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Integration of analytic differential systems with singularities and some applications to real submanifolds of C^n *

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1. Introduction.

A module D of analytic vector fields on $\boldsymbol{R}^n$ defines at each $y \in \boldsymbol{R}^n$ a subspace $D(y)=\{X(y): X \in D\}$ of the tangent space to $\boldsymbol{R}^n$ at y. A real-analytic submanifold M of $\boldsymbol{R}^n$ is an *integral manifold of* D if

$$T_y M = D(y) \qquad \text{for all} \quad y \in M, \tag{1.1}$$

where $T_y M$ is the tangent space to M at y. In [6] Nagano proved that if D is closed under the Lie bracket then through each point there passes a unique integral manifold of D. This result extends the classical Frobenius theorem, which assumes in addition that $D(y)$ has constant dimension. In dropping this hypothesis, Nagano relies on the analyticity. The classical theorem also holds in the C^∞ category and in [6] Nagano gives a simple C^∞ counterexample to his result.

This paper contains 1) a new proof of Nagano's theorem in a formulation which describes the integral manifold directly in terms of D, 2) a sharpened form of the theorem giving necessary and sufficient conditions at p for the existence of an integral manifold through p, and 3) some applications of these results to the local geometry of real-analytic submanifolds of a complex manifold. In particular, it is shown that a point p on a real-analytic CR submanifold M is not of finite weight [1] if and only if there is a complex submanifold of M of maximum dimension through p.

The proofs given here are almost entirely algebraic and make no use of differential equations. They appear to be new even in the classical case where $D(y)$ has constant dimension. Besides a simple and standard majorization argument and advanced calculus, one needs only the standard Weierstrass division theorem [3, Satz 1, p. 23]. All definitions are within the real-analytic category,

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and unless otherwise stated are to be interpreted as germs at a fixed point p of $\boldsymbol{R}^n$ (recall that there is a standard interpretation of (1.1) for germs). The ring of (germs at p of) real-analytic functions is denoted C^ω and the C^ω-module of real-analytic vector fields is denoted $T\boldsymbol{R}^n$.

2. Integration Theorems.

A submanifold M defines the ideal $\boldsymbol{I}_M$ of all functions in C^ω which vanish on M, and $M\supset N$ if and only if $\boldsymbol{I}_M\subset\boldsymbol{I}_N$. If $\boldsymbol{I}$ is an ideal in C^ω then the variety $V(\boldsymbol{I})$ is the set of points where all functions in $\boldsymbol{I}$ vanish. The ideal generated by functions $f_{l+1}, \cdots, f_n$ is denoted $(f_{l+1}, \cdots, f_n)$. If these functions are independent at p then $M=V(f_{l+1}, \cdots, f_n)$ if and only if

$$\boldsymbol{I}_M=(f_{l+1}, \cdots, f_n). \tag{2.1}$$

Of course, in this case M has dimension l. A vector field $X\in T\boldsymbol{R}^n$ is *tangent to* M if $X(y)\in T_yM$ for all $y\in M$. It is elementary that this is equivalent to $Xf_j\in\boldsymbol{I}_M$ for $j=l+1, \cdots, n$. In view of (2.1) and the Leibniz rule $X(fg)=fXg+gXf$,

$$X \textit{ is tangent to } M \textit{ if and only if } \boldsymbol{I}_M \textit{ is closed under } X \tag{2.2}$$

(which, of course, means that $Xf\in\boldsymbol{I}_M$ for every $f\in\boldsymbol{I}_M$). Since C^ω is Noetherian D is finitely generated. Choose now a set $X_1, \cdots, X_k$ of generators for D, and for each k-index $\alpha=(\alpha_1, \cdots, \alpha_k)$ of nonnegative integers let $X^\alpha=X_1^{\alpha_1}\cdots X_k^{\alpha_k}$. In X^α the X_j's always occur in serial order, and as usual X^0 is defined by $X^0f=f$.

If M is an integral manifold of D, (1.1) and (2.2) show that $\boldsymbol{I}_M$ is invariant under D. Therefore, $\boldsymbol{I}_M$ is contained in

$$\boldsymbol{I}_D=\{g\in C^\omega:\ X^\alpha g(0)=0 \quad \text{for all} \quad \alpha\}. \tag{2.3}$$

It follows from the Leibniz rule that $\boldsymbol{I}_D$ is an ideal in C^ω. If D is bracket-closed, the Leibniz rule and Lemma 2.3 below show that $\boldsymbol{I}_D$ is independent of a choice of generators for D. This ideal provides an invariant means for utilizing the basic principle that an analytic function is determined by its derivatives at a fixed point. The following is equivalent to Nagano's theorem 1 in [6].

THEOREM 2.1. *If D is a bracket-closed submodule of $T\boldsymbol{R}^n$ then $V(\boldsymbol{I}_D)$ is the unique integral manifold of D through p.*

PROOF. In order that $V(\boldsymbol{I}_D)$ is a manifold, there must exist independent functions $f_{l+1}, \cdots, f_n$ such that

$$\boldsymbol{I}_D=(f_{l+1}, \cdots, f_n), \tag{2.4}$$

and if $V(\boldsymbol{I}_D)$ is an integral manifold then $l=\dim D(p)$ must hold. It will be shown first that there exist such functions, and later that $V(\boldsymbol{I}_D)$ is indeed an integral manifold.

It may be assumed that $p=0$. In terms of analytic coordinates $x=(x_1, \cdots, x_n)$ each $X_i=\sum_{j=1}^{n} a_{ij}\partial/\partial x_j$, where the matrix (a_{ij}) has rank $l=\dim D(0)$ at 0. By a reindexing, it may be assumed that $(a_{ij})_{i,j=1}^{l}$ is non-singular at 0. Since elementary row operations over C^ω can be achieved by linear combination of the X_i, it may be assumed that the generators X_i of D have the form

$$X_i=\partial/\partial x_i+\sum_{j=l+1}^{n} a_{ij}\partial/\partial x_j\,, \qquad i=1, \cdots, l, \quad \text{and}$$
$$X_i=\sum_{j=l+1}^{n} a_{ij}\partial/\partial x_j\,, \qquad i=l+1, \cdots, k\,, \tag{2.5}$$

and $a_{ij}(0)=0$, $i, j>l$. This has the usual geometric interpretation that an integral manifold is spread over the $x_1, \cdots, x_l$ coordinate space. Since it must be the graph of a function of these variables it is natural to seek defining functions of the form

$$f_i=x_i-g_i(x_1, \cdots, x_l)\,, \qquad i=l+1, \cdots, n\,, \tag{2.6}$$

where each g_i is analytic.

An analytic function depending only on $x_1, \cdots, x_l$ will be written $g=\sum'_\alpha a_\alpha x^\alpha$, where the prime means that summation extends only over those α such that $\alpha_{l+1}=\cdots=\alpha_n=0$. By (2.5) such a function satisfies $X_i g=\partial g/\partial x_i$, $i\leqq l$, so that

$$X^\alpha g(0)=\partial^{|\alpha|} g/\partial x^\alpha(0)=a_\alpha \alpha\,! \tag{2.7}$$

if $\alpha_{l+1}=\cdots=\alpha_n=0$.

Therefore, in order that (2.6) defines functions in $\boldsymbol{I}_D$ it is necessary that

$$g_i=\sum'(\alpha\,!)^{-1}X^\alpha(x_i)(0)x^\alpha\,. \tag{2.8}$$

LEMMA 2.2. *The series* (2.8) *converges.*

This will be proved later.

In the classical Frobenius theorem $D(y)$ has constant dimension, which is equivalent to $k=l$. In this case membership of f_i in $\boldsymbol{I}_D$ is immediate from (2.6), (2.7), and (2.8). When $k>l$ this fact also depends on the property that

$$\textit{for any } f,\ X_i f(0)=0 \quad \textit{if} \quad i>l\,, \tag{2.9}$$

and on Lemma 2.3 below. A *polynomial* is any linear combination over C^ω of operators of the form X^α as defined above. If α and β are multiindices then $\alpha+\beta$ is their usual vector sum and $|\alpha|=\alpha_1+\cdots+\alpha_n$. For $1\leqq j\leqq l$, $e^j=(0, \cdots, 0, 1, 0, \cdots, 0)$ (the 1 in the j-th place), and $\alpha\pm e^j$ is written $\alpha\pm j$.

LEMMA 2.3. *If D is bracket closed then for any α, β $X^\alpha X^\beta-X^{\alpha+\beta}$ is a polynomial of degree less than $|\alpha+\beta|$.*

This is also proved later. It is now used to show by induction on $|\gamma|$ that

$$X^{\gamma}f_j(0)=0\,, \qquad j=l+1, \cdots, n\,. \tag{2.10}$$

Relations (2.7) and (2.9) imply the case $|\gamma|=1$. Now assume $m>1$ and (2.10) holds for all $|\gamma|<m$, and suppose $|\gamma|=m$. If $\gamma_{l+1}=\cdots=\gamma_n=0$ then (2.10) holds as shown above. Otherwise $\gamma=\beta+i$ for some $i>l$ and Lemma 2.3 with $\alpha=e^i$ shows that $X^{\gamma}=X_iX^{\beta}+$(polynomial of degree$<m$). Thus (2.10) follows for γ by the induction hypothesis and (2.9). This proves that each $f_i\in \boldsymbol{I}_D$ and hence

$$(f_{l+1}, \cdots, f_n)\subset \boldsymbol{I}_D\,. \tag{2.11}$$

To show (2.4), suppose that f is any function in C^{ω}. By the Weierstrass Division Theorem [3] $f=a_{l+1}f_{l+1}+r_{l+1}$ where r_{l+1} is a polynomial in x_{l+1} of degree less than the order of the zero $x_{l+1}\to f_{l+1}(0, \cdots, 0, x_{l+1}, 0, \cdots, 0)=x_{l+1}$ has at 0. In other words, r_{l+1} is independent of x_{l+1}. So is f_{l+2}, and hence r_{l+1} may be divided by f_{l+2} in the subring of functions independent of x_{l+1} to obtain $f=a_{l+1}f_{l+1}+a_{l+2}f_{l+2}+r_{l+2}$, where r_{l+2} is independent of x_{l+1} and x_{l+2}. Continuing in this way will achieve

$$f=a_{l+1}f_{l+1}+\cdots+a_nf_n+r_n \tag{2.12}$$

where r_n depends only on $x_1, \cdots, x_l$. If also $f\in \boldsymbol{I}_D$, then so does r_n by (2.11). By (2.7), $r_n=0$. This proves (2.4), and hence $M=V(\boldsymbol{I}_D)=V(f_{l+1}, \cdots, f_n)$ is a manifold.

Next, it is claimed that

$$\boldsymbol{I}_D \textit{ is closed under } D\,. \tag{2.13}$$

This is proved by an easy induction on Lemma 2.3, this time with $|\beta|=1$. It follows that M is an integral manifold, for (2.2) and (2.13) imply that each vector field in D is tangent to M. Hence $D(y)\subset T_yM$ for all y on M near 0. This implies (1.1) since $\dim D(y)\geqq \dim D(0)=l=\dim T_yM$ for all y near 0.

It remains to prove the uniqueness. If N is any integral manifold through 0, (1.1) implies that $D\subset TN$, the module of vector fields tangent to N. Therefore $\boldsymbol{I}_M=\boldsymbol{I}_D\supset \boldsymbol{I}_{TN}\supset \boldsymbol{I}_N$, in which the containments are obvious from the definition of $\boldsymbol{I}_{TN}$ as in (2.3) (with TN replacing D and any choice of generators for TN). It follows that $M\subset N$, and since both manifolds have dimension l they are equal. This completes the proof of Theorem 2.1.

PROOF OF LEMMA 2.2. This simple majorization argument is given in detail (and greater generality) in [4], so it is only sketched here. There exist $\rho>0$ and $M>0$ such that the geometric series $G(t)=M(1-(t/\rho))^{-1}$ majorizes f and all a_{ij} at 0, and such that $X_j(x_i)$ is majorized by ΔG at 0, where Δ is the operator $\Delta=G(t)\,d/dt$. It follows inductively that each $X^{\alpha}(x_i)$ is majorized by $\Delta^{|\alpha|}G=M^{|\alpha|+1}\rho^{-|\alpha|}(1\cdot 3\cdot 5\cdots(2|\alpha|-1))(1-(t/\rho))^{2|\alpha|+1}$. Therefore $|(X^{\alpha}(x_i))(0)|\leqq M^{|\alpha|+1}\rho^{-|\alpha|}(1\cdot 3\cdot 5\cdots(2|\alpha|-1))$, and the ratio test now gives convergence of (2.8).

PROOF OF LEMMA 2.3. If $|\alpha+\beta|=2$ the lemma is either trivial ($\alpha=0$ or $\beta=0$), or is just a restatement of the bracket closure hypothesis, which is equivalent to

$$X_iX_j-X_jX_i=\sum_{q=1}^{k} a_{ij}^q X_q. \tag{2.14}$$

Assume $m\geqq 2$ and the lemma holds for all α, β with $|\alpha+\beta|\leqq m$. It follows that if $|\alpha|+q\leqq m$ and P is a polynomial of degree $\leqq q$ then $X^\alpha P$ and PX^α are polynomials of degree $\leqq|\alpha|+q$. This is obvious for PX^α, and for $X^\alpha P$ a simple induction on the Leibniz rule will account for the differentiations of the coefficients of P by X^α.

Now suppose that $|\alpha+\beta|=m+1$. It may be assumed that $\beta\neq 0$.

The case $|\beta|=1$: Then $\beta=e^j$. Let i be the largest integer r such that $\alpha_r\neq 0$. If $i\leqq j$ then $X^\alpha X^\beta=X^\alpha X_j=X^{\alpha+j}=X^{\alpha+\beta}$. Otherwise, by (2.14), $X^\alpha X^j=X^{\alpha-i}X_iX_j=X^{\alpha-i}X_jX_i+X^{\alpha-i}P$ where P is a polynomial of degree 1. Since $|\alpha-i|=m-1$ the inductive hypothesis yields $X^{\alpha-i}X_j=X^{\alpha-i+j}+Q$, where Q is a polynomial of degree at most $m-1$. Thus $X^\alpha X_j=X^{\alpha-i+j}X_i+QX_i+X^{\alpha-i}P=X^{\alpha+j}$ +(polynomial of degree $\leqq m$) because $i>j$ and the inductive assumption applies to $QX_i+X^{\alpha-i}P$. This establishes the case $|\beta|=1$.

The general case $|\beta|\geqq 1$: Then $\beta=\gamma+e^j$, where j is the largest integer r such that $\beta_r\neq 0$, and $|\alpha+\gamma|=m$. Hence $X^\alpha X^\beta=X^\alpha X^\gamma X_j=X^{\alpha+\gamma}X_j+PX_j$, where P is a polynomial of degree at most $m-1$. By Case 1 applied to $X^{\alpha+\gamma}X_j$ and the inductive assumption applied to PX_j, $X^\alpha X^\beta=X^{\alpha+\gamma+j}+Q=X^{\alpha+\beta}+Q$, where Q is a polynomial of degree at most m. This proves Lemma 2.3.

The condition (1.1) can be replaced by

$$TM=D+O(M) \tag{2.15}$$

where $O(M)$ denotes all vector fields vanishing on M and TM is the module of vector fields tangent to M. Condition (2.15) appears stronger than (1.1) (after all $\mathrm{sp}_{\boldsymbol{R}}\{x(d/dx)\}=\mathrm{sp}_{\boldsymbol{R}}\{x^2d/dx\}$ for each x in $\boldsymbol{R}$ but xd/dx is not in the C^ω-module spanned by $x^2(d/dx)$), but in fact it is equivalent. For certainly (2.15) implies (1.1) and if (1.1) holds then clearly $D\subset TM$. If $X\in TM$ there exist by (2.5) functions $c_1,\cdots,c_l$ in C^ω such that $Y=X-\sum_{i=1}^{l}c_iX_i$ is free from terms involving $\partial/\partial x_1,\cdots,\partial/\partial x_l$. Since Y is also tangent to M, it is easy to see that $Y\in O(M)$, and (2.15) is proved.

If p is fixed in Theorem 2.1, the bracket-closure is not necessary. For example, let $X=\partial/\partial x$ and $Y=\partial/\partial y+xz\partial/\partial z$ in $\boldsymbol{R}^3$ with coordinates (x, y, z). Then $V(z)$ is an integral manifold through O of $D=\mathrm{sp}\{X, Y\}$, but $[X, Y]=z\partial/\partial z\notin D$. Let $\tilde{D}$ denote the smallest bracket-closed submodule containing a given submodule D.

COROLLARY 2.4. *There exists an integral manifold of D through p if and* *only if*

$$\tilde{D}(p)=D(p), \tag{2.16}$$

and an integral manifold through p is unique.

PROOF. If M is any manifold the module TM is bracket-closed. Thus if M is an integral manifold then $D\subset TM$ implies

$$D\subset\tilde{D}\subset TM=D+O(M) \tag{2.17}$$

which in turn implies (2.16). It also shows that M is an integral manifold of $\tilde{D}$, which implies its uniqueness by that part of Theorem 2.1.

For the converse, apply Theorem 2.1 to $\tilde{D}$ and note that its integral manifold M through p satisfies $T_yM=\tilde{D}(y)\supset D(y)$ for all y on M near p. Because of (2.16) $T_pM=D(p)$. Hence for y on M near p, $\dim D(y)\geqq\dim T_yM$, and consequently $T_yM=D(y)$. This proves the Corollary.

In the example above $[X, [X, Y]]=0=[Y, [X, Y]]$ so $\tilde{D}=\mathrm{sp}\{X, Y, [X, Y]\}$ and $\tilde{D}(0)=\mathrm{sp}_{\boldsymbol{R}}\{\partial/\partial x, \partial/\partial y\}=D(0)$.

3. Existence of complex submanifolds of a real-analytic submanifold.

Let $C_C^\omega=C^\omega+iC^\omega$ be the (Noetherian) ring of complex-valued real-analytic functions and $\boldsymbol{CTR}^n=\boldsymbol{TR}^n+i\boldsymbol{TR}^n$ the complexified space of vector fields. If M is a real analytic submanifold then $\boldsymbol{I}_M$ now denotes the ideal of all functions in C_C^ω vanishing on M, and $\boldsymbol{CTM}=\boldsymbol{TM}+i\boldsymbol{TM}$ the complexified space of vector fields tangent to M. It is clear that $\boldsymbol{CTM}$ is a bracket-closed module over C_C^ω and that it is defined by (2.2).

There is a natural conjugation $Z=X+iY\rightarrow\bar{Z}=X-iY$ on $\boldsymbol{CTR}^n$, where X and Y are in $\boldsymbol{TR}^n$, and a submodule D of $\boldsymbol{CTR}^n$ is *real* if $D=\bar{D}$. Defining $\mathrm{Re}\,D=D\cap\boldsymbol{TR}^n$ it is easy to see that D is real if and only if $D=\mathrm{Re}\,D+i\,\mathrm{Re}\,D$, and that a real D is bracket closed if and only if $\mathrm{Re}\,D$ is bracket closed. Let $\tilde{D}$ be the bracket closure of D in $\boldsymbol{CTR}^n$, and $\widetilde{\mathrm{Re}\,D}$ the bracket closure of $\mathrm{Re}\,D$ in $\boldsymbol{TR}^n$ (as in Section 2). If D is real then it is easily seen that $\tilde{D}=\widetilde{\mathrm{Re}\,D}+i\,\widetilde{\mathrm{Re}\,D}$, so that $\tilde{D}$ is also real and $\mathrm{Re}\,\tilde{D}=\widetilde{\mathrm{Re}\,D}$.

A submanifold M is an *integral manifold* of D if

$$\boldsymbol{CTM}=D+O(M), \tag{3.1}$$

where $O(M)$ is the C_C^ω module of vector fields vanishing on M. Since $\boldsymbol{CTM}$ and $O(M)$ are real, so is D if (3.1) holds. It is easy to see that (3.1) holds if and only if $TM=\mathrm{Re}\,D+\mathrm{Re}\,O(M)$, so that M is an integral manifold of D if and only if it is for $\mathrm{Re}\,D$.

Therefore the extension of Theorem 2.1 to submodules of $\boldsymbol{CTR}^n$ is the following.

THEOREM 3.1. *If a real submodule D of $\boldsymbol{CTR}^n$ is bracket closed then it has a unique integral manifold through p.*

Moreover, by the above remarks, Corollary 2.4 is true verbatim for real submodules of $\boldsymbol{CTR}^n$, where $\tilde{D}$ is interpreted as above.

If $\boldsymbol{R}^{2n}$ is the real space underlying $\boldsymbol{C}^n$, one writes $\boldsymbol{CTC}^n$ for $\boldsymbol{CTR}^{2n}$. As usual, it carries another complex structure induced from $\boldsymbol{C}^n$, whose action is denoted by J. The module of *complex tangent* vector fields to a submanifold M is $H=\boldsymbol{CTM}\cap J\boldsymbol{CTM}$, the largest J-invariant submodule of $\boldsymbol{CTM}$. The submanifold M is complex if and only if $H=\boldsymbol{CTM}$; i.e. $\boldsymbol{CTM}$ is J-invariant. Since $O(M)$ is J-invariant, it follows from (3.1) that

$$\text{an integral manifold of } D \text{ is complex if } D \text{ is } J\text{-invariant.} \tag{3.2}$$

It is shown in [2] that if M is any submanifold then $N_2=\{X\in H:[X, H]\subset H\}$ is real, J-invariant, and bracket closed (N_2 can often be viewed as the null-space of the E.E. Levi form of M). It is also shown in [2] that there exists a chain (which in general is strictly decreasing) $N_2\supset N_3\supset\cdots\supset S_0=\bigcap_{j=2}^{\infty}N_j$ of real, J-invariant, bracket closed submodules. The following is thus a consequence of Theorem 3.1.

THEOREM 3.2. *A real analytic submanifold M has a chain of successively finer partitions by the complex integral manifolds of the submodules N_j.*

It is possible for M to have a complex submanifold larger than the integral manifold of N_2. In [7], Sommer showed that $M=\{|z_1|^2+|z_2|^2-|z_3|^2-1=0\}$ has 1-dimensional complex submanifolds but $N_2=0$. It is therefore of interest to seek an integral manifold of H through a fixed point p on M. In case M is CR, this would be the largest complex submanifold of M through p. The answer is immediate from Corollary 2.4 and (3.2).

THEOREM 3.3. *There exists a (unique) integral manifold of H through $p\in M$ if and only if $H(p)=\tilde{H}(p)$. This integral manifold is a complex submanifold of M.*

All statements of Theorem 3.3 but the last are proved above. If S is an integral manifold of H through p then $H\subset\boldsymbol{CTM}$ implies $\boldsymbol{I}_S=\boldsymbol{I}_H\supset\boldsymbol{I}_{\boldsymbol{CTM}}=\boldsymbol{I}_M$, so that $S\subset M$. It is easy to see that $\tilde{H}$ is the module spanned over C_C^ω by the set H' of all repeated commutators of elements of H. Therefore $\tilde{H}(p)=H(p)$ if and only if $H'(p)=H(p)$. This is a kind of infinitely high-order flatness condition at p. It means that p is not a point of finite type in the sense of Bloom and Graham [1] or Kohn [5], and it causes behavior in contrast with the case of finite type. Thus if M is a real analytic hypersurface satisfying Theorem 3.3 there exists a neighborhood U of p such that the integral manifold of H through z is the zero set of a holomorphic function f. Then $1/f$ is a holomorphic function on $U-M$ which cannot be continued across p. The local Levi problem [5, p. 528] is thus solvable near p in this case. However, such a point is clearly never a local peak point [5, p. 540].

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