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Construction of a solution of random transport equation with boundary condition

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§ 0. Introduction.

Concerning the analysis of wave propagation in random media, S. Ogawa [7] introduced a new type of partial differential equation of first order with a random coefficient:

$$\begin{aligned}(0.1)\qquad & \frac{\partial u}{\partial t}(t, x; \omega)+\{\dot{B}_t(\omega)+b(t, x)\}\frac{\partial u}{\partial x}(t, x; \omega) \\ & =c(t, x)u(t, x; \omega)+d(t, x), \\ & (t, x)\in[0, T]\times R^1, \qquad T<\infty,\end{aligned}$$

where $\dot{B}_t(\omega)$ is the white noise. He constructed a solution of Cauchy problem of equation (0.1) with given initial data

$$(0.2)\qquad u(0, x; \omega)=\phi(x).$$

His main tools are a stochastic integral which he defined and the concept of the differentiation $\dfrac{\partial X_t}{\partial B_t}$ of a stochastic process X_t with respect to the Brownian motion B_t.

Here, in this paper, we consider a natural extension of his equation:

$$\begin{aligned}(0.3)\qquad & \frac{\partial u}{\partial t}(t, x; \omega)+\sum_{i,j=1}^{d}\{a_{ij}(t, x)\dot{B}_t^j(\omega)+b_i(t, x)\}\frac{\partial u}{\partial x_i}(t, x; \omega) \\ & =c(t, x)u(t, x; \omega)+d(t, x), \\ & (t, x)\in[0, T]\times G,\end{aligned}$$

with initial data (0.2) and boundary conditions at ∂G, where G is a given region in R^d ($d\geqq 1$) and $\dot{B}_t(\omega)=\{\dot{B}_t^j(\omega)\}_{j=1}^d$ is the d-dimensional white noise. More precisely, we construct a solution of the equation (0.3) for

(i) the case of multidimensional bounded domain G with Dirichlet boundary condition in § 3, § 4, and for

(ii) the case of half line $G=(0, \infty)$ with Neumann boundary condition in § 5.

Our main tool for the case (i) is a time reversed process $Y_t(\omega)$ of a diffusion process $X_t(\omega)$ which is determined by a stochastic differential equation:

$$\begin{cases} dX_t=a(t, X_t)\circ dB_t+b(t, X_t)dt \\ X_0=x, \end{cases} \tag{0.4}$$

where $a(t, x)=\{a_{ij}(t, x)\}_{1\le i,j\le d}$, $b(t, x)=\{b_i(t, x)\}_{1\le i\le d}$

and $a(t, X_t)\circ dB_t$ is a symmetric differential (K. Itô [5]). S. Ogawa defined a stochastic differential $d^{1/2}B_t$ of order $\frac{1}{2}$ of Brownian motion. In our case, this differential is equivalent to the symmetric differential. The equation (0.3) can be considered as an equation for the transport along a path of X_t and the time reversed process Y_t plays the same role as a characteristic curve by means of which we solve a non random partial differential equation of type,

$$\frac{\partial u}{\partial t}(t, x)+\sum_{i=1}^{d} A_i(t, x)\frac{\partial u}{\partial x_i}(t, x)=c(t, x)u(t, x)+d(t, x), \tag{0.5}$$
$$(t, x)\in[0, T]\times G.$$

Here, the time reversed process Y_t is defined in an intrinsic way different from that obtained by transforming the transition probabilities.

For the case (ii), we treat the simplest equation:

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}\dot{B}_t=0, \quad (t, x)\in[0, T]\times(0, \infty) \tag{0.6}$$

and show that, among reflecting Brownian motions, that defined by A. V. Skorohod [4] is available for our problem. We also consider averaged boundary value problem of equation (0.6).

I wish to express my thanks to Professor T. Ueno for his continual encouragement and valuable suggestions. I also thank Professor S. Ogawa for introducing me to random transport problems.

§ 1. The regularity of a solution of stochastic differential equation with respect to parameters.

Here, we state some results of Y. N. Blagovescenskii and M. I. Freidlin [1] in a convenient form (Theorem 1.1, 1.2) and extend them a little (Theorem 1.1′). Let $\{B_t(\omega)\}_{t\ge0}=\{(B_t^i(\omega))_{i=1}^d\}_{t\ge0}$ be a d-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_{[r,t]}$ be the smallest σ-field generated by $\{B_s(\omega)-B_r(\omega);\ r\leqq s\leqq t\}$ and the set of all P-null sets. When $r=0$, we write simply $\mathcal{F}_t$ in place of $\mathcal{F}_{[0,t]}$.

For $r\in[0, T]$, we consider a stochastic differential equation:

$$(1.1)\qquad \begin{cases} dx_t(\omega)=a(t, x_t(\omega))dB_t(\omega)+b(t, x_t(\omega))dt\,, & t\in[r, T] \\ x_r(\omega)=x\,, \quad x\in R^d\,, & \end{cases}$$

where $a(t, x)=\{a_{ij}(t, x)\}_{1\leq i,j\leq d}$ is a matrix and $b(t, x)=\{b_i(t, x)\}_{1\leq i\leq d}$ is a vector defined on $[0, T]\times R^d$. We denote by $x(r, t, x;\omega)$ or $x(r, t, x)$, $0\leqq r\leqq t\leqq T$, $x\in R^d$, a solution $x_t(\omega)$ of equation (1.1).

THEOREM 1.1 [1]. *Suppose that components of the matrix $a(t, x)$ and of the vector $b(t, x)$ are bounded measurable functions and satisfy the Lipschitz condition uniformly in x. Then, for each $r\in[0, T]$, there exists a random function $x(r, t, x)$ which satisfies equation* (1.1) *and is continuous in $(t, x)\in[r, T]\times R^d$ with probability* 1.

We consider the following function spaces for a non-negative integer k,

$$C_b^{0,k}([0, T]\times R^d, S)=\{f(t, x)\in C^{0,k}([0, T]\times R^d);\ f \text{ is } S\text{-valued and } D^\alpha f \text{ is bounded for each } \alpha=(\alpha_1, \cdots, \alpha_d) \text{ with } |\alpha|\leqq k\},$$

where
$$D^\alpha=\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_d}\right)^{\alpha_d} \quad\text{and}\quad |\alpha|=\sum_{i=1}^d \alpha_i$$

for $\alpha=(\alpha_1, \cdots, \alpha_d)$ with non-negative integers α_i $(1\leqq i\leqq d)$,

$$C_0^{0,k}([0, T]\times R^d, S)=\{f(t, x)\in C^{0,k}([0, T]\times R^d);\ f \text{ is } S\text{-valued and has a compact support in } [0, T]\times R^d\}\,,$$

where
$$S=R^d\otimes R^d \quad\text{or}\quad R^d\,.$$

THEOREM 1.2 [1]. *If $a(t, x)$ and $b(t, x)$ belong to classes $C_b^{0,k+1}([0, T]\times R^d, R^d\otimes R^d)$ and $C_b^{0,k+1}([0, T]\times R^d, R^d)$, respectively, then for almost all ω and all $\alpha=(\alpha_1, \cdots, \alpha_d)$, $|\alpha|\leqq k$, there exist derivatives $D^\alpha x(r, t, x)$ which are continuous in (t, x) and satisfy*

$$D^\alpha x(r, t, x)=D^\alpha x+\int_r^t D^\alpha\{a(s, x(r, s, x))\}\,dB_s+\int_r^t D^\alpha\{b(s, x(r, s, x))\}\,ds\,.$$

For the proof of Theorem 1.1, following estimate is available, that is, for each positive integer n, there exists a positive constant C_n which is independent of r and

$$(1.2)\qquad E[|x(r, t_1, x)-x(r, t_2, y)|^{2n}] \leqq C_n(|x-y|^{2n}+|t_1-t_2|^n)\,,\quad t_1, t_2\in[r, T]\,,\quad x, y\in R^d\,.$$

Theorem 1.1 follows from this estimate and A. N. Kolmogorov's theorem which is stated below.

THEOREM (Kolmogorov). *Let* $x_\mu(\omega)$ *be an* R^d*-valued separable random field defined for* $\mu \in R^m$. *Then, in order that* $x_\mu(\omega)$ *be continuous in* μ *with probability* 1, *it suffices that for certain* $\gamma>0$, $\varepsilon>0$, *the inequality*

$$E[|x_\mu(\omega)-x_{\mu'}(\omega)|^\gamma] \leqq C|\mu-\mu'|^{m+\varepsilon}$$

should hold.

we can extend Theorem 1.1 as below.

THEOREM 1,1'. *Under the same condition as in Theorem* 1.1, *there exists a random function* $x(r, t, x)$ *which satisfies equation* (1.1) *and is continuous in* (r, t, x), $0 \leqq r \leqq t \leqq T$, $x \in R^d$, *with probability* 1.

PROOF. It is well-known that under the condition of Theorem 1.1 equation (1.1) has a unique solution in the following sense, if there exist two solutions $x_1(r, t, x)$ and $x_2(r, t, x)$, then

$$x_1(r, t, x)=x_2(r, t, x) \quad \text{a.s.}$$

for each r, t, x with $0 \leqq r \leqq t \leqq T$, $x \in R^d$. Noting this fact, we can show the equality,

$$x(r_1, t, x)=x(r_2, t, x(r_1, r_2, x)) \quad \text{a.s.} \tag{1.3}$$

for each r_1, r_2, t, x with $0 \leqq r_1 \leqq r_2 \leqq t \leqq T$, $x \in R^d$. Since $x(r, t, x)$ is independent of $\mathcal{F}_r$, it follows from the estimate (1.2) that for $\mathcal{F}_r$-measurable random variables $\alpha(\omega)$, $\beta(\omega)$,

$$E[|x(r, t, \alpha(\omega))-x(r, t, \beta(\omega))|^{2n}|\mathcal{F}_r] \leqq C_n|\alpha(\omega)-\beta(\omega)|^{2n} \quad \text{a.s.} \tag{1.4}$$

For r_1, r_2, t with $0 \leqq r_1 \leqq r_2 \leqq t \leqq T$, using (1.2), (1.3) and (1.4),

$$\begin{aligned} E[|x(r_1, t, x)-x(r_2, t, x)|^{2n}] & \\ &= E[|x(r_2, t, x(r_1, r_2, x))-x(r_2, t, x)|^{2n}] \\ &\leqq C_n E[|x(r_1, r_2, x)-x|^{2n}] \\ &\leqq C_n|r_1-r_2|^n. \end{aligned} \tag{1.5}$$

By the estimates (1.2) and (1.5), we have

$$E[|x(r_1, t_1, x)-x(r_2, t_2, y)|^{2n}] \leqq C'_n(|x-y|^{2n}+|r_1-r_2|^n+|t_1-t_2|^n),$$

$$r_1, r_2 \in [0, T], \qquad t_1 \in [r_1, T], \qquad t_2 \in [r_2, T], \qquad x, y \in R^d.$$

Hence Kolmogorov's theorem completes the proof of the theorem.

§ 2. A time reversed process.

In this section, we assume that two functions $a(t, x)$ and $b(t, x)$ belong to classes $C_0^{0,3}([0, T]\times R^d, R^d\otimes R^d)$ and $C_0^{0,2}([0, T]\times R^d, R^d)$, respectively. Let $X(r, t, x)$ $(0\leqq r\leqq t\leqq T,\ x\in R^d)$ be a solution of a stochastic differential equation:

$$(2.1)\qquad \begin{cases} dX_t=a(t, X_t)\circ dB_t+b(t, X_t)dt, & t\in[r, T] \\ X_r=x, & x\in R^d. \end{cases}$$

REMARK 2.1. The equation (2.1) is equivalent to (2.1)$'$,

$$(2.1)'\qquad dX_t=a(t, X_t)dB_t+\bar{b}(t, X_t)dt,$$

where $\bar{b}(t, x)=b(t, x)+\frac{1}{2}(a'a)(t, x)$ and $(a'a)$ is a vector with components $(a'a)_i=\sum_{k,j=1}^{d}\frac{\partial a_{ij}}{\partial x_k}a_{kj}$. Hence, the equation (2.1) has a unique solution.

By the uniqueness and the continuity of $X(r, t, x)$ from the argument in § 1, we have

$$(2.2)\qquad X(r, t, x)=X(s, t, X(r, s, x)) \quad \text{for all } r, s, t, x \text{ with } 0\leqq r\leqq s\leqq t\leqq T,\ x\in R^d \text{ with probability 1.}$$

Noting that $X(r, t, x)$ is differentiable in x by Theorem 1.2, we set

$$J(r, t, x)=\det\left(\frac{\partial X_j}{\partial x_i}(r, t, x)\right)_{1\leq i,j\leq d},$$

the Jacobian of $X(r, t, x)$, where X_j is the j-th component of X.

LEMMA 2.1. *$J(r, t, x)$ is positive for each r, t, x with $0\leqq r\leqq t\leqq T$, $x\in R^d$.*

PROOF. By Theorem 1.2, $\frac{\partial X_j}{\partial x_i}(r, t, x)$ $(1\leqq i, j\leqq d)$ are continuous in x and satisfy

$$\frac{\partial X_j}{\partial x_i}(r, t, x)=\delta_{ij}+\sum_{k,l=1}^{d}\int_r^t\frac{\partial a_{jk}}{\partial x_l}(s, X(r, s, x))\frac{\partial X_l}{\partial x_i}(r, s, x)\circ dB_s^k$$
$$+\sum_{l=1}^{k}\int_r^t\frac{\partial b_j}{\partial x_l}(s, X(r, s, x))\frac{\partial X_l}{\partial x_i}(r, s, x)ds,$$

where δ_{ij} is Kronecker's δ.

Then, by the definition of J, we have

$$(2.3)\qquad J(r, t, x)=1+\sum_{i=1}^{d}\int_r^t\{J(r, s, x)(\operatorname{div} a^i)(s, X(r, s, x))\}\circ dB_s^i$$
$$+\int_r^t J(r, s, x)(\operatorname{div} b)(s, X(r, s, x))ds,$$

where a^i is the i-th column vector of matrix a. Since the solution is given by

$$J(r, t, x)=\exp\left\{\sum_{i=1}^{d}\int_r^t(\operatorname{div} a^i)(s, X(r, s, x))\circ dB_s^i\right.$$

$$\left.+\int_r^t(\operatorname{div} b)(s, X(r, s, x))ds\right\},$$

$J(r, t, x)$ is positive.

LEMMA 2.2.(1) *For each r and t, $0\leqq r\leqq t\leqq T$, $X(r, t, x)$ is a homeomorphism of R^d onto R^d.*

PROOF. Since $a(t, x)$ and $b(t, x)$ have compact supports in $[0, T]\times R^d$, there exists positive $N=N(\omega)$ such that

$$X(r, t, x)=x \qquad \text{for all } \ r, t, x \ \text{ with } \ 0\leqq r\leqq t\leqq T, |x|>N.$$

Therefore, for an unbounded sequence $\{x_n\}_{n=1}^{\infty}$ of points in R^d, $\{X(r, t, x_n)\}_{n=1}^{\infty}$ is also an unbounded sequence. We have the conclusion by Lemma 2.1 and Carathéodory [**2**].

DEFINITION 2.1. We define

$$Y(r, t, y)=X^{-1}(r, t, \cdot)(y) \qquad \text{for each } \ r, t, y \ \text{ with } \ 0\leqq r\leqq t\leqq T, y\in R^d,$$

where X^{-1} is the inverse function of $X(r, t, x)$ as a function of x. We call Y the time reversed process of X.

PROPOSITION 2.1. *The random function $Y(r, t, y)$ satisfies,*

(i) $Y(s, t, y)=X(r, s, Y(r, t, y))$ *for* r, s, t, y *with* $0\leqq r\leqq s\leqq t\leqq T$, $y\in R^d$.

(ii) $Y(r, t, y)=Y(r, s, Y(s, t, y))$ *for* r, s, t, y *with* $0\leqq r\leqq s\leqq t\leqq T$, $y\in R^d$.

(iii) $Y(r, t, y)$ *is continuous in* (r, t, y).

PROOF. (i) Since $y=X(r, t, Y(r, t, y))=Y(r, t, X(r, t, y))$ and $X(r, t, x)=X(s, t, X(r, s, x))$, we have

$$\begin{aligned} Y(s, t, y)&=Y(s, t, X(r, t, Y(r, t, y)))\\ &=Y(s, t, X(s, t, X(r, s, Y(r, t, y))))\\ &=X(r, s, Y(r, t, y)). \end{aligned}$$

(ii) The second assertion follows from (i) and the definition of Y.

(iii) By (i), $Y(r, t, y)=X(0, r, Y(0, t, y))$, and using (ii)

(1) The author thanks a referee for pointing out an error in the proof of this lemma.

$$Y(0,\ t,\ y)=Y(0,\ t,\ Y(t,\ T,\ X(t,\ T,\ y)))$$
$$=Y(0,\ T,\ X(t,\ T,\ y)),$$

and hence, $Y(r, t, y)=X(0, r, Y(0, T, X(t, T, y)))$.
Since $X(r, t, x)$ is continuous in (r, t, x) and $Y(0, T, y)$ is continuous in y, $Y(r, t, y)$ is also continuous in (r, t, y).

Let ∂ be a point which does not belong to R^d. For a bounded region G of R^d, we consider a stopped process $Y_G(r, t, y)$ of $Y(r, t, y)$ in $\bar{G}$ as follows.

$$Y_G(r,\ t,\ y)=\begin{cases} Y(r,\ t,\ y) & \text{for} \quad r\in[\max(0,\ \sigma(t,\ y)),\ t] \\ \partial & \text{for} \quad r\in[0,\ \sigma(t,\ y)) \quad \text{if} \quad \sigma(t,\ y)>0 \\ \partial & \text{for} \quad r=0-\ , \end{cases}$$

where

$$\sigma(t,\ y)=\sigma(t,\ y\ ;\ \omega)=\begin{cases} \max\{s\ ;\ Y(s,\ t,\ y)\notin G\ ,\ s\in[0,\ t]\} \\ 0- \quad \text{if the set in the bracket is empty}\,. \end{cases}$$

Hereafter, we shall use the following notations (i)$\sim$(iii).

(i) For each r and t, $0\leqq r\leqq t\leqq T$, we set

$$G_X(r,\ t)=G_X(r,\ t\ ;\ \omega)=\{x\in G\ ;\ X(r,\ s,\ x)\in G \quad \text{for all} \quad s\in[r,\ t]\}\ ,$$

$$G_Y(r,\ t)=G_Y(r,\ t\ ;\ \omega)=\{y\in G\ ;\ Y(s,\ t,\ y)\in G \quad \text{for all} \quad s\in[r,\ t]\}\ .$$

(ii) For each $t\in[0, T]$ and $h\in[0, T-t]$, we set

$$\tilde{G}(t)=\tilde{G}(t\ ;\ \omega)=\{y\in G\ ;\ \sigma(t,\ y)\geqq 0\}=G-G_Y(0,\ t)\,,$$

$$\tilde{G}(t,\ h)=\tilde{G}(t,\ h\ ;\ \omega)=\{Y_G(t,\ t+h,\ y)\ ;\quad y\in\tilde{G}(t+h)\}\cap G\,.$$

(iii) For positive ε and $D\subset G$, we set

$$(D)^{\varepsilon}=\{x\in G\ ;\ |x-y|<\varepsilon \text{ for some } y\in D\}\ .$$

LEMMA 2.3. *For each r and t, $0\leqq r\leqq t\leqq T$, we have*

$$G_X(r,\ t)=\{Y(r,\ t,\ y)\ ;\quad y\in G_Y(r,\ t)\}\ .$$

PROOF. We obtain the conclusion by Proposition 2.1 immediately.

LEMMA 2.4. *For each $t\in[0, T]$ and $h\in[0, T-t]$, $\tilde{G}(t, h)$ is included in $\tilde{G}(t)$. For any $\varepsilon>0$, there exists $h_0=h_0(\omega)>0$ such that*

$$\tilde{G}(t)-\tilde{G}(t,\ h)\subset(\partial G)^{\varepsilon} \qquad \textit{for all} \quad h\in[0,\ h_0]\,.$$

PROOF. Since

$$\tilde{G}(t,\ h)=\{y\in G\ ;\ Y(s,\ t+h,\ y)\notin G \text{ for some } s\in[0,\ t) \quad \text{and}$$

$$Y(s,\ t+h,\ y)\in G \text{ for all } s\in[t,\ t+h]\}$$

$$\cup\{y\in G\,;\ Y(s,\ t+h,\ y)\notin G \text{ for some } s\in[t,\ t+h]\}$$

$$=\tilde{G}_1(t+h)\cup\tilde{G}_2(t+h)\,,$$

we have

$$\begin{aligned}\tilde{G}(t,\ h)&=[\{Y_G(t,\ t+h,\ y)\,;\ y\in\tilde{G}_1(t+h)\}\cup\{Y_G(t,\ t+h,\ y)\,;\ y\in\tilde{G}_2(t+h)\}]\cap G\\&=\{Y_G(t,\ t+h,\ y)\,;\ y\in\tilde{G}_1(t+h)\}\\&=\{x\in\tilde{G}(t)\,;\ X(t,\ s,\ x)\in G \text{ for some } s\in[t,\ t+h]\}\,,\end{aligned}$$

therefore we see that $\tilde{G}(t,\ h)$ is included in $\tilde{G}(t)$. Since $X(t,\ s,\ x)$ is continuous in $(t,\ s,\ x)$ and G is bounded, we have the second statement easily.

§ 3. Dirichlet problem.

Let G be a bounded region in R^d with boundary ∂G of class C^3. We consider an initial-boundary value problem of a random transport equation:

$$\frac{\partial u}{\partial t}+\sum_{i,j=1}^{d}(a_{ij}(t,\ x)\dot{B}_t^j+b_i(t,\ x))\frac{\partial u}{\partial x_i}=c(t,\ x)u+d(t,\ x)\,,\qquad (t,\ x)\in[0,\ T]\times G\,, \tag{3.1}$$

$$\lim_{t\to 0}u(t,\ x\,;\ \omega)=\phi(x)\qquad \text{for}\quad x\in G\,, \tag{3.2}$$

$$\lim_{x(\in G)\to\xi}u(t,\ x\,;\ \omega)=\psi(\xi)\qquad \text{for}\quad (t,\ \xi)\in(0,\ T]\times\partial G\,, \tag{3.3}$$

for given functions a, b, c, d, ϕ and ψ which satisfy,

(i) $a(t,\ x)\in C^{0,3}([0,\ T]\times\bar{G},\ R^d\otimes R^d)$, and

$$\sum_{i,j,k=1}^{d}a_{ik}(t,\ \xi)a_{jk}(t,\ \xi)\nu_i(\xi)\nu_j(\xi)>0\qquad \text{for each}\quad (t,\ \xi)\in(0,\ T]\times\partial G\,,$$

where $\nu=(\nu_1,\ \cdots,\ \nu_d)$ is the inward normal,

(ii) $b(t,\ x)\in C^{0,2}([0,\ T]\times\bar{G},\ R^d)$,

(iii) $c(t,\ x)$ and $d(t,\ x)\in C([0,\ T]\times\bar{G},\ R^1)$,

(iv) $\phi(x)\in C(\bar{G})$, $\quad\psi(\xi)\in C(\partial G)$.

We denote by Q the set of all $\{\mathcal{F}_t\}$-quasi martingales, that is, $Q_t\in Q$ is a measurable process written as a sum of a square integrable $\{\mathcal{F}_t\}$-martingale and an $\{\mathcal{F}_t\}$-adapted process which has a sample path of bounded variation.

DEFINITION 3.1. If a real valued random function $u=u(t, x; \omega)$, $(t, x, \omega) \in [0, T]\times G\times\Omega$, satisfies the following (u.1)$\sim$(u.5), then u is called a solution of equation (3.1) with conditions (3.2) and (3.3).

(u.1) $u(t, \cdot; \omega)\in L^1(G)$ for each (t, ω),

(u.2) $\int_G u(t, x; \omega)w(x)dx\in Q$ for all $w\in C_0^\infty(G)$,

(u.3) for all $v(t)\in C_0^\infty((0, T))$ and $w(x)\in C_0^\infty(G)$, the function u satisfies the following equality,

$$\int_G dx\int_0^T\left\{w(x)\frac{dv}{dt}+v(t)\sum_{i=1}^d\frac{\partial}{\partial x_i}(b_i(t, x)w(x))\right\}u(t, x; \omega)dt$$

$$+\sum_{j=1}^d\int_0^T\left\{v(t)\int_G u(t, x; \omega)\times\sum_{i=1}^d\frac{\partial}{\partial x_i}(a_{ij}(t, x)w(x))dx\right\}\circ dB_t^j$$

$$+\int_G w(x)dx\int_0^T\{c(t, x)u(t, x; \omega)+d(t, x)\}v(t)dt$$

$$=0 \quad \text{a.s.},$$

(u.4) $u(t, x; \omega)\to\phi(x)$ a.s. as $t\downarrow 0$, for each $x\in G$,

(u.5) $u(t, x; \omega)\to\psi(\xi)$ a.s. as $x(\in G)\to\xi$,

for each $(t, \xi)\in(0, T]\times\partial G$.

We extend the domain of functions $a(t, x)$ and $b(t, x)$ to outside of region G so that new functions belong to $C_0^{0,3}([0, T]\times R^d, R^d\otimes R^d)$ and $C_0^{0,2}([0, T]\times R^d, R^d)$, respectively. For these new functions, let $X(r, t, x)$ be the solution of the stochastic differential equation (2.1), and consider the time reversed process $Y(r, t, y)$ of X and the stopped process $(Y_G(r, t, y), \sigma(t, y))$ of Y in $\bar{G}$ defined in § 2. Clearly, $Y_G(r, t, y)$ is determined independently of the way of extension of $a(t, x)$ and $b(t, x)$. We define at the point ∂,

(i) $c(t, \partial)=d(t, \partial)=\phi(\partial)=\psi(\partial)=0$ for $t\in[0, T]$,

(ii) $Y_G(r, t, \partial)=\partial$ and $\sigma(t, \partial)=t$ for $t\in[0, T]$, $r\in[0, t]$.

THEOREM 3.1. *The random transport equation* (3.1) *with initial condition* (3.2) *and boundary condition* (3.3) *has a next solution.*

$$u(t, x; \omega) \tag{3.4}$$

$$=\{\phi(Y_G(0, t, x))+\psi(Y_G(\sigma(t, x), t, x))\}\cdot\exp\left\{\int_0^t c(s, Y_G(s, t, x))ds\right\}$$

$$+\int_0^t d(s, Y_G(s, t, x))\cdot\exp\left\{\int_s^t c(r, Y_G(r, t, x))dr\right\}ds.$$

§ 4. The proof of Theorem 3.1.

We prepare three lemmas for the proof of Theorem 3.1. But, since the proofs of first two lemmas are quite elementary, we shall omit the details. We consider following conditions (i) and (ii) for a family of Borel sets $\{G_t\}_{t\in[0,T]}=\{G_t(\omega)\}_{t\in[0,T]}$;

(i) the indicator function $\chi_{G_t}(x)$ is $\mathcal{F}_t$-measurable for each (t, x),

(ii) $d(t, t+h; \omega)=\sup_{x\in G_t\Delta G_{t+h}}\inf_{y\in\partial G_t}|x-y|\to 0$ as $h\downarrow 0$ uniformly in $t\in[0, T)$ a.s., where $G_t\Delta G_{t+h}$ is the symmetric difference of two sets.

We also consider following conditions (iii)$\sim$(v) for an R^d-valued function $f(t, x; \omega)=\{f_i(t, x; \omega)\}_{i=1}^d$ and condition (vi) for a real valued function $g(t, x; \omega)$, $(t, x; \omega)\in[0, T]\times G\times\Omega$;

(iii) $f_i(t, x; \omega)\in Q$ for each x,

(iv) the total variation on $[0, T]$ of bounded variation part of f_i belongs to $L^1(G\times\Omega)$ for each i, $1\leqq i\leqq d$,

(v) $\sup_{t\in[0,T]}\left\{\|f_i\|_{L^2(G\times\Omega)}+\left\|\frac{d[M^i, B^i]}{dt}\right\|_{L^2(G\times\Omega)}\right\}<\infty$,

where M_t^i is the martingale part of f_i and $\frac{d[M^i, B^i]}{dt}$ is the Radon-Nikodym derivative of quadratic variation $[M_t^i, B_t^i]$ (K. Itô [5]). S. Ogawa called this process B-derivative of f_i,

(vi) $g\in L^1([0, T]\times G\times\Omega)$ and is an $\{\mathcal{F}_t\}$-adapted process for each x.

LEMMA 4.1. *We assume that a family $\{G_t\}_{t\in[0,T]}$ of subsets of a bounded region G satisfies conditions* (i) *and* (ii). *We also assume that, for functions $f(t, x; \omega)$ and $g(t, x; \omega)$ which satisfy conditions* (iii)$\sim$(v) *and* (vi), *respectively, there exists $\varepsilon=\varepsilon(\omega)>0$ such that*

$$f_i(t, x; \omega)=g(t, x; \omega)=0 \qquad \text{for} \quad t\in[0, T] \quad \text{and} \quad x\in(\partial G_t)^\varepsilon. \tag{4.1}$$

If we set

$$h(t, x; \omega)=\sum_{i=1}^d\int_0^t f_i(s, x; \omega)\circ dB_s^i+\int_0^t g(s, x; \omega)ds,$$

then $\int_{G_t}h(t, x; \omega)dx$ belongs to Q and the equality

$$d\left\{\int_{G_t}h(t, x; \omega)dx\right\}=\sum_{i=1}^d\left\{\int_{G_t}f_i(t, x; \omega)dx\right\}\circ dB_t^i+\left\{\int_{G_t}g(t, x; \omega)dx\right\}dt$$

holds.

OUTLINE OF THE PROOF. We may note that the condition (4.1) implies that when we consider the differential form $d\left\{\int_{G_t}h(t, x; \omega)dx\right\}$, we can treat

G_t like immovable and the conditions (iii)$\sim$(vi) guarantee the exchange of order of integral with respect to dB_t^i and dx.

LEMMA 4.2. *The random function* $u(t, x; \omega)$ *defined in* (3.4) *satisfies the initial condition* (3.2).

PROOF. Since $Y(r, t, y)$ is continuous in (r, t, y), we can easily prove that the function $u(t, x; \omega)$ satisfies the condition (u.4) of Definition 3.1.

LEMMA 4.3. *The random function* $u(t, x; \omega)$ *defined in* (3.4) *satisfies the Dirichlet boundary condition* (3.3).

PROOF. We prove that each $\xi\in\partial G$ is a regular point of $Y(r, t, x)$, that is,

$$\tilde{\sigma}(t, \xi)=t \qquad \text{for each} \quad t\in[0, T],$$

where $\quad \tilde{\sigma}(t, \xi)=\tilde{\sigma}(t, \xi; \omega)=\sup\{r; Y(r, t, \xi)\notin\bar{G}, r\in\{0-\}\cup[0, t]\}$.

Since $\xi\in\partial G$ is a regular point of the process $X(r, \cdot, \xi)$ (we can easily extend the result in A. Friedman [**3**] in tha case of a stochastic differential equation with time-dependent coefficients) and $X(r, t, x)$ is continuous in (r, t, x), we have

$$\sigma_X(r, \xi)=r \qquad \text{for all} \quad (r, \xi)\in[0, T]\times\partial G \quad \text{a.s.,} \tag{4.2}$$

where $\quad \sigma_X(r, \xi)=\inf\{t; X(r, t, \xi)\notin\bar{G}, t\in[r, T]\}$.

If, for some (t, ξ), $Y(r, t, \xi)\in\partial G$ for all $r\in[0, t]$, then for $r<t$,

$$\sigma_X(r, Y(r, t, \xi))\geqq t,$$

and by (4.2),

$$\sigma_X(r, Y(r, t, \xi))=r,$$

so, we have

$$P(Y(r, t, \xi)\in\partial G \text{ for all } r\in[0, t])=0. \tag{4.3}$$

Since G is a bounded region with smooth boundary ∂G, we can take two sequences $\{\varepsilon_n\}_{n=1}^{\infty}$ and $\{D_n\}_{n=1}^{\infty}$ such that

(i) $\varepsilon_1>\varepsilon_2>\cdots\downarrow 0$

(ii) D_n are regions with smooth boundaries which satisfy,

$$D_n\subset G, \qquad G-D_n\subset(\partial G)^{\varepsilon_n}, \quad \text{and}$$

$$\sum_{i,j,k=1}^{d} a_{ik}(t, \eta)a_{jk}(t, \eta)\nu_i(\eta)\nu_j(\eta)>0 \qquad \text{for each} \quad \eta\in\partial D_n,$$

where $\nu(\eta)=(\nu_1(\eta), \cdots, \nu_d(\eta))$ is the inward normal.

By the same reason to get (4.2), we have

$$\sigma_{X,n}(r, \eta)=r \qquad \text{for all} \quad n=1, 2, \cdots \quad \text{and} \quad (r, \eta)\in\{\partial G\cup\partial D_n\}, \quad \text{a.s.,} \tag{4.4}$$

where $\quad \sigma_{X,n}(r, \eta)=\inf\{t; X(r, t, \xi)\notin\bar{G}-D_n, t\in[r, T]\}$.

There exists some $n=n(\omega)$, by (4.3), such that

$$\tilde{\sigma}_n(t, \xi)>0 \qquad \text{for each} \quad (t, \xi)\in[0, T]\times\partial G,$$

where $\quad \tilde{\sigma}_n(t, \xi)=\sup\{r;\ Y(r, t, \xi)\notin \bar{G}-D_n,\ r\in\{0-\}\cup[0, t]\}$.

Since $\quad \sigma_{X,n}(\tilde{\sigma}_n(t, \xi),\ Y(\tilde{\sigma}_n(t, \xi), t, \xi))\geqq t$,

and $\quad \sigma_{X,n}(\tilde{\sigma}_n(t, \xi),\ Y(\tilde{\sigma}_n(t, \xi), t, \xi))=\tilde{\sigma}_n(t, \xi)\leqq t$,

we get $\quad \tilde{\sigma}_n(t, \xi)=t$.

Therefore $\tilde{\sigma}(t, \xi)=t$, namely, $\xi\in\partial G$ is a regular point of Y. By the continuity of the process $Y(r, t, y)$ in (r, t, y), we see that the random function $u(t, x;\omega)$ defined in (3.4) satisfies the boundary condition (u.5) of Definition 3.1.

PROOF OF THEOREM 3.1. We set

$$I_1(t, x)=\phi(Y_G(0, t, x))\exp\left\{\int_0^t c(s, Y_G(s, t, x))ds\right\},$$

$$I_2(t, x)=\int_0^t d(s, Y_G(s, t, x))\exp\left\{\int_s^t c(r, Y_G(r, t, x))dr\right\}ds,$$

$$I_3(t, x)=\psi(Y_G(\sigma(t, x), t, x))\exp\left\{\int_0^t c(s, Y_G(s, t, x))ds\right\}.$$

STEP 1. First we consider $\int_G I_1(t, x)w(x)dx$ for $w(x)\in C_0^\infty(G)$. Substituting x by $X(0, t, y)$ in the integrand, and by the fact that Y is the time reversed process of X, we have

$$\begin{aligned}
&\int_G I_1(t, x)w(x)dx \\
&=\int_{G_Y(0,t)} I_1(t, x)w(x)dx \\
&=\int_{G_X(0,t)}\phi(y)\exp\left\{\int_0^t c(s, X(0, s, y))ds\right\}w(X(0, t, y))J(0, t, y)dy,
\end{aligned}$$

where $G_X(0, t)$ and $G_Y(0, t)$ are defined in §2. By (2.3) and Itô's formula for symmetric differentials,

$$\begin{aligned}
&d\left[\exp\left\{\int_0^t c(s, X(0, s, y))ds\right\}w(X(0, t, y))J(0, t, y)\right] \\
&=\exp\left\{\int_0^t c(s, X(0, s, y))ds\right\}J(0, t, y)\left[\sum_{i,j=1}^d \frac{\partial}{\partial x_i}(a_{ij}w)(t, X(0, t, y))\circ dB_t^j\right. \\
&\qquad\left.+\left\{\sum_{i=1}^d \frac{\partial}{\partial x_i}(b_i w)+(cw)\right\}(t, X(0, t, y))dt\right].
\end{aligned}$$

We can apply Lemma 4.1 in this case, since $\partial G_X(0, t)$ is included in $\{y\in G; X(0, t, y)\in\partial G\}$ and $X(0, t, y)$ is continuous in (t, y) and $w(x)$ has a compact support in G. Then, we obtain

$$d\Big[\int_G I_1(t, x)w(x)dx\Big]$$

$$=\sum_{j=1}^d\Big[\int_{G_X(0,t)}\phi(y)\exp\Big\{\int_0^t c(s, X(0, s, y))ds\Big\}J(0, t, y)$$

$$\times\sum_{i=1}^d\frac{\partial}{\partial x_i}(a_{ij}w)(t, X(0, t, y))dy\Big]\circ dB_t^j$$

$$+\Big[\int_{G_X(0,t)}\phi(y)\exp\Big\{\int_0^t c(s, X(0, s, y))ds\Big\}J(0, t, y)$$

$$\times\Big\{\sum_{i=1}^d\frac{\partial}{\partial x_i}(b_iw)+cw\Big\}(t, X(0, t, y))dy\Big]dt,$$

substituting y by $Y(0, t, x)$, as an $\mathcal{F}_t$-measurable substitution,

$$=\sum_{j=1}^d\Big[\int_G\phi(Y_G(0, t, x))\exp\Big\{\int_0^t c(s, Y_G(s, t, x))ds\Big\}$$

$$\times\sum_{i=1}^d\frac{\partial}{\partial x_i}(a_{ij}w)(t, x)dx\Big]\circ dB_t^j+\Big[\int_G\phi(Y_G(0, t, x))$$

$$\times\exp\Big\{\int_0^t c(s, Y_G(s, t, x))ds\Big\}\cdot\Big\{\sum_{i=1}^d\frac{\partial}{\partial x_i}(b_iw)+cw\Big\}(t, x)dx\Big]dt.$$

Hence,

$$d\Big[v(t)\int_G I_1(t, x)w(x)dx\Big]=\sum_{j=1}^d\Big[v(t)\int_G I_1(t, x)\sum_{i=1}^d\frac{\partial}{\partial x_i}(a_{ij}w)(t, x)dx\Big]\circ dB_t^j$$

$$+\Big[v(t)\int_G I_1(t, x)\Big\{\sum_{i=1}^d\frac{\partial}{\partial x_i}(b_iw)+cw\Big\}(t, x)dx+\frac{dv}{dt}(t)\int_G I_1(t, x)w(x)dx\Big]dt,$$

$$\text{for}\quad v(t)\in C_0^\infty((0, T)).$$

Thus, we have

$$\int_G dx\int_0^T\Big\{w(x)\frac{dv}{dt}(t)+v(t)\sum_{i=1}^d\frac{\partial}{\partial x_i}(b_i(t, x)w(x))\Big\}I_1(t, x)dt \tag{4.5}$$

$$+\sum_{j=1}^d\int_0^T\Big\{v(t)\int_G I_1(t, x)\sum_{i=1}^d\frac{\partial}{\partial x_i}(a_{ij}(t, x)w(x))dx\Big\}\circ dB_t^j$$

$$+\int_G w(x)dx\int_0^T c(t, x)I_1(t, x)dt=0.$$

STEP 2. Concerning the second term $I_2(t, x)$, a similar way as in Step 1 implies

$$\begin{aligned}(4.6)\qquad &\int_G dx\int_0^T\Big\{w(x)\frac{dv}{dt}(t)+v(t)\sum_{i=1}^d\frac{\partial}{\partial x_i}(b_i(t, x)w(x))\Big\}I_2(t, x)dt\\ &+\sum_{j=1}^d\int_0^T\Big\{v(t)\int_G I_2(t, x)\sum_{i=1}^d\frac{\partial}{\partial x_i}(a_{ij}(t, x)w(x))dx\Big\}\circ dB_t^j\\ &+\int_G w(x)dx\int_0^T\Big\{c(t, x)I_2(t, x)+d(t, x)\Big\}dt=0\end{aligned}$$

$$\text{for each}\quad v(t)\in C_0^\infty((0, T))\quad\text{and each}\quad w(x)\in C_0^\infty(G).$$

STEP 3. For simplicity, we set

$$\xi(t, x)=\xi(t, x\,;\,\omega)=Y_G(\sigma(t, x), t, x)\in\partial G\cup\{\partial\}\,.$$

For each sufficiently small $h>0$ which depends on ω and each $x\in\operatorname{supp}(w)$,

$$\xi(t+h, x)=\xi(t, Y(t, t+h, x)),$$

holds, since $Y(r, t, x)$ is continuous in (r, t, x) and function $w(x)$ has a compact support in G and the relation in (ii) of Proposition 2.1 holds. Therefore, we have

$$\begin{aligned}\int_G I_3(t+h, x)w(x)dx&=\int_{\tilde G(t+h)}\phi(\xi(t+h, x))\exp\Big\{\int_0^{t+h}c(s, Y_G(s, t+h, x))ds\Big\}w(x)dx\\ &=\int_{\tilde G(t+h)}\phi(\xi(t, Y(t, t+h, x))\exp\Big\{\int_0^{t+h}c(s, Y_G(s, t+h, x))ds\Big\}w(x)dx\,,\end{aligned}$$

substituting x by $X(t, t+h, y)$,

$$\begin{aligned}&=\int_{\tilde G(t,h)}\phi(\xi(t, y))\exp\Big\{\int_0^t c(s, Y_G(s, t, y))ds+\int_t^{t+h}c(s, X(t, s, y))ds\Big\}\\ &\qquad\times w(X(t, t+h, y))J(t, t+h, y)dy\,,\end{aligned}$$

where $\tilde G(t+h)$ and $\tilde G(t, h)$ are defined in §2. Using Lemma 2.4, we can take $h_0(\omega)>0$ such that

$$w(X(t, t+h, y))=0\qquad\text{for each}\quad (h, y)\in[0, h_0(\omega)]\times\{G(t)-G(t, h)\}\,.$$

Therefore, for each $h\in[0, h_0(\omega)]$,

$$\begin{aligned}&\int_G\{I_3(t+h, x)-I_3(t, x)\}\,w(x)dx\\ &=\int_{\tilde G(t)}\phi(\xi(t, y))\exp\Big\{\int_0^t c(s, Y_G(s, t, y))ds\Big\}\end{aligned}$$

$$\times\Big[\exp\Big\{\int_t^{t+h} c(s,\,X(t,\,s,\,y))ds\Big\}w(X(t,\,t+h,\,y))J(t,\,t+h,\,y)-w(y)\Big]dy$$

$$=\int_{\tilde{G}(t)}\phi(\xi(t,\,y))\exp\Big\{\int_0^t c(s,\,Y_G(s,\,t,\,y))ds\Big\}dy\int_t^{t+h}\exp\Big\{\int_t^\tau c(s,\,X(t,\,s,\,y))ds\Big\}$$

$$\times J(t,\,\tau,\,y)\Big[\sum_{i,j=1}^d\frac{\partial}{\partial x_i}(a_{ij}w)(\tau,\,X(t,\,\tau,\,y))\circ dB_\tau^j$$

$$+\Big\{\sum_{i=1}^d\frac{\partial}{\partial x_i}(b_iw)+(cw)\Big\}(\tau,\,X(t,\,\tau,\,y))d\tau\Big]$$

$$=\sum_{j=1}^d\int_t^{t+h}\Big[\int_{\tilde{G}(t,\tau-t)}\phi(\xi(t,\,y))\exp\Big\{\int_0^t c(s,\,Y_G(s,\,t,\,y))ds$$

$$+\int_t^\tau c(s,\,X(t,\,s,\,y))ds\Big\}J(t,\,\tau,\,y)\sum_{i=1}^d\frac{\partial}{\partial x_i}(a_{ij}w)(\tau,\,X(t,\,\tau,\,y))\Big]\circ dB_\tau^j$$

$$+\int_t^{t+h}d\tau\int_{\tilde{G}(t,\tau-t)}\phi(\xi(t,\,y))\exp\Big\{\int_0^t c(s,\,Y_G(s,\,t,\,y))$$

$$+\int_t^\tau c(s,\,X(t,\,s,\,y))ds\Big\}J(t,\,\tau,\,y)\Big\{\sum_{i=1}^d\frac{\partial}{\partial x_i}(b_iw)+(cw)\Big\}(\tau,\,X(t,\,\tau,\,y))dy\,,$$

substituting y by $Y(t,\,\tau,\,x)$,

$$=\sum_{j=1}^d\int_t^{t+h}\Big[\int_{\tilde{G}(\tau)}\phi(\xi(\tau,\,x))\exp\Big\{\int_0^\tau c(s,\,Y_G(s,\,\tau,\,x))ds\Big\}\sum_{i=1}^d\frac{\partial}{\partial x_i}(a_{ij}w)(\tau,\,x)dx\Big]\circ dB_\tau^j$$

$$+\int_t^{t+h}d\tau\int_{\tilde{G}(\tau)}\phi(\xi(\tau,\,x))\exp\Big\{\int_0^\tau c(s,\,Y_G(s,\,\tau,\,x))ds\Big\}\Big\{\sum_{i=1}^d\frac{\partial}{\partial x_i}(b_iw)+(cw)\Big\}(\tau,\,x)dx$$

$$=\sum_{j=1}^d\int_t^{t+h}\Big[\int_G I_3(\tau,\,x)\sum_{i=1}^d\frac{\partial}{\partial x_i}(a_{ij}w)(\tau,\,x)dx\Big]\circ dB_\tau^j$$

$$+\int_t^{t+h}d\tau\int_G I_3(\tau,\,x)\Big\{\sum_{i=1}^d\frac{\partial}{\partial x_i}(b_iw)+cw\Big\}(\tau,\,x)dx\,.$$

Hence,

$$d\Big[\int_G I_3(t,\,x)w(x)dx\Big]$$

$$=\sum_{j=1}^d\Big[\int_G I_3(t,\,x)\sum_{i=1}^d\frac{\partial}{\partial x_i}(a_{ij}w)(t,\,x)dx\Big]\circ dB_t^j$$

$$+\Big[\int_G I_3(t,\,x)\Big\{\sum_{i=1}^d\frac{\partial}{\partial x_i}(b_iw)+cw\Big\}(t,\,x)dx\Big]dt\,.$$

Thus, in the same way as in Step 1, we have

$$(4.7)\qquad \int_G dx\int_0^T\left\{w(x)\frac{dv}{dt}(t)+v(t)\sum_{i=1}^d\frac{\partial}{\partial x_i}(b_i(t, x)w(x))\right\}I_3(t, x)dt$$

$$+\sum_{j=1}^d\int_0^T\left\{v(t)\int_G I_3(t, x)\sum_{i=1}^d\frac{\partial}{\partial x_i}(a_{ij}(t, x)w(x))dx\right\}\circ dB_t^j$$

$$+\int_G w(x)dx\int_0^T c(t, x)I_3(t, x)dt=0$$

for each $v(t)\in C_0^\infty((0, T))$ and each $w(x)\in C_0^\infty(G)$.

STEP 4. The equalities (4.5), (4.6) and (4.7) indicate that the function $u(t, x\,;\,\omega)$ satisfies the condition (u.3) of Definition 3.1. We complete the proof by Lemma 4.2 and 4.3.

§ 5. One-dimensional Neumann problem.

We consider the random transport equation with initial condition and Neumann boundary condition in the half line,

$$(5.1)\qquad \frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}\dot{B}_t=0, \qquad (t, x)\in[0, T]\times(0, \infty),$$

$$(5.2)\qquad \lim_{t\downarrow 0}u(t, x\,;\,\omega)=\phi(x), \qquad x\in(0, \infty),$$

$$(5.3)\qquad \frac{\partial u}{\partial \nu}(t\,;\,\omega)=\lim_{x\downarrow 0}\frac{1}{x}\{u(t, x\,;\,\omega)-u(t, 0\,;\,\omega)\}=0,\ t\in(0, T],$$

for $\phi(x)\in C([0, \infty))\cap C^1((0, \infty))$.

We interpret equation (5.1) with conditions (5.2) and (5.3) in the same way as in § 3. We set

$$Y(r, t, x)=x+B_r-B_t+\zeta(r, t, x), \qquad x\in[0, \infty), \qquad 0\leqq r\leqq t\leqq T,$$

where $$\zeta(r, t, x)=[-\inf\{x+B_s-B_t\,;\, s\in[r, t]\}]\vee 0,$$

and let B_t be one-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, P)$.

THEOREM 5.1. *The random transport equation* (5.1) *with initial-boundary conditions* (5.2), (5.3) *has a next solution,*

$$(5.4)\qquad u(t, x\,;\,\omega)=\phi(Y(0, t, x)).$$

PROOF. It is clear that $u(t, x\,;\,\omega)$ in (5.4) satisfies the initial condition (5.2). Setting $\rho(t)=\rho(t\,;\,\omega)=-\inf\{B_r\,;\, r\in[0, t]\}$, we have

$$\frac{\partial\zeta}{\partial\nu}(0, t\,;\,\omega)=\begin{cases} -1 & \text{if } t \text{ satisfies } B_t+\rho(t)>0 \\ 0 & \text{if } t \text{ satisfies } B_t+\rho(t)=0. \end{cases}$$

Hence we have $\frac{\partial u}{\partial \nu}(t)=0$ a.s. for each $t>0$, because $P(B_t+\rho(t)=0)=0$ and

$$\frac{\partial u}{\partial \nu}(t)=\phi'(-B_t+\zeta(0, t, 0))\times\left\{1+\frac{\partial \zeta}{\partial \nu}(0, t)\right\} \qquad \left(\phi'=\frac{d\phi}{dx}\right).$$

Now, we prove that $u(t, x; \omega)$ satisfies the following relation:

$$\int_0^T dt\int_0^\infty w(x)v'(t)u(t, x; \omega)dx+\int_0^T v(t)\int_0^\infty w'(x)u(t, x; \omega)dx\circ dB_t \tag{5.5}$$
$$=0 \qquad \left(v'=\frac{dv}{dt},\ w'=\frac{dw}{dx}\right)$$

for each $v(t)\in C_0^\infty((0, T))$ and each $w(x)\in C_0^\infty((0, \infty))$.

The left hand side of (5.5) is

$$\int_0^T v'(t)dt\left\{\int_{B_t+\rho(t)}^\infty \phi(x-B_t)w(x)dx+\int_0^{B_t+\rho(t)}\phi(\rho(t))w(x)dx\right\}$$
$$+\int_0^T v(t)\left\{\int_{B_t+\rho(t)}^\infty \phi(x-B_t)w'(x)dx+\int_0^{B_t+\rho(t)}\phi(\rho(t))w'(x)dx\right\}\circ dB_t .$$

Here,

$$\int_{B_t+\rho(t)}^\infty \phi(x-B_t)w(x)dx=\int_{\rho(t)}^\infty \phi(y)w(y+B_t)dy .$$

Noting that the process $\rho(t)$ is increasing, we have

$$d\left[v(t)\int_{\rho(t)}^\infty \phi(y)w(y+B_t)dy\right]$$
$$=-\phi(\rho(t))w(B_t+\rho(t))v(t)d\rho(t)+v(t)\int_{\rho(t)}^\infty \phi(y)w'(y+B_t)dy\circ dB_t$$
$$+v'(t)\int_{\rho(t)}^\infty \phi(y)w(y+B_t)dydt .$$

Since $d\rho(t)\neq 0$ implies $B_t+\rho(t)=0$, the first term of the right hand side of the above equality is 0. Hence, we have

$$\int_0^T v(t)\int_{B_t+\rho(t)}^\infty \phi(x-B_t)w'(x)dx\circ dB_t \tag{5.6}$$
$$+\int_0^T v'(t)dt\int_{B_t+\rho(t)}^\infty \phi(x-B_t)w(x)dx=0 .$$

In the same way,

$$d\left[v(t)\left(\int_0^{B_t+\rho(t)} w(x)dx\right)\phi(\rho(t))\right]$$

$$=\phi(\rho(t))\left(\int_0^{B_t+\rho(t)} w(x)dx\right)v'(t)dt+v(t)w(B_t+\rho(t))\phi(\rho(t))\{\circ dB_t+d\rho(t)\}$$

$$+v(t)\left(\int_0^{B_t+\rho(t)} w(x)dx\right)\phi'(\rho(t))d\rho(t)$$

$$=\phi(\rho(t))\left(\int_0^{B_t+\rho(t)} w(x)dx\right)v'(t)dt+v(t)w(B_t+\rho(t))\phi(\rho(t))\circ dB_t .$$

Therefore, we have

$$\int_0^T v'(t)dt\int_0^{B_t+\rho(t)}\phi(\rho(t))w(x)dx + \int_0^T v(t)\int_0^{B_t+\rho(t)}\phi(\rho(t))w'(x)dx\circ dB_t=0 . \tag{5.7}$$

Combining the equalities (5.6) and (5.7), we can show that the function $u(t, x; \omega)$ in (5.4) satisfies the relation (5.5).

Now, we introduce a problem of the random transport equation (5.1) with an initial condition (5.2) and an averaged Neumann boundary condition (5.3)$'$,

$$\frac{\partial \bar{u}}{\partial \nu}(t)=0, \qquad t\in(0, T], \tag{5.3$'$}$$

$$\text{where} \qquad \bar{u}(t, x)=E[u(t, x; \omega)] .$$

DEFINITION 5.1. A random function $u(t, x; \omega)$ is called a solution of (5.1) with (5.2), (5.3)$'$ if $u(t, x; \omega)$ satisfies the conditions (u.1) and (u.2) of Definition 3.1 and,

($\bar{\text{u}}$) there exists a real valued random function $U(v; \omega)$, $(v, \omega)\in\Phi([0, T])\times\Omega$, which satisfies,

1) $E[U(v; \omega)]=0 \quad$ for all $\quad v(t)\in\Phi([0, T])$,

2) $$\int_0^\infty w(x)dx\int_0^T v'(t)u(t, x; \omega)dt+\int_0^T v(t)\int_0^\infty w'(x)u(t, x; \omega)dx\circ dB_t$$
$$+v(0)\int_0^\infty w(x)\phi(x)dx=w(0)U(v; \omega)$$

for all $(v, w)\in\Phi([0, T])\times\Phi([0, \infty))$,

where $\Phi([0, T])=\{v(t)\in C_0^\infty([0, T]); \quad \text{supp}(v)\subset[0, T)\}$,

$\Phi([0, \infty))=\{w(x)\in C_0^\infty([0, \infty)); \quad w'(x)\in C_0^\infty((0, \infty))\}$.

We can prove that a function u defined by

$$u(t, x; \omega)=\phi(Z(0, t, x)),$$

$$\text{where} \quad Z(r, t, x)=|x+B_r-B_t|,$$

is not a solution of the initial-boundary value problem of the random transport equation (5.1), (5.2) and (5.3), but a solution of the initial-averaged boundary value problem of the random transport equation (5.1), (5.2) and (5.3)′.

REMARK 5.1. On the uniqueness of solution of random transport equation with boundary condition, we shall study in forthcoming paper.

Appendix.

A time reversed diffusion process (general case).

We assume that $a(t, x)$ and $b(t, x)$ belong to $C^{0,3}([0, T]\times R^d, R^d\otimes R^d)$ and $C^{0,2}([0, T]\times R^d, R^d)$, respectively. Let $X(r, t, x)$ be a solution of a stochastic differential equation:

$$dX_t=a(t, X_t)\circ dB_t+b(t, X_t)dt, \qquad t\in[r, \tau_X(r, x)), \tag{A.1}$$

with initial condition

$$X_r=x,$$

where $\tau_X(r, x)$ $(r<\tau_X(r, x)\leqq T)$ is an explosion time of $X(r, t, x)$, if $\tau_X(r, x)<T$. Let $f(x)$ be a function defined on R^d which satisfies,

(i) $f(x)=1$ for $x;\ |x|\leqq 1$,

(ii) $f(x)=0$ for $x;\ |x|\geqq 2$,

(iii) $f(x)\in C^3(R^d)$.

We define, for $n=1, 2, \cdots$,

$$a^{(n)}(t, x)=f\left(\frac{x}{n}\right)a(t, x),$$

$$b^{(n)}(t, x)=f\left(\frac{x}{n}\right)b(t, x).$$

Let $\{X_n(r, t, x)\}_{n=1}^{\infty}$ be the solutions of (A.1) with $a(t, x)$ and $b(t, x)$ replaced by $a^{(n)}(t, x)$ and $b^{(n)}(t, x)$, respectively, and let

$$\tau^{(n)}(r, x)=\tau^{(n)}(r, x; \omega)=\begin{cases} \inf\{t;\ |X^{(n)}(r, t, x)|\geqq n,\ t\in[r, T]\} \\ T \text{ if the set in the bracket is empty.}\end{cases}$$

Since $\{\tau^{(n)}(r, x)\}_{n=1}^{\infty}$ are lower semi-continuous in (r, x) and

$$P\{X^{(n)}(r, t, x)=X(r, t, x) \text{ for } t\in[r, \tau^{(n)}(r, x))\}=1$$

$$\text{for each } (r, x)\in[0, T]\times R^d,$$

we see that

$$X^{(n)}(r, t, x)=X(r, t, x) \tag{A.2}$$

$$\text{for } (r, x)\in[0, T]\times R^d,\ t\in[r, \tau^{(n)}(r, x)) \text{ a.s.}$$

Hence,

$$\lim_{n\to\infty}\tau^{(n)}(r, x)=\tau_X(r, x) \quad \text{for } (r, x)\in[0, T]\times R^d. \tag{A.3}$$

By Lemma 2.2, $\{X^{(n)}(r, t, \cdot)\}_{n=1}^{\infty}$ are homeomorphisms of R^d onto R^d for each $r, t, 0\leqq r\leqq t\leqq T$.

Here, we define for each $r, t, 0\leqq r\leqq t\leqq T$,

$$R_X(r, t)=R_X(r, t\,;\omega)=\{x\,;\ t<\tau_X(r, x)\},$$

$$R_Y(r, t)=R_Y(r, t\,;\omega)=\{X(r, t, x)\,;\ x\in R_X(r, t)\}.$$

Since $\{X^{(n)}(r, t, \cdot)\}_{n=1}^{\infty}$ are homeomorphisms of R^d into R^d, by equalities (A.2) and (A.3), $X(r, t, \cdot)$ is a bijection of $R_X(r, t)$ onto $R_Y(r, t)$ for each $r, t, 0\leqq r\leqq t\leqq T$.

DEFINITION A.1. For each $r, t, y, 0\leqq r\leqq t\leqq T, y\in R^d\cup\{\partial\}$, we set

$$Y(r, t, y)=\begin{cases} X^{-1}(r, t, \cdot)(y) & \text{if } y\in R_Y(r, t) \\ \partial & \text{if } y\notin R_Y(r, t), \end{cases}$$

where X^{-1} is the inverse mapping of $X(r, t, x)$ as a function of x.

LEMMA A.1. *If $Y(r, t, y)=\partial$, then $Y(s, t, y)=\partial$ for each $s\in[0, r]$.*

PROOF. Since, if $y=\partial$, the statement is obvious, we assume $y\in R^d$. If the conclusion is not true, there exists $s\in[0, r]$ and $x\in R^d$ such that $Y(s, t, y)=x$, namely, $y=X(s, t, x)$. By the equality,

$$X(s, t, x)=X(r, t, X(s, r, x)),$$

we have

$$Y(r, t, y)=X(s, r, x)\in R^d.$$

This contradicts to the assumption.

DEFINITION A.2. For each $(t, y)\in[0, T]\times R^d$, let

$$\tau_Y(t, y)=\tau_Y(t, y\,;\omega)=\begin{cases} \sup\{r\,;\ Y(r, t, y)=\partial,\ r\in[0, t]\} \\ 0- \text{ if the set in the bracket is empty.} \end{cases}$$

PROPOSITION A.1. *$Y(r, t, y)$ satisfies following* (i) *and* (ii). (i) *$Y(r, t, y)=Y(r, s, Y(s, t, y))$ for $r, s, t, y, 0\leqq r\leqq s\leqq t\leqq T, y\in R^d\cup\{\partial\}$,* (ii) *$Y(r, t, y)$ is continuous in $r\in(\tau_Y(t, y), t]$ for each $(t, y)\in[0, T]\times R^d$, where $(0-, t]=[0, t]$.*

The proof is similar to that of Proposition 2.1.

LEMMA A.2.[2] *For each* $t\in[0, T]$, $\bigcup_{r\in[0,t)} R_Y(r, t)=R^d$.

PROOF. We write simply $\pi_r(\cdot)=X(r, t, \cdot)$ for $r\in[0, t]$. If the conclusion is not true, there exists a point $x_0\in R^d$ which does not belong to $\bigcup_{r\in[0,t)} R_Y(r, t)$. Consider a unit sphere $V(x_0)$ with center x_0 in R^d. We denote by i the identity mapping of $V(x_0)$ into R^d and by g a retraction of $R^d-\{x_0\}$ onto $V(x_0)$ defined by

$$g: x\longrightarrow \frac{x-x_0}{|x-x_0|}.$$

Since $\pi_r(x)$ is continuous in (r, x), $\pi_{t|V(x_0)}\simeq g\circ\pi_r\circ i$ (homotopic) for $r\in[0, t)$. If we denote by $[z]$ a generator of $H_{d-1}(V(x_0))\cong Z^1$, then, by $\pi_{t|V(x_0)}=$identity, we have

$$(g\circ\pi_r\circ i)_*([z])=[z].$$

On the other hand, since

$$(g\circ\pi_r\circ i)_*: H_{d-1}(V(x_0)) \xrightarrow{i_*} H_{d-1}(R^d) \xrightarrow{(\pi_r)_*} H_{d-1}(R^d-\{0\}) \xrightarrow{g_*} H_{d-1}(V(x_0)),$$

and $H_{d-1}\{R^d\}=\{0\}$, we have

$$(g\circ\pi_r\circ i)_*([z])=0.$$

This is a contradiction.

LEMMA A.3. *For each* r, t, $0\leqq r\leqq t\leqq T$, $\bigcup_{s\in[0,r)} R_Y(s, t)=R_Y(r, t)$.

PROOF. It is obvious that $\bigcup_{s\in[0,r)} R_Y(s, t)\subset R_Y(r, t)$. Conversely, for $y\in R_Y(r, t)$, there exists $x\in R^d$ such that

$$X(r, t, x)=y. \tag{A.4}$$

Since $R^d=\bigcup_{s\in[0,r)} R_Y(s, r)$ by Lemma A.2, there exists $s\in[0, r)$ such that $x\in R_Y(s, r)$, namely, there exists $z\in R^d$ such that

$$X(s, r, z)=x. \tag{A.5}$$

By (A.4) and (A.5), we have

$$y=X(r, t, X(s, r, z))=X(s, t, z)\in R_Y(s, t)\subset \bigcup_{s\in[0,r)} R_Y(s, t).$$

PROPOSITION A.2. *If* $\tau_Y(t, y)\geqq 0$, *then* $\tau_Y(t, y)$ *is the explosion time of* $Y(r, t, y)$, *that is,*

$$\lim_{r\downarrow\tau_Y(t,y)} |Y(r, t, y)|=\infty.$$

PROOF. If the assertion is not true, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ and $x_0\in R^d$ such that

(2) The author thanks A. Gyoja for his advice on the proof of this lemma.

$$r_n \downarrow \tau_Y(t, y) \quad \text{and} \quad Y(r_n, t, y) \to x_0 \quad \text{as} \quad n \to \infty .$$

If we set $x_n = Y(r_n, t, y)$, then $y = X(r_n, t, x_n)$.
Since $X(r, t, x)$ is continuous in (r, t, x),

$$X(r_n, t, x_n) \to X(\tau_Y(t, y), t, x_0) \quad \text{as} \quad n \to \infty .$$

Therefore $y = X(\tau_Y(t, y), t, x_0)$ and hence $y \in R(\tau_Y(t, y), t)$. By Lemma A.3, there exists a positive ε such that

$$y \in R(\tau_Y(t, y) - \varepsilon, t) .$$

This contradicts to the definition of $\tau_Y(t, y)$.

REMARK A.1. In the case of unbounded region G in § 4, we can obtain a similar result, noting Proposition A.2 and that the test function $w \in C_0^\infty(G)$ has a compact support.

THEOREM A.1. *For each $(t, y) \in [0, T] \times (R^d \cup \{\partial\})$, $Y(r, t, y)$ is a time reversed Markov process with respect to $r \in [0, t]$, that is,*

$$\text{(A.6)} \qquad P[Y(r, t, y) \in A \,|\, Y(r_1, t, y) = y_1, \cdots, Y(r_n, t, y) = y_n]$$
$$= P[Y(r, r_1, y_1) \in A] \quad a.s.,$$

for $0 \leqq r \leqq r_1 \leqq \cdots \leqq r_n \leqq t$, $y, y_1, \cdots, y_n \in R^d \cup \{\partial\}$ and a Borel subset A of $R^d \cup \{\partial\}$.

PROOF. By Proposition A.1, $Y(r, t, y) = Y(r, r_1, Y(r_1, t, y))$. Since random function $Y(r, r_1, x)$ is $\mathcal{F}_{[r, r_1]}$-measurable and $\{Y(r_i, t, y)\}_{i=1}^n$ are $\mathcal{F}_{[r_1, t]}$-measurable, $Y(r, r_1, x)$ is independent of $\{Y(r_i, t, y)\}_{i=1}^n$. Therefore we have (A.6).

We investigate the generator of time reversed process $Y(r, t, y)$ in a temporally homogeneous case, that is,

$$\text{(A.7)} \qquad a(t, x) = a(x) \quad \text{and} \quad b(t, x) = b(x) .$$

We consider the following function space,

$$C_\infty(R^d \cup \{\partial\}) = \{f(x) \in C(R^d \cup \{\partial\})\,;\ \lim_{|x| \to \infty} f(x) = 0 \quad \text{and} \quad f(\partial) = 0\}$$

with norm $\|f\|_\infty = \sup_{x \in R^d} |f(x)|$.

REMARK A.2. Under the condition (A.7), if we set

$$T_t f(x) = E[f(X(0, t, x))] \qquad \text{for} \quad f \in C_\infty(R^d \cup \{\partial\}) ,$$

then the system of linear operators $\{T_t\}_{t \geqq 0}$ is a semigroup on $C_\infty(R^d \cup \{\partial\})$, that is,

$$T_t T_s = T_{t+s} , \qquad T_0 = \text{the identity}, \qquad \|T_t\|_\infty \leqq 1, \qquad \lim_{t \downarrow 0} \|T_t f - f\|_\infty = 0$$

for all $f \in C_\infty(R^d \cup \{\partial\})$, and $T_t f \geqq 0$ for all $f \geqq 0$.

The generator $\mathcal{G}_X$ of $\{T_t\}$ is defined by

$$\mathcal{G}_X f = \text{s-}\lim_{t\to\infty} \frac{1}{t}(T_t f - f) \quad \text{for} \quad f \in \mathcal{D}(\mathcal{G}_X),$$

where

$$\mathcal{D}(\mathcal{G}_X) = \{f \in C_\infty(R^d \cup \{\partial\});\ \text{there exists s-}\lim_{t\to 0} \frac{1}{t}(T_t f - f)\},$$

and

$$\text{s-}\lim_{n\to\infty} f_n = f \quad \text{means} \quad \|f_n - f\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

As is well-known,

$$(\mathcal{G}_X f)(x) = \frac{1}{2} \sum_{i,j,k=1}^{d} a_{jk}(x) \frac{\partial}{\partial x_j}\left(a_{ik}(x) \frac{\partial f}{\partial x_i}(x)\right) + \sum_{i=1}^{d} b_i(x) \frac{\partial f}{\partial x_i}(x)$$

$$\text{for} \quad f \in \mathcal{D}(\mathcal{G}_X) \cap C^2(R^d).$$

We set

$$(\tilde{T}_r f)(X) = E[f(Y(T-r, T, x))] \quad \text{for} \quad f \in C_\infty(R^d \cup \{\partial\}), \quad r \in [0, T].$$

PROPOSITION A.3. *$\{\tilde{T}_r\}_{r\in[0,T]}$ is a semigroup in the sense in Remark* A.2.

PROOF. First we note that, since $Y(r, t, x)$ is continuous in x, by Lebesgue's dominated convergence theorem, $E[f(Y(r, t, x))]$ is continuous in x, especially $\tilde{T}_r f \in C(R^d)$.

For $w(x) \in C_0^\infty(R^d)$,

$$\text{(A.7)} \qquad \begin{aligned} &\int_{R^d} E[f(Y(t-r, t, x))] w(x) dx \\ &= E\left[\int_{R^d} f(Y(t-r, t, x)) w(x) dx\right] \\ &= E\left[\int_{R^d} f(y) w(X(t-r, t, y)) J(t-r, t, y) dy\right] \\ &= E\left[\int_{R^d} f(y) w(X(T-r, T, y)) J(T-r, T, y) dy\right] \\ &= \int_{R^d} \tilde{T}_r f(x) w(x) dx, \end{aligned}$$

where $J(r, t, x)$ is the Jacobian of $X(r, t, x)$ and we used the fact that $X(r, t, x)$ is temporally homogeneous. Since $E[f(Y(t-r, t, x))]$ and $\tilde{T}_r f(x)$ are continuous in x, we have

$$E[f(Y(t-r, t, x))] = \tilde{T}_r f(x) \qquad r \in [0, t].$$

By virtue of this equality and Proposition A.1, (i), we have

$$\tilde{T}_{r_1}\tilde{T}_{r_2}=\tilde{T}_{r_1+r_2} \quad \text{for} \quad r_1, r_2 \geqq 0 \quad \text{with} \quad r_1+r_2 \leqq T\,.$$

It is easy to show that $\tilde{T}_r f \in C_\infty(R^d \cup \{\partial\})$ for $f \in C_\infty(R^d \cup \{\partial\})$, $\tilde{T}_0$=the identity, $\|\tilde{T}_r\|_\infty \leqq 1$, and $\tilde{T}_r f \geqq 0$ for $f \geqq 0$. The strong continuity of $\{\tilde{T}_r\}_{r\geqq 0}$ follows from the continuity of $Y(r, t, y)$ in (r, t, y).

We consider the generator $\mathcal{G}_Y$ of $\{\tilde{T}_r\}_{r\in[0,T]}$ with domain $\mathcal{D}(\mathcal{G}_Y)$ in the same way as in Remark A.2.

THEOREM A.2. *For* $f \in \mathcal{D}(\mathcal{G}_Y)$,

$$(\mathcal{G}_Y f)(x)=\frac{1}{2}\sum_{i,j,k=1}^{d} a_{jk}(x)\frac{\partial}{\partial x_j}\Big(a_{ik}(x)\frac{\partial f}{\partial x_i}(x)\Big)-\sum_{i=1}^{d} b_i(x)\frac{\partial f}{\partial x_i}(x)$$

in distribution's sense.

PROOF. For $f \in \mathcal{D}(\mathcal{G}_Y)$ and $w \in C_0^\infty(R^d)$,

$$\begin{aligned}
&\int_{R^d}\mathcal{G}_Y f(y)w(y)dy \\
&=\lim_{h\downarrow 0}\int_{R^d}\frac{1}{h}\{E[f(Y(T-h, T, y))]-f(y)\}\, w(y)dy \\
&=\lim_{h\downarrow 0}\frac{1}{h}\Big\{E\Big[\int_{R^d}f(Y(T-h, T, y))w(y)dy\Big]-\int_{R^d}f(y)w(y)dy\Big\} \\
&=\lim_{h\downarrow 0}\frac{1}{h}\Big\{E\Big[\int_{R^d}f(x)w(X(T-h, T, x))J(T-h, T, x)dx\Big] \\
&\qquad\qquad -\int_{R^d}f(x)w(x)dx\Big\}\,.
\end{aligned}$$

We can easily check that $J(r, t, x)$ also satisfies equality (2.3), that is,

$$dJ_t=J_t\Big(\sum_{i,j=1}^{d}\frac{\partial a_{ij}}{\partial x_i}(X_t)\circ dB_t^j+\sum_{i=1}^{d}\frac{\partial b_i}{\partial x_i}(X_t)dt\Big)\,.$$

Hence, by Itô's formula,

$$\begin{aligned}
d(w(X_t)J_t)&=J_t\sum_{i,j=1}^{d}\frac{\partial w}{\partial x_i}(X_t)\{a_{ij}(X_t)\circ dB_t^j+b_i(X_t)dt\} \\
&\quad +J_t w(X_t)\Big\{\sum_{i,j=1}^{d}\frac{\partial a_{ij}}{\partial x_i}(X_t)\circ dB_t^j+\sum_{i=1}^{d}\frac{\partial b_i}{\partial x_i}(X_t)dt\Big\} \\
&=J_t\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}(a_{ij}w)(X_t)\circ dB_t^j+J_t\sum_{i=1}^{d}\frac{\partial}{\partial x_i}(b_i w)(X_t)dt \\
&=J_t\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}(a_{ij}w)(X_t)dB_t^j+\frac{1}{2}J_t\Big\{\sum_{i,j,k=1}^{d}\frac{\partial}{\partial x_j}a_{jk}\frac{\partial}{\partial x_i}(a_{ik}, w)\Big\}(X_t)dt
\end{aligned}$$

$$+J_t\sum_{i=1}^{d}\frac{\partial}{\partial x_i}(b_i w)(X_t)dt\,.$$

Since $J(T-h, T-h, x)=1$, we have

$$E[w(X(T-h, T, x))J(T-h, T, x)]$$
$$=w(x)+E\Big[\int_{T-h}^{T}\Big\{\frac{1}{2}\sum_{i,j,k=1}^{d}\frac{\partial}{\partial x_j}(a_{jk}\frac{\partial}{\partial x_i}(a_{ik}w))$$
$$+\sum_{i=1}^{d}\frac{\partial}{\partial x_i}(b_i w)\Big\}(X(T-h, s, x))J(T-h, s, x)ds\Big].$$

Therefore,

$$\int_{R^d}\mathcal{G}_Y f(y)w(y)dy$$
$$=\lim_{h\downarrow 0}\frac{1}{h}\int_{T-h}^{T}ds\int_{R^d}f(x)E\Big[\Big\{\frac{1}{2}\sum_{i,j,k=1}^{d}\frac{\partial}{\partial x_j}(a_{jk}\frac{\partial}{\partial x_i}(a_{ik}w))$$
$$+\sum_{i=1}^{d}\frac{\partial}{\partial x_i}(b_i w)\Big\}(X(T-h, s, x))J(T-h, s, x)\Big]dx$$
$$=\int_{R^d}f(x)\Big\{\frac{1}{2}\sum_{i,j,k=1}^{d}\frac{\partial}{\partial x_j}(a_{jk}\frac{\partial}{\partial x_i}(a_{ik}w))(x)+\sum_{i=1}^{d}\frac{\partial}{\partial x_i}(b_i w)(x)\Big\}dx$$
$$={}_{\mathcal{D}'(R^d)}\Big\langle\frac{1}{2}\sum_{i,j,k=1}^{d}a_{ik}\frac{\partial}{\partial x_i}(a_{jk}\frac{\partial f}{\partial x_j}(x))-\sum_{i=1}^{d}b_i(x)\frac{\partial f}{\partial x_i}(x),\ w\Big\rangle_{C_0^\infty(R^d)}.$$

This implies the conclusion.

REMARK A.3. H. Omori [**9**] showed the fact corresponding to our Lemma 2.2 on a compact manifold. P. Malliavin [**6**] also studied the time reversed processes.

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