C_2 we have that

$$|B_\eta[v, v]-\lambda(v, v)|\geqq C(\|v\|_m+|\lambda|^{1/2}\|v\|_0)^2 \tag{5.5}$$

for $\lambda\in\Lambda$, $v\in\mathring{H}_m(S_r(x^0))$, η such that $|\eta|\leqq C_2|\lambda|^{1/2m}$. According to Lax-Milgram theorem, $A_\eta-\lambda$ has a bounded inverse defined in the whole of $H_{-m}(S_r(x^0))$. From (5.5) we have for $\lambda\in\Lambda$

$$\begin{aligned}
&\text{(i)} && \|(A_\eta-\lambda)^{-1}\|_{(-m,m)_0}\leqq C\,, \\
&\text{(ii)} && \|(A_\eta-\lambda)^{-1}\|_{(-m,0)_0}\leqq C|\lambda|^{-1/2}\,, \\
&\text{(iii)} && \|(A_\eta-\lambda)^{-1}\|_{(0,m)_0}\leqq C|\lambda|^{-1/2}\,, \\
&\text{(iv)} && \|(A_\eta-\lambda)^{-1}\|_{(0,0)_0}\leqq C|\lambda|^{-1}\,.
\end{aligned} \tag{5.6}$$

Let K_λ^η be the kernel of $(A_\eta-\lambda)^{-1}$. From Lemma 3.2 and (5.6) we have for $x, y\in S_r(x^0)$, $\lambda\in\Lambda$

$$|K_\lambda^\eta(x, y)|\leqq C|\lambda|^{n/2m-1} \tag{5.7}$$

where C is a constant independent of x^0, r. We note that

$$K_\lambda^{2,r}(x, y)=e^{<x-y,\eta>}K_\lambda^\eta(x, y)\,.$$

From (5.7), setting $\eta=-C_2|\lambda|^{1/2m}(x-y)/|x-y|$, we have the present lemma.

q. e. d.

Let $e_{2,r}(x, y, t)$ be the spectral function of $A_{2,r}$.

LEMMA 5.2. *There exists a constant C independent of x^0, r such that for any $t>1$*

$$|e_{2,r}(x^0, x^0, t)-c(x^0)t^{n/2m}|\leqq C\frac{1}{r}t^{(n-1)/2m} \tag{5.8}$$

where

$$c(x^0)=(2\pi)^{-n}\int_{\sum_{|\alpha|=|\beta|=m}a_{\alpha\beta}(x^0)\xi^{\alpha+\beta}<1}d\xi\,.$$

PROOF. Let $\tilde{A}_{2,r}(x, D)u=\sum_{|\alpha|,|\beta|\leq m}D^\beta(\tilde{a}_{\alpha\beta}(x)D^\alpha u)$. By definition of $A_{2,r}$ we have $D(A_{2,r})\supset C_0^\infty(S_r(x^0))$ and for any $u\in C_0^\infty(S_r(x^0))$, $\tilde{A}_{2,r}(x, D)u=A_{2,r}u$. From Lemma 5.1 we see that $A_{2,r}$ satisfies the assumption of the main theorem of [**11**]. Noting (5.3) and Remark 2.2 of [**11**], from the main theorem of [**11**] we have the present lemma.

LEMMA 5.3. *There exists a constant C independent of x^0, r and λ such that*

$$|K_\lambda^{2,r}(x^0, x^0)-c'(x^0)(-\lambda)^{n/2m-1}|\leqq C\frac{1}{r}|\lambda|^{(n-1)/2m}/d(\lambda) \qquad \text{for } |\lambda|\geqq 1 \tag{5.9}$$

where

$$c'(x^0)=\frac{(n\pi/2m)}{\sin(n\pi/2m)}c(x^0)\,.$$

PROOF. We note that

$$K_\lambda^{2,r}(x^0, x^0)=\int_0^\infty\frac{de_{2,r}(x^0, x^0, t)}{t-\lambda}\,,$$

$$c'(x^0)(-\lambda)^{n/2m-1}=c(x^0)\int_0^\infty \frac{d(t^{n/2m})}{t-\lambda}.$$

Hence we have

$$I=|K_\lambda^{2,r}(x^0, x^0)-c'(x^0)(-\lambda)^{n/2m-1}|$$

$$\leqq\left|\int_0^\infty \frac{de_{2,r}(x^0, x^0, t)}{t-\lambda}-c(x^0)\int_0^\infty \frac{d(t^{n/2m})}{t-\lambda}\right|$$

$$\leqq\int_0^\infty \frac{|e_{2,r}(x^0, x^0, t)-c(x^0)t^{n/2m}|}{|t-\lambda|^2}dt.$$

Using (5.5), we have

$$I\leqq C\frac{1}{r}\int_0^\infty \frac{t^{(n-1)/2m}}{|t-\lambda|^2}dt$$

$$=C\frac{1}{r}|\lambda|^{(n-1)/2m-1}\int_0^\infty \frac{t^{(n-1)/2m}}{|t-\lambda/|\lambda||^2}dt.$$

Setting $\lambda/|\lambda|=e^{i\phi}$, we have

$$\int_0^\infty \frac{t^{(n-1)/2m}}{|t-e^{i\phi}|^2}dt=\int_0^\infty \frac{t^{(n-1)/2m}}{t^2+2t\cdot\cos(\pi-\phi)+1}dt$$

$$=\frac{\sin\{(n-1)(\pi-\phi)/2m\}\cdot\pi}{\sin(\pi-\phi)\cdot\sin\{(n-1)\pi/2m\}}$$

$$\leqq C|\lambda|/d(\lambda).$$

q. e. d.

6. Estimates of resolvent kernels ——2.

In this section we shall estimate the difference between the resolvent kernels of $A_{1,r}$ and those of $A_{2,r}$.

LEMMA 6.1. *There exists a constant C independent of* x^0, r *and* λ *such that for* $0<r<r^0\delta(x^0)$, $|\lambda|>1$

$$|K_\lambda^{1,r}(x^0, x^0)-K_\lambda^{2,r}(x^0, x^0)|$$
$$\leqq C\left(\frac{|\lambda|}{d(\lambda)}\right)^2|\lambda|^{(n/2m-1)}\{r^\tau+r^{\tau'}|\lambda|^{-1/2m}+|\lambda|^{-2/2m}\}. \tag{6.1}$$

PROOF. We note that for $x\in S_r(x^0)$, $0<r<r^0\delta(x^0)$

$$|a_{\alpha\beta}(x)-\tilde{a}_{\alpha\beta}(x)|\leqq Cr^\tau, \quad \text{if } |\alpha|=|\beta|=m. \tag{6.2}$$

$$|a_{\alpha\beta}(x)-\tilde{a}_{\alpha\beta}(x)|\leqq Cr^{\tau'}, \quad \text{if } |\alpha|+|\beta|=2m-1. \tag{6.3}$$

For $f\in L^2(S_r(x^0))$ we set $u=(A_{1,r}-\lambda)^{-1}f-(A_{2,r}-\lambda)^{-1}f$. Then we have

$$
\begin{aligned}
&-B[u, u]+\lambda(u, u) \\
&\quad=B[(A_{2,r}-\lambda)^{-1}f, u]-B_2[(A_{2,r}-\lambda)^{-1}f, u] \\
&\quad=\sum_{|\alpha|+|\beta|\geqq 2m-1}\int_{S_r(x^0)}\{a_{\alpha\beta}(x)-\tilde{a}_{\alpha\beta}(x)\}D^\alpha((A_{2,r}-\lambda)^{-1}f)\overline{D^\beta u}\,dx \\
&\qquad+\sum_{|\alpha|+|\beta|\leqq 2m-2}\int_{S_r(x^0)}a_{\alpha\beta}(x)D^\alpha((A_{2,r}-\lambda)^{-1}f)\overline{D^\beta u}\,dx \\
&\qquad-\lambda^0\int_{S_r(x^0)}(A_{2,r}-\lambda)^{-1}f\cdot\bar{u}\,dx\,.
\end{aligned}
$$

Hence, using (6.2), (6.3), we have

$$
\begin{aligned}
&|B[u, u]-\lambda(u, u)| \\
&\quad\leqq C\{r^\tau\|(A_{2,r}-\lambda)^{-1}f\|_m\|u\|_m+r^{\tau'}\sum_{k=0}^{1}\|(A_{2,r}-\lambda)^{-1}f\|_{m-k}\|u\|_{m-1+k} \\
&\qquad+\sum_{k=0}^{2}\|(A_{2,r}-\lambda)^{-1}f\|_{m-k}\|u\|_{m-2+k}\}\,.
\end{aligned}
$$

Since $(A_{1,r}-\lambda)^{-1}f$, $(A_{2,r}-\lambda)^{-1}f\in\mathring{H}_m(S_r(x^0))$, there exists a constant C independent of r, x^0 and λ such that for $j=1, 2$

$$
\begin{aligned}
&\sum_{k=0}^{j}\|(A_{2,r}-\lambda)^{-1}f\|_{m-k}\|u\|_{m-j+k} \\
&\quad\leqq C|\lambda|^{-j/2m}(\|(A_{2,r}-\lambda)^{-1}f\|_m+|\lambda|^{1/2}\|(A_{2,r}-\lambda)^{-1}f\|_0)(\|u\|_m+|\lambda|^{1/2}\|u\|_0)\,.
\end{aligned}
$$

Hence we have

$$|B[u, u]-\lambda(u, u)|\leqq C\{r^\tau+r^{\tau'}|\lambda|^{-1/2m}+|\lambda|^{-2/2m}\}Q$$

where

$$Q=(\|(A_{2,r}-\lambda)^{-1}f\|_m+|\lambda|^{1/2}\|(A_{2,r}-\lambda)^{-1}f\|_0)(\|u\|_m+|\lambda|^{1/2}\|u\|_0)\,.$$

The present lemma can be proved just as Lemma 6.2 of [4], based upon this inequality. q. e. d.

From (4.1), (4.2), (5.9), (6.1), we have for $0<r<r^0\delta(x^0)$

$$
\begin{aligned}
&|K_\lambda(x^0, x^0)-c'(x^0)(-\lambda)^{n/2m-1}| \\
&\quad\leqq C\frac{|\lambda|^{n/2m}}{d(\lambda)}\Big\{C_p\Big(\frac{|\lambda|^{1-1/2m}}{\delta(x^0)d(\lambda)}\Big)^p+C_q\Big(\frac{|\lambda|^{1-1/2m}}{r\,d(\lambda)}\Big)^q+r^\tau\frac{|\lambda|}{d(\lambda)} \qquad (6.4)\\
&\qquad+r^{\tau'}\frac{|\lambda|^{1-1/2m}}{d(\lambda)}+\frac{|\lambda|^{1-2/2m}}{d(\lambda)}+\frac{1}{r}|\lambda|^{-1/2m}\Big\}, \qquad |\lambda|\geqq 1\,.
\end{aligned}
$$

7. Proof of the main theorem.

Let $e(x, y, t)$ be the spectral function of A.

LEMMA 7.1. *For any* $\theta\in(0, \tau/(\tau+2))$ *there exists a constant* C_{θ} *independent of* x *and* t *such that*

$$|e(x, x, t)-c(x)t^{n/2m}|\leqq C_{\theta}t^{(n-\theta)/2m}\delta(x)^{-\theta} \tag{7.1}$$

for any $x\in\Omega$ *and* $t>1$.

PROOF. Now we follow the method of Agmon [2]. Let $L(\xi)$ be an oriented curve in the complex plane from $\bar{\xi}$ to $\xi=t+i\mu$ not intersecting $[0, \infty)$. Then, for $t>0$, $\mu>0$ we have

$$\begin{aligned}&\left|e(x, x, t)-\frac{1}{2\pi i}\int_{L(\xi)}c'(x)(-z)^{n/2m-1}dz\right|\\&\leqq\frac{1}{2\pi}\left|\int_{L(\xi)}\{K_{z}(x, x)-c'(x)(-z)^{n/2m-1}\}dz\right|\\&\quad+(1+\pi^{-2})^{1/2}\cdot\mu\cdot|K_{\xi}(x, x)|\\&=I_{1}+I_{2}.\end{aligned}$$

In proving (7.1) we may assume without loss of generality that $t^{1/2m}\delta(x)>1$. We take $\mu=\mu_{x}(t)=t(t^{1/2m}\delta(x))^{-\theta}$ and

$$L(\xi)=\{z=t+iu:\mu_{x}(t)\leqq|u|\leqq t\}\cup\{z:|z|=\sqrt{2}t,\ \mathrm{Re}\,z\leqq t\}.$$

Using (6.4) we have

$$\begin{aligned}I_{2}&\leqq C\mu_{x}(t)\{|K_{\xi}(x, x)-c'(x)(-\xi)^{n/2m-1}|+|c'(x)||\xi|^{n/2m-1}\}\\&\leqq C\mu_{x}(t)\Big[\frac{|\xi|^{n/2m}}{\mu_{x}(t)}\Big\{C_{p}\Big(\frac{|\xi|^{1-1/2m}}{\delta(x)\mu_{x}(t)}\Big)^{p}+C_{q}\Big(\frac{|\xi|^{1-1/2m}}{r\mu_{x}(t)}\Big)^{q}+r^{\tau}\frac{|\xi|}{\mu_{x}(t)}\\&\quad+r^{\tau'}\frac{|\xi|^{1-1/2m}}{\mu_{x}(t)}+\frac{|\xi|^{1-2/2m}}{\mu_{x}(t)}+\frac{1}{r}|\xi|^{-1/2m}\Big\}+|\xi|^{n/2m-1}\Big].\end{aligned}$$

Noting that $t\leqq|\xi|\leqq\sqrt{2}t$, we get

$$\begin{aligned}I_{2}\leqq C\,t^{n/2m}\{&C_{p}(t^{1/2m}\delta(x))^{-(1-\theta)p}+C_{q}t^{-(1-\theta)q/2m}\cdot r^{-q}\cdot\delta(x)^{\theta q}\\&+r^{\tau}t^{\theta/2m}\delta(x)^{\theta}+r^{\tau'}t^{(-1+\theta)/2m}\delta(x)^{\theta}+t^{(-2+\theta)/2m}\delta(x)^{\theta}\\&+r^{-1}t^{-1/2m}+t^{-\theta/2m}\delta(x)^{-\theta}\}.\end{aligned}$$

On the other hand, again by (6.4) we have

$$I_1\leqq C\int_{L(\xi)}\frac{|z|^{n/2m}}{d(z)}\Big\{C_p\Big(\frac{|z|^{1-1/2m}}{\delta(x)d(z)}\Big)^p+C_q\Big(\frac{|z|^{1-1/2m}}{r\,d(z)}\Big)^q+r^\tau\frac{|z|}{d(z)}$$
$$+r^{\tau'}\frac{|z|^{1-1/2m}}{d(z)}+\frac{|z|^{1-2/2m}}{d(z)}+\frac{1}{r}|z|^{-1/2m}\Big\}|dz|$$
$$\leqq C\Big\{\int_{\mu_x(t)}^t+\int_{\substack{|z|=\sqrt{2}t\\ \mathrm{Re}\,z\leqq t}}\Big\}$$
$$=I_{1,1}+I_{1,2}\,.$$

We note that

$$\int_{\mu_x(t)}^t\frac{1}{u^{p+1}}du\leqq C\,t^{-p}(t^{1/2m}\delta(x))^{\theta p}\,,$$
$$\int_{\mu_x(t)}^t\frac{1}{u}du\leqq C\log(t^{1/2m}\delta(x))\,.$$

Hence we have

$$I_{1,1}\leqq C\,t^{n/2m}\{C_p(t^{1/2m}\delta(x))^{-(1-\theta)p}+C_qt^{-(1-\theta)q/2m}r^{-q}\delta(x)^{\theta q}$$
$$+r^\tau t^{\theta/2m}\delta(x)^\theta+r^{\tau'}t^{(-1+\theta)/2m}\delta(x)^\theta+t^{(-2+\theta)/2m}\delta(x)^\theta$$
$$+\frac{1}{r}t^{-1/2m}\log(t^{1/2m}\delta(x))\Big\}$$

and

$$I_{1,2}\leqq C\,t^{n/2m}\Big\{C_p(t^{1/2m}\delta(x))^{-p}+C_q(t^{1/2m}r)^{-q}+r^\tau$$
$$+r^{\tau'}t^{-1/2m}+t^{-2/2m}+\frac{1}{r}t^{-1/2m}\Big\}\,.$$

1st case: $(t^{1/2m}\delta(x))^{-2\theta/\tau}<r^0\delta(x)$.

Then, setting $r=(t^{1/2m}\delta(x))^{-2\theta/\tau}$, we have

$$I_{1,1}\leqq C\,t^{n/2m}\{C_p(t^{1/2m}\delta(x))^{-(1-\theta)p}+C_qt^{\frac{(-1+\theta+2\theta/\tau)q}{2m}}\delta(x)^{(\theta+2\theta/\tau)q}$$
$$+t^{-\theta/2m}\delta(x)^{-\theta}+t^{-1/2m}(t^{1/2m}\delta(x))^{\theta(1-2\tau'/\tau)}$$
$$+t^{\frac{-1+2\theta/\tau}{2m}}\delta(x)^{2\theta/\tau}\cdot\log(t^{1/2m}\delta(x))\}\,.$$

We note that $(t^{1/2m}\delta(x))^{\theta(1-2\tau'/\tau)}=(t^{1/2m}\delta(x))^{2\theta/\tau}$ and

$$(t^{1/2m}\delta(x))^{2\theta/\tau}\log(t^{1/2m}\delta(x))\leqq C(t^{1/2m}\delta(x))^{1-\theta}\,.$$

Hence, taking $p=\theta/(1-\theta)$, $q=\dfrac{\theta}{1-\theta-2\theta/\tau}$, we have

$$I_{1,1} \leqq C\, t^{(n-\theta)/2m} \delta(x)^{-\theta}. \tag{7.2}$$

By the same way, we have the same estimate for I_2, $I_{1,2}$.

2nd case: $(t^{1/2m}\delta(x))^{-2\theta/\tau} \geqq r^0 \delta(x)$.

Then we have

$$\delta(x) \leqq C\, t^{\frac{-2\theta}{2m(2\theta+\tau)}}. \tag{7.3}$$

Setting $r = r^0\delta(x)$, we have

$$\begin{aligned} I_{1,1} \leqq C\, t^{n/2m} \{ & C_p(t^{1/2m}\delta(x))^{-(1-\theta)p} + C_q(t^{1/2m}\delta(x))^{-(1-\theta)q} \\ & + t^{\theta/2m}\delta(x)^{\theta+\tau} + t^{(-1+\theta)/2m}\delta(x)^{\tau'+\theta} + t^{(-2+\theta)/2m}\delta(x)^{\theta} \\ & + (t^{1/2m}\delta(x))^{-1} \cdot \log (t^{1/2m}\delta(x)) \}. \end{aligned}$$

Hence, noting (7.3), taking $p = q = \theta/(1-\theta)$, we get (7.2). Moreover we get the same estimate for I_2, $I_{1,2}$ by the same way. Hence we have

$$\left| e(x, x, t) - \frac{1}{2\pi i} \int_{L(\xi)} c'(x)(-z)^{n/2m-1} dz \right| \leqq C\, t^{(n-\theta)/2m} \delta(x)^{-\theta}.$$

Finally noting that

$$\begin{aligned} & \left| \frac{1}{2\pi i} \int_{L(\xi)} (-z)^{n/2m-1} dz - t^{n/2m} \frac{\sin (n\pi/2m)}{n\pi/2m} \right| \\ & \leqq C\, t^{(n-\theta)/2m} \delta(x)^{-\theta}, \end{aligned}$$

we obtain the desired estimate. q. e. d.

If (2.1) is satisfied for θ in Lemma 7.1, then integrating (7.1) over Ω we immediately obtain the asymptotic formula for $N(t)$ described in the main theorem.

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