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The Gauss map of a submanifold in a Euclidean space

By Yosio MUTŌ

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§ 1. Introduction.

It is well-known that the Gauss map is an excellent device in classical differential geometry of curves and surfaces in a Euclidean three-space. The idea of Gauss map can be generalized to the case of an m-dimensional submanifold M' in a Euclidean n-space E^n. In this case the image lies in the Grassmann manifold $G(m, n-m)$. It is conceivable that the Gauss map in this sense will be an even more useful device in the differential geometry of submanifolds in a Euclidean n-space. The Gauss map was generalized one step further by M. Obata [2] to the case of an m-dimensional submanifold in V where V denotes one of the following Riemannian manifolds of dimension N: (i) An N-sphere S^N of radius a, (ii) A Euclidean N-space, (iii) A hyperbolic N-space of curvature $-1/a^2$. Then the image lies in $Q=G(N)/G(m)\times O(N-m)$ (for $G(N)$ see [2]).

On the other hand E. A. Ruh and J. Vilms [3] studied neighborhoods of the Gauss map in the first generalized sense and obtained the following theorems.

THEOREM A. *If M is immersed with parallel mean curvature vector into E^n, then the Gauss map is harmonic.*

THEOREM B. *The Gauss map of a minimal submanifold in E^n is harmonic.*

The present author also took great interest in some problems concerning the Gauss map. In the present paper the following subjects are treated.

(i) Gauss-critical submanifolds, or Gauss-critical immersions.

(ii) Gauss-critical submanifolds with respect to which the Gauss map is homothetic.

(iii) Submanifolds M' in E^n such that the sectional curvature of the Grassmann manifold $G(m, n-m)$ in the tangent planes of the Gauss image totally vanishes.

(ii) is considered since submanifolds with homothetic Gauss map have some characteristic properties. In this respect the following theorems of M. Obata are suggestive [2].

THEOREM C. *Let x be an isometric immersion of an Einstein space into V.*

Then x is pseudo-umbilical if and only if the Gauss map is conformal.

THEOREM D. *Let x be a pseudo-umbilical immersion of a Riemannian manifold M into a V. Then the Gauss map is conformal if and only if M is Einsteinian. In the case* $\dim M>2$, *the Gauss map is homothetic if and only if M is Einsteinian.*

Let M be a compact orientable C^∞ manifold of dimension m and i an immersion of M into an n-dimensional Euclidean space E^n. Then we get a submanifold (iM, g_i) where g_i is the Riemannian metric induced naturally from the standard Riemannian metric on E^n. From the Gauss map of iM into $G(m, n-m)$ we define *the Gauss map associated with the immersion i* and denote it by $\Gamma_i: M\to G(m, n-m)$, so that $\Gamma_i(M)$ is the Gauss image of iM. As studied by K. Leichtweiss [1] and Y.-C. Wong [5] the Grassmann manifold $G(m, n-m)$ bears a standard Riemannian metric $\tilde{g}$ such that $(G(m, n-m), \tilde{g})$ is a symmetric space. Then we have $\Gamma_i: M\to(G(m, n-m), \tilde{g})$. We consider in the present paper only the case where Γ_i is regular, hence the second fundamental form of iM does not vanish. We also assume constantly that M is C^∞.

Let G_i be the Riemannian metric on $\Gamma_i(M)$ induced from $\tilde{g}$. Then we can define $\mathrm{Vol}^*(\Gamma_i(M), G_i)$. Let $\mathcal{I}_M$ be the space of all immersions i of M into E^n such that Γ_i is regular. Then $\mathrm{Vol}^*(\Gamma_i(M), G_i)$ defines a mapping $\mathrm{Vol}^*: \mathcal{I}_M\to \boldsymbol{R}$. A critical point of this mapping Vol^* will be called a *Gauss-critical immersion* and, if i is such an immersion, the submanifold iM is called a *Gauss-critical submanifold* and is denoted *GCS*. The equation of a *GCS* and the following theorem are obtained where Γ_i is called a homothetic mapping when the Gauss map $\Gamma: (iM, g_i)\to(\Gamma_i(M), G_i)$ is homothetic.

THEOREM I. *Let M be a given compact orientable manifold of dimension m. If $i: M\to E^n$ is a Gauss-critical immersion and at the same time (iM, g_i) is an Einsteinian submanifold and moreover Γ_i is a homothetic mapping, then the components of the mean curvature vector of iM are eigenfunctions of the Laplacian on (iM, g_i) belonging to an eigenvalue λ. If $i: M\to E^n$ is an immersion such that (iM, g_i) is an Einsteinian submanifold, Γ_i is a homothetic mapping and moreover the components of the mean curvature vector of iM are eigenfunctions of the Laplacian on (iM, g_i) belonging to one and the same eigenvalue, then i is a Gauss-critical immersion.*

In this theorem we consider a critical point of the mapping $\mathrm{Vol}^*: \mathcal{I}_M\to R$, hence i moves in $\mathcal{I}_M$ and the integral represents the volume, whereas in the theorem of Ruh and Vilms the immersion considered is such that the associated Gauss map is a critical point of the energy integral. So to say, in our theorem we consider nothing but the Gauss maps but in the theorem of Ruh and Vilms Gauss maps are compared with other maps.

The sectional curvature of $(G(m, n-m), \tilde{g})$ in a section σ lying in a tangent plane of $\Gamma_i(M)$ is called the *sectional curvature of the Grassmann manifold in the Gauss map* Γ_i or shortly the *sectional curvature of the Gauss map* Γ_i and is denoted by $K_{\Gamma,i}(\sigma)$. A necessary and sufficient condition that $K_{\Gamma,i}(\sigma)$ totally vanish is obtained and some examples are given. The following theorem is proved.

THEOREM II. *Let M be a manifold of dimension m and Γ_i the Gauss map associated with an immersion $i: M \to E^n$. Assume Γ_i to be regular. If $n<2m$, then the sectional curvature of the Gauss map Γ_i cannot totally vanish.*

In § 2 we introduce local coordinates in a neighborhood U of a point Π_0 of $G(m, n-m)$ and get the formulas of the curvature tensor and the sectional curvature of the Grassmann manifold. Such formulas greatly facilitate subsequent calculations. In § 3 we introduce the Gauss map Γ_i and define the Riemannian metric G_i associated with an immersion i. In § 4 the equation of a *GCS* is obtained and Theorem I is proved. § 5 is devoted to the study of sectional curvature of the Gauss map. In § 6 some examples including T^2 in E^4 and the Veronese surface in E^5 are given.

The present paper is intended rather as a preliminary one and in a forthcoming paper some properties of Γ_i which is homothetic and the image $\Gamma_i(M)$ is totally geodesic in $(G(m, n-m), \tilde{g})$ will be studied.

§ 2. The Grassmann manifold $G(m, n-m)$.

We consider here a suitable neighborhood U of a point Π_0 of a Grassmann manifold $G(m, n-m)$ and introduce local coordinates valid in U. For this purpose we fix in E^n an orthonormal frame

$$e_1, e_2, \cdots, e_m, e_{m+1}, \cdots, e_n$$

where the vectors e_α $(\alpha=1, \cdots, m)$ lie in Π_0 and the vectors e_x $(x=m+1, \cdots, n)$ are normal to Π_0. Let Π be a point in U and (f_α, f_x) an orthonormal frame where f_α lie in Π and f_x are normal to Π. Then we can put

$$f_\alpha = \xi_{\alpha\beta} e_\beta + \xi_{\alpha y} e_y , \qquad f_x = \xi_{x\beta} e_\beta + \xi_{xy} e_y$$

where, here and in the sequel, the summation convention is adopted for repeated indices in each term and the range of indices are as follows,

$$\alpha, \beta, \gamma, \delta, \cdots = 1, \cdots, m ; \qquad x, y, z, u, \cdots = m+1, \cdots, n .$$

Though the frame (e_α, e_x) is fixed, there remains some freedom in taking the frame (f_α, f_x) and we can take

$$f_\alpha' = \gamma_{\alpha\beta} f_\beta , \qquad f_x' = \gamma_{xy} f_y$$

in stead of f_α and f_x where $(\gamma_{\alpha\beta})$ and (γ_{xy}) are orthogonal matrices. We can choose them such that

$$\gamma_{\alpha\beta}\xi_{\beta\gamma}=\gamma_{\gamma\beta}\xi_{\beta\alpha}, \qquad \gamma_{xy}\xi_{yz}=\gamma_{zy}\xi_{yx}$$

since there exist for any square matrix A an orthogonal matrix R and a symmetric matrix S satisfying $A=RS$. This means that we can take a frame (f_α, f_x) such that

$$\langle f_\alpha, e_\beta\rangle=\langle f_\beta, e_\alpha\rangle, \qquad \langle f_x, e_y\rangle=\langle f_y, e_x\rangle.$$

Then taking U suitably we can put

$$\begin{aligned} f_\alpha&=(\delta_{\alpha\beta}+\xi_{\alpha\beta})e_\beta+\xi_{\alpha y}e_y, \\ f_x&=\xi_{x\beta}e_\beta+(\delta_{xy}+\xi_{xy})e_y \end{aligned} \tag{2.1}$$

where $\xi_{\alpha\beta}=\xi_{\beta\alpha}$, $\xi_{xy}=\xi_{yx}$ and $|\xi_{\alpha\beta}|$, $|\xi_{x\beta}|$, $|\xi_{\alpha y}|$, $|\xi_{xy}|$ are smaller than a certain number $\varepsilon>0$. As (f_α, f_x) is an orthonormal frame we get

$$\begin{aligned} &\xi_{\alpha\gamma}+\frac{1}{2}(\xi_{\alpha\beta}\xi_{\gamma\beta}+\xi_{\alpha y}\xi_{\gamma y})=0, \\ &\xi_{x\alpha}+\xi_{\alpha x}+\xi_{\alpha\beta}\xi_{x\beta}+\xi_{\alpha y}\xi_{xy}=0, \\ &\xi_{xz}+\frac{1}{2}(\xi_{x\beta}\xi_{z\beta}+\xi_{xy}\xi_{zy})=0. \end{aligned} \tag{2.2}$$

If U is such that ε is sufficiently small, we can solve (2.2) and get

$$\begin{aligned} &\xi_{\alpha\beta}=-\frac{1}{2}\xi_{\alpha y}\xi_{\beta y}+O(\varepsilon^3), \\ &\xi_{x\alpha}=-\xi_{\alpha x}+O(\varepsilon^3), \\ &\xi_{xy}=-\frac{1}{2}\xi_{\alpha x}\xi_{\alpha y}+O(\varepsilon^3). \end{aligned} \tag{2.3}$$

This proves that $m(n-m)$ numbers $\xi_{\alpha x}$ can be used as local coordinates in U.

Let us study Riemannian geometry of $G(m, n-m)$ with the use of such local coordinates (see [**1**], [**5**]).

Let $\Pi(\xi)$ be a point of U whose local coordinates are $\xi_{\alpha x}$ and $\Pi(\xi+d\xi)$ a point whose local coordinates are $\xi_{\alpha x}+d\xi_{\alpha x}$. Then the distance $ds=d(\Pi(\xi), \Pi(\xi+d\xi))$ is given, according to Leichtweiss, by

$$ds^2=\sum_{\alpha,x}\langle df_\alpha, f_x\rangle^2=\langle df_\alpha, f_x\rangle\langle df_\alpha, f_x\rangle$$

where

$$df_\alpha=d\xi_{\alpha\beta}e_\beta+d\xi_{\alpha y}e_y.$$

Hence we get

$$ds^2=\xi_{x\beta}\xi_{x\gamma}d\xi_{\alpha\beta}d\xi_{\alpha\gamma}+2(\delta_{xy}+\xi_{xy})\xi_{x\beta}d\xi_{\alpha\beta}d\xi_{\alpha y}$$
$$+(\delta_{xy}+\xi_{xy})(\delta_{xz}+\xi_{xz})d\xi_{\alpha y}d\xi_{\alpha z}\,.$$

Let us denote the components of $\tilde{g}$ with respect to the local coordinates $\xi_{\alpha x}$ by $\tilde{g}_{\beta y,\alpha x}$ so that

$$ds^2=\tilde{g}_{\beta y,\alpha x}d\xi_{\beta y}d\xi_{\alpha x}\,.$$

Then we get

$$\tilde{g}_{\beta y,\alpha x}=\xi_{z\delta}\xi_{z\gamma}\frac{\partial\xi_{\varepsilon\delta}}{\partial\xi_{\beta y}}\frac{\partial\xi_{\varepsilon\gamma}}{\partial\xi_{\alpha x}}$$
$$+(\delta_{zy}+\xi_{zy})\xi_{z\gamma}\frac{\partial\xi_{\beta\gamma}}{\partial\xi_{\alpha x}}+(\delta_{zx}+\xi_{zx})\xi_{z\gamma}\frac{\partial\xi_{\alpha\gamma}}{\partial\xi_{\beta y}}$$
$$+(\delta_{zy}+\xi_{zy})(\delta_{zx}+\xi_{zx})\delta_{\beta\alpha}$$

which becomes on account of (2.2), (2.3)

$$\tilde{g}_{\beta y,\alpha x}=\delta_{yx}\delta_{\beta\alpha}+\xi_{\alpha y}\xi_{\beta x}+O(\varepsilon^4)\,. \tag{2.4}$$

For the contravariant components we get

$$\tilde{g}^{\beta y,\alpha x}=\delta_{yx}\delta_{\beta\alpha}-\xi_{\alpha y}\xi_{\beta x}+O(\varepsilon^4)\,. \tag{2.5}$$

From (2.4) and (2.5) we get for the Christoffel symbols

$$\left\{ {\gamma z \atop \beta y,\ \alpha x} \right\}=\frac{1}{2}(-\delta_{\gamma\alpha}\delta_{zy}\xi_{\beta x}-\delta_{\gamma\beta}\delta_{zx}\xi_{\alpha y}+\delta_{\gamma\alpha}\delta_{yx}\xi_{\beta z} \tag{2.6}$$
$$+\delta_{\beta\alpha}\delta_{zx}\xi_{\gamma y}+\delta_{\beta\alpha}\delta_{zy}\xi_{\gamma x}+\delta_{\gamma\beta}\delta_{yx}\xi_{\alpha z})+O(\varepsilon^3).$$

The curvature tensor and the Ricci tensor are given by

$$\tilde{K}_{\delta u,\gamma z,\beta y,\alpha x}=(\delta_{\delta\alpha}\delta_{\gamma\beta}-\delta_{\delta\beta}\delta_{\gamma\alpha})\delta_{uz}\delta_{yx} \tag{2.7}$$
$$+\delta_{\delta\gamma}\delta_{\beta\alpha}(\delta_{ux}\delta_{zy}-\delta_{uy}\delta_{zx})+O(\varepsilon^2)\,,$$

$$\tilde{K}_{\gamma z,\beta y}=(n-2)\delta_{\gamma\beta}\delta_{zy}+O(\varepsilon^2)\,. \tag{2.8}$$

These are not invariant expressions. But we get from (2.4), (2.8) $\tilde{K}_{BA}=(n-2)\tilde{g}_{BA}$ which is valid for any local coordinates ξ^A $(A=1,\cdots,m(n-m))$ of $G(m,n-m)$.

With the use of (2.7) we can calculate the sectional curvature $K(\sigma)$ of $(G(m,n-m),\tilde{g})$. Let (u,v) be a pair of orthonormal tangent vectors of $G(m,n-m)$ at a point Π_0 and denote their components by $u^{\alpha x}$, $v^{\alpha x}$. Let $\sigma=\sigma(u,v)$ be a 2-plane spanned by u and v. Then, for the sectional curvature $K(\sigma(u,v))$, we get from (2.7)

$$K(\sigma(u,v))=v^{\alpha y}v^{\alpha x}u^{\beta y}u^{\beta x}-v^{\beta y}v^{\alpha x}u^{\alpha y}u^{\beta x}$$
$$+v^{\beta x}v^{\alpha x}u^{\beta y}u^{\alpha y}-v^{\beta y}v^{\alpha x}u^{\beta x}u^{\alpha y},$$

where $u^{\alpha x}u^{\alpha x}=v^{\alpha x}v^{\alpha x}=1$, $u^{\alpha x}v^{\alpha x}=0$. But we have identities

$$(u^{\beta x}v^{\alpha x}-v^{\beta x}u^{\alpha x})(u^{\beta y}v^{\alpha y}-v^{\beta y}u^{\alpha y})$$
$$=2(u^{\beta x}u^{\beta y}v^{\alpha x}v^{\alpha y}-u^{\beta x}u^{\alpha y}v^{\alpha x}v^{\beta y}),$$
$$(u^{\beta y}v^{\beta x}-v^{\beta y}u^{\beta x})(u^{\alpha y}v^{\alpha x}-v^{\alpha y}u^{\alpha x})$$
$$=2(u^{\beta y}u^{\alpha y}v^{\beta x}v^{\alpha x}-u^{\beta y}u^{\alpha x}v^{\beta x}v^{\alpha y}).$$

Hence we get

$$K(\sigma(u,v))=\frac{1}{2}(P^{\beta\alpha}P^{\beta\alpha}+Q^{yx}Q^{yx}) \tag{2.9}$$

where

$$\begin{aligned} P^{\beta\alpha}&=u^{\beta x}v^{\alpha x}-v^{\beta x}u^{\alpha x},\\ Q^{yx}&=u^{\alpha y}v^{\alpha x}-v^{\alpha y}u^{\alpha x}. \end{aligned} \tag{2.10}$$

§ 3. The Gauss map of a submanifold of a Euclidean space.

Let i be an immersion of an m-dimensional manifold M into an n-space E^n and assume that the image iM is given in each suitable neighborhood V of M by

$$x^h=x^h(y^1,\cdots,y^m)$$

where x^h $(h=1,\cdots,n)$ are rectangular coordinates and y^κ $(\kappa=1,\cdots,m)$ local coordinates of M in V. Define $B_\lambda{}^h$ by $B_\lambda{}^h=\partial x^h/\partial y^\lambda$. At each point $p\in V$ and for each λ $(\lambda=1,\cdots,m)$ the vector b_λ whose components are $B_\lambda{}^h(p)$ is a tangent vector of iM at ip. The tangent plane $iM_{ip}=i(M_p)$ can be taken, after a suitable parallel displacement, as a point $\Gamma(p)$ of the Grassmann manifold $G(m,n-m)$ and from this fact we get naturally a mapping $\Gamma: iM\to G(m,n-m)$, namely, $\Gamma_i: M\to G(m,n-m)$. Γ is called a Gauss map and Γ_i a *Gauss map associated with the immersion i*. In order to avoid possible difficulties we consider only the case of regular mapping.

Let us take at each point ip where $p\in V$ an orthonormal frame (f_α,f_x) of E^n such that f_α $(\alpha=1,\cdots,m)$ are vectors in $i(M_p)$ and f_x $(x=m+1,\cdots,n)$ are vectors normal to $i(M_p)$. The components $f_\alpha{}^h$ of f_α satisfy

$$f_\alpha{}^h=\gamma_\alpha{}^\lambda B_\lambda{}^h \tag{3.1}$$

where the matrix $(\gamma_\alpha{}^\lambda)$ is such that

(3.2) $$\gamma_\beta{}^\mu \gamma_\alpha{}^\lambda g_{\mu\lambda}=\delta_{\beta\alpha}, \qquad g_{\mu\lambda}=B_\mu{}^h B_\lambda{}^h,$$

where $g_{\mu\lambda}$ are the coefficients of the first fundamental form of iM, hence the components of the Riemannian metric g_i induced on iM. The coefficients of the second fundamental form of iM are

$$H_{\mu\lambda}{}^h=\partial_\mu B_\lambda{}^h-\left\{ {\kappa \atop \mu\lambda} \right\} B_\kappa{}^h, \qquad \partial_\mu=\partial/\partial y^\mu$$

where $\left\{ {\kappa \atop \mu\lambda} \right\}$ is the Christoffel symbol derived from $g_{\mu\lambda}$.

(f_α, f_x) determines a point of $G(m, n-m)$. The distance between the two points (f_α, f_x) and $(f_\alpha+df_\alpha, f_x+df_x)$ is denoted by $d\sigma$ and is given by

(3.3) $$d\sigma^2=\sum_{\alpha,x}\langle df_\alpha, f_x\rangle^2.$$

From (3.1) we get, assuming (f_α, f_x) to be a field in V,

$$df_\alpha{}^h=(\nabla_\mu \gamma_\alpha{}^\lambda B_\lambda{}^h+\gamma_\alpha{}^\lambda H_{\mu\lambda}{}^h)dy^\mu$$

where, here and in the sequel, ∇AB means $(\nabla A)B$ and ∇ covariant differentiation with respect to the Riemannian metric g_i. Let $C_x{}^h$ be the components of the normal vector field f_x. Then, as we have $\langle df_\alpha, f_x\rangle=\gamma_\alpha{}^\lambda H_{\mu\lambda}{}^h C_x{}^h dy^\mu$, we get

$$d\sigma^2=g^{\lambda\kappa}H_{\nu\lambda}{}^h H_{\mu\kappa}{}^h dy^\nu dy^\mu,$$

hence

(3.4) $$d\sigma^2=G_{\mu\lambda}dy^\mu dy^\lambda$$

where $G_{\mu\lambda}$ is defined by

(3.5) $$G_{\mu\lambda}=H_{\mu\sigma}{}^h H_{\lambda\rho}{}^h g^{\sigma\rho}.$$

(3.5) is the first fundamental tensor G_i of the submanifold $\Gamma_i(M)$ of $G(m, n-m)$. As we assume Γ_i to be regular, G_i is a Riemannian metric on $\Gamma_i(M)$.

If M is compact we have the following integral,

(3.6) $$\int_M (\mathfrak{G}_i/\mathfrak{g}_i)^{1/2}\mu(g_i)$$

where $\mathfrak{G}_i=\det(G_{\mu\lambda})$, $\mathfrak{g}_i=\det(g_{\mu\lambda})$ and $\mu(g_i)$ is the volume form of (iM, g_i). The integrand of (3.6) is the volume form of $(\Gamma_i(M), G_i)$. But, as (3.6) does not always give the volume of $(\Gamma_i(M), G_i)$, we call this integral Vol*$(\Gamma_i(M), G_i)$.

§ 4. Gauss-critical submanifolds.

DEFINITION. Let $\mathcal{I}_M$ be the space of all immersions i of M into E^n such that Γ_i is regular. Then Vol*$(\Gamma_i(M), G_i)$ defines a mapping Vol*$: \mathcal{I}_M \to R$. A critical point i of this mapping will be called a *Gauss-critical immersion,* the corresponding submanifold iM a *Gauss-critical submanifold* and denoted *GCS.*

Let us consider an immersion $i(t)$ depending on a parameter t such that the points of $i(t)M$ can be expressed by differentiable functions in the form

$$x^h=x^h(t, y^\kappa)$$

if $t\in(-\varepsilon, \varepsilon)$ and the points are in some coordinate neighborhood of M. Then we have for each $t\in(-\varepsilon, \varepsilon)$ the Gauss map $\Gamma_{i(t)}: M\to(\Gamma_{i(t)}(M), G_{i(t)})$. Let $\mathrm{Vol}^*(t)$ denote the volume* of $(\Gamma_{i(t)}(M), G_{i(t)})$. Then we have

$$\frac{d}{dt}\mathrm{Vol}^*(t)=\frac{1}{2}\int_M(\mathfrak{G}_i/\mathfrak{g}_i)^{1/2}(G^{-1})^{\mu\lambda}\frac{\partial}{\partial t}G_{\mu\lambda}\mu(g_i)$$

where $i=i(t)$ and $((G^{-1})^{\mu\lambda})$ is the inverse matrix of $(G_{\mu\lambda})$.

REMARK. Though $g_{\mu\lambda}$ change with t they are not contained in the integrand ultimately, for the latter can be written $\sqrt{\mathfrak{G}_i}\, dy^1\wedge \cdots \wedge dy^m$ locally.

From (3.5) we get

$$\frac{\partial G_{\mu\lambda}}{\partial t}=\frac{\partial H_{\mu\sigma}{}^h}{\partial t}H_{\lambda\rho}{}^h g^{\sigma\rho}+H_{\mu\sigma}{}^h\frac{\partial H_{\lambda\rho}{}^h}{\partial t}g^{\sigma\rho}$$
$$-H_{\mu\sigma}{}^h H_{\lambda\rho}{}^h g^{\sigma\nu}g^{\rho\kappa}\frac{\partial g_{\nu\kappa}}{\partial t}, \tag{4.1}$$

where

$$\frac{\partial H_{\mu\lambda}{}^h}{\partial t}=\partial_\mu\partial_\lambda\frac{\partial x^h}{\partial t}-\left\{{\kappa \atop \mu\lambda}\right\}\partial_\kappa\frac{\partial x^h}{\partial t}-\frac{\partial}{\partial t}\left\{{\kappa \atop \mu\lambda}\right\}\partial_\kappa x^h,$$

$$\frac{\partial}{\partial t}\left\{{\kappa \atop \mu\lambda}\right\}=\frac{1}{2}(\nabla_\mu D_\lambda{}^\kappa+\nabla_\lambda D_\mu{}^\kappa-\nabla^\kappa D_{\mu\lambda}),$$

$$D_{\mu\lambda}=\frac{\partial}{\partial t}g_{\mu\lambda},\qquad D_\mu{}^\kappa=D_{\mu\lambda}g^{\lambda\kappa}.$$

Now we define the vector field of deformation X^h by

$$X^h=\frac{\partial x^h}{\partial t} \tag{4.2}$$

and put $t=0$. Then we have

$$D_{\mu\lambda}=\partial_\mu X^h\partial_\lambda x^h+\partial_\lambda X^h\partial_\mu x^h$$

and

$$\left(\frac{\partial H_{\mu\lambda}{}^h}{\partial t}\right)_0=\nabla_\mu\nabla_\lambda X^h$$
$$-\nabla^\kappa x^h(\nabla_\mu\nabla_\lambda X^i\nabla_\kappa x^i+\nabla_\kappa X^i H_{\mu\lambda}{}^i). \tag{4.3}$$

Substituting (4.3) into (4.1) we get

$$\left(\frac{\partial G_{\mu\lambda}}{\partial t}\right)_0 = H_\lambda{}^{\sigma h}\nabla_\mu\nabla_\sigma X^h + H_\mu{}^{\sigma h}\nabla_\lambda\nabla_\sigma X^h$$
$$-H_\mu{}^{\sigma h}H_\lambda{}^{\rho h}(B_\rho{}^i\nabla_\sigma X^i + B_\sigma{}^i\nabla_\rho X^i)\,.$$

Hence we have

$$\left(\frac{d}{dt}\mathrm{Vol}^*(t)\right)_0 = \int_M (\mathfrak{G}_i/\mathfrak{g}_i)^{1/2}[(G^{-1})^{\nu\mu}H_\nu{}^{\lambda h}\nabla_\mu\nabla_\lambda X^h$$
$$-(G^{-1})^{\nu\mu}H_\nu{}^{\lambda i}H_\mu{}^{\kappa i}B_\lambda{}^h\nabla_\kappa X^h]\mu(g_i)$$

and after repeated integration by parts we obtain

$$\left(\frac{d}{dt}\mathrm{Vol}^*(t)\right)_0 = \int_M [\nabla_\mu\nabla_\lambda\{(\mathfrak{G}_i/\mathfrak{g}_i)^{1/2}(G^{-1})^{\nu\mu}H_\nu{}^{\lambda h}\}$$
$$+\nabla_\kappa\{(\mathfrak{G}_i/\mathfrak{g}_i)^{1/2}(G^{-1})^{\nu\mu}H_\nu{}^{\lambda i}H_\mu{}^{\kappa i}B_\lambda{}^h\}]X^h\mu(g_i)\,. \tag{4.4}$$

From (4.4) we see that, if (iM, g_i) satisfies

$$\nabla_\mu\nabla_\lambda\{(\mathfrak{G}_i/\mathfrak{g}_i)^{1/2}(G^{-1})^{\nu\mu}H_\nu{}^{\lambda h}\}$$
$$+\nabla_\lambda\{(\mathfrak{G}_i/\mathfrak{g}_i)^{1/2}(G^{-1})^{\nu\mu}H_\nu{}^{\lambda i}H_\mu{}^{\kappa i}B_\kappa{}^h\} = 0\,, \tag{4.5}$$

then (iM, g_i) is a Gauss-critical submanifold.

Thus we have proved the following lemma.

LEMMA 4.1. *Let (M', g') be an m-dimensional compact submanifold of E^n such that $\Gamma : (M', g') \to G(m, n-m)$ is a regular mapping. Then (M', g') is a GCS if and only if* (4.5) *is satisfied.*

As an immediate application we get the following theorem.

THEOREM 4.2. *An immersion i of an m-dimensional manifold M into E^n such that iM is contained in a subspace E^{m+1} of E^n is a Gauss-critical immersion.*

PROOF. As iM is contained in E^{m+1}, we can take a coordinate system in E^n such that $x^{m+2} = \cdots = x^n = 0$ on iM. The second fundamental tensor takes the form

$$H_{\mu\lambda}{}^h = h_{\mu\lambda}N^h$$

where N^h is a unit normal vector satisfying $N^\xi = 0$ $(\xi = m+2, \cdots, n)$. Let $C_\xi{}^h$ be $n-m-1$ orthonormal constant vectors normal to E^{m+1}. From

$$N^h\nabla_\lambda N^h = 0\,, \qquad C_\xi{}^h\nabla_\lambda N^h = 0$$

we get

$$\nabla_\lambda N^h = -h_\lambda{}^\alpha B_\alpha{}^h$$

where $h_\lambda{}^\kappa = g^{\alpha\kappa}h_{\lambda\alpha}$. If we define $k^{\mu\lambda}$ by $k^{\alpha\lambda}h_{\alpha\mu} = \delta_\mu^\lambda$ we get

$$(G^{-1})^{\mu\lambda} = k^{\mu\alpha}k_\alpha{}^\lambda$$

from $G_{\mu\lambda}=h_{\mu\alpha}h_{\lambda}{}^{\alpha}$. We also obtain $\sqrt{\mathfrak{G}_i/\mathfrak{g}_i}=\det(h_{\beta}{}^{\alpha})$ which is assumed not to vanish. Hence we get

$$\sqrt{\mathfrak{G}_i/\mathfrak{g}_i}\,(G^{-1})^{\nu\mu}H_{\nu}{}^{\lambda h}=\det(h_{\beta}{}^{\alpha})k^{\mu\lambda}N^{h}\,,$$

$$\sqrt{\mathfrak{G}_i/\mathfrak{g}_i}\,(G^{-1})^{\nu\mu}H_{\nu}{}^{\lambda i}H_{\mu}{}^{\kappa i}B_{\kappa}{}^{h}=\det(h_{\beta}{}^{\alpha})g^{\kappa\lambda}B_{\kappa}{}^{h}\,.$$

As we have

$$\nabla_{\lambda}k^{\mu\lambda}=-k^{\mu\sigma}k^{\lambda\rho}\nabla_{\lambda}h_{\sigma\rho}\,,\qquad \nabla_{\lambda}\log\det(h_{\beta}{}^{\alpha})=k_{\alpha}{}^{\beta}\nabla_{\lambda}h_{\beta}{}^{\alpha}$$

and $\nabla_{\nu}h_{\mu\lambda}=\nabla_{\mu}h_{\nu\lambda}$, we get

$$\nabla_{\lambda}\{\sqrt{\mathfrak{G}_i/\mathfrak{g}_i}(G^{-1})^{\nu\mu}H_{\nu}{}^{\lambda h}\}=-\det(h_{\beta}{}^{\alpha})g^{\mu\rho}B_{\rho}{}^{h}\,,$$

hence (4.5) is satisfied.

COROLLARY 4.3. *A totally umbilical immersion in a Euclidean space is a Gauss-critical immersion.*

DEFINITION. If $\Gamma_i: M\to(\Gamma_i(M), G_i)$ satisfies

$$G_{\mu\lambda}=c^2 g_{\mu\lambda}$$

with a constant $c>0$, Γ_i is said to be *homothetic* or M is said to be homothetic to $(\Gamma_i(M), G_i)$ by Γ_i. This is equivalent to saying that (iM, g_i) is homothetic to $(\Gamma_i(M), G_i)$ by Γ.

If Γ_i is homothetic, (4.5) is equivalent to

$$\nabla_{\mu}\nabla_{\lambda}H^{\mu\lambda h}+mc^2H^{h}=0 \tag{4.6}$$

where

$$H^{h}=\frac{1}{m}H_{\mu}{}^{\mu h}$$

is the mean curvature vector. On the other hand we get from the Ricci identity

$$\nabla_{\lambda}H^{\mu\lambda h}-m\nabla^{\mu}H^{h}=K^{\mu\lambda}B_{\lambda}{}^{h} \tag{4.7}$$

where $K_{\mu\lambda}$ is the Ricci tensor of (iM, g_i). From (4.6) and (4.7) we get

$$m\nabla_{\mu}\nabla^{\mu}H^{h}+\nabla_{\mu}(K^{\mu\lambda}B_{\lambda}{}^{h})+mc^2H^{h}=0\,.$$

If (iM, g_i) is an Einstein manifold we get

$$\nabla_{\mu}\nabla^{\mu}H^{h}+(c^2+K/m)H^{h}=0 \tag{4.8}$$

where K is the scalar curvature. This proves the following theorem.

THEOREM 4.4. *Let M be a given compact orientable manifold of dimension m. If $i: M\to E^n$ is a Gauss-critical immersion and at the same time (iM, g_i) is an Einsteinian submanifold and moreover Γ_i is a homothetic mapping, then the* *components of the mean curvature vector of iM are eigenfunctions of the Laplacian on (iM, g_i) belonging to an eigenvalue. If $i: M\to E^n$ is an immersion such that (iM, g_i) is an Einsteinian submanifold, Γ_i is a homothetic mapping and moreover the components of the mean curvature vector of iM are eigenfunctions of the Laplacian on (iM, g_i) belonging to an eigenvalue, i is a Gauss-critical immersion.*

§ 5. Sectional curvature of the Gauss map of a submanifold of E^n.

As in § 3 we assume the equation of an immersion $i: M\to E^n$ to be

$$x^h = x^h(y^1, \cdots, y^m)$$

in some neighborhood V of M containing the point $y^\kappa=0$. We take a fixed orthonormal frame (e_α, e_x) of E^n at $y^\kappa=0$ and a field of orthonormal frame (f_α, f_x) satisfying $(f_\alpha, f_x)_0=(e_\alpha, e_x)$ and $\langle f_\alpha, e_\beta\rangle=\langle f_\beta, e_\alpha\rangle$, $\langle f_x, e_y\rangle=\langle f_y, e_x\rangle$. Then taking a matrix $(\gamma_\alpha{}^\kappa)$ satisfying (3.1) we get

$$\gamma_\alpha{}^\lambda B_\lambda{}^h e_\beta{}^h = \gamma_\beta{}^\lambda B_\lambda{}^h e_\alpha{}^h . \tag{5.1}$$

Indices $\alpha, \beta, \gamma, \cdots = 1, \cdots, m$ are used to the vectors of the frame tangent to iM, while $\kappa, \lambda, \mu, \cdots = 1, \cdots, m$ are used in connection with local coordinates of V.

Now let us turn our attention to the Gauss map Γ_i. A point (y^κ) of $V\subset M$ is mapped into a point of $G(m, n-m)$ whose local coordinates are $\xi_{\alpha x}$ where

$$\xi_{\alpha x} = \gamma_\alpha{}^\kappa B_\kappa{}^h e_x{}^h \tag{5.2}$$

since we have adopted the frame (f_α, f_x) as mentioned above. From (5.2) we get

$$\frac{\partial \xi_{\alpha x}}{\partial y^\lambda} = (\nabla_\lambda \gamma_\alpha{}^\kappa B_\kappa{}^h + \gamma_\alpha{}^\kappa H_{\lambda\kappa}{}^h) e_x{}^h . \tag{5.3}$$

DEFINITION. Let M be an m-dimensional manifold, p a point of M and Γ_i the Gauss map associated with an immersion $i: M\to E^n$. Assume Γ_i to be regular. Let σ be a 2-plane in the tangent m-plane of $\Gamma_i(M)$ at $\Gamma_i(p)$. The corresponding sectional curvature of $(G(m, n-m), \tilde{g})$ is called the *sectional curvature of the Gauss map* Γ_i at p and is denoted $K_{\Gamma,i}(\sigma)$.

In order to find properties of $K_{\Gamma,i}(\sigma)$ we take a coordinate neighborhood V of M such that $p\in V$ and such that $y^\kappa=0$ at p. Then the tangent m-plane of $\Gamma_i(M)$ at $\Gamma_i(p)$ is spanned by m vectors $\xi_{(\lambda)}$ $(\lambda=1, \cdots, m)$ whose components are

$$(\partial_\lambda \xi_{\alpha x})_0 = \gamma_\alpha{}^\kappa(0) H_{\lambda\kappa}{}^h(0) e_x{}^h . \tag{5.4}$$

Let $u_{(\alpha)}$ $(\alpha=1, \cdots, m)$ be an orthonormal frame in $T(\Gamma_i(M))$ at $\Gamma_i(p)$ with respect to the Riemannian metric $G_{\mu\lambda}$. Then we can put

$$u_{(\alpha)}=\tau_\alpha{}^\lambda\xi_{(\lambda)}\,. \tag{5.5}$$

Now we use the symbol $\langle\,,\rangle_{\tilde g}$ or $\langle\,,\rangle_G$ for an inner product with respect to the Riemannian metric $\tilde g$ or G_i. Then we get from (2.4) and (5.4)

$$\begin{aligned}\langle u_{(\beta)}, u_{(\alpha)}\rangle_{\tilde g}&=\langle u_{(\beta)}, u_{(\alpha)}\rangle_G\\ &=\tau_\beta{}^\mu\tau_\alpha{}^\lambda(\partial_\mu\xi_{\gamma x})_0(\partial_\lambda\xi_{\gamma x})_0\\ &=\tau_\beta{}^\mu\tau_\alpha{}^\lambda\gamma_\gamma{}^\omega(0)H_{\mu\omega}{}^i(0)e_x{}^i\gamma_\gamma{}^\nu(0)H_{\lambda\nu}{}^h(0)e_x{}^h\\ &=\tau_\beta{}^\mu\tau_\alpha{}^\lambda g^{\omega\nu}(0)H_{\mu\omega}{}^i(0)H_{\lambda\nu}{}^i(0)\\ &=\tau_\beta{}^\mu\tau_\alpha{}^\lambda G_{\mu\lambda}(0)\,.\end{aligned}$$

Hence $\tau_\alpha{}^\lambda$ must satisfy

$$\tau_\beta{}^\mu\tau_\alpha{}^\lambda G_{\mu\lambda}(0)=\delta_{\beta\alpha}\,.$$

In order to get the formula of $K_{\Gamma,i}(\sigma)$ for the 2-plane spanned by $u_{(\alpha)}$ and $u_{(\beta)}$, we must calculate

$$\begin{aligned}P^{\delta\gamma}_{(\beta)(\alpha)}&=u^{\delta x}_{(\beta)}u^{\gamma x}_{(\alpha)}-u^{\delta x}_{(\alpha)}u^{\gamma x}_{(\beta)}\,,\\ Q^{yx}_{(\beta)(\alpha)}&=u^{\gamma y}_{(\beta)}u^{\gamma x}_{(\alpha)}-u^{\gamma y}_{(\alpha)}u^{\gamma x}_{(\beta)}\end{aligned} \tag{5.6}$$

and substitute them into the formula (2.9). We get, substituting (5.4) and (5.5) into (5.6),

$$P^{\delta\gamma}_{(\beta)(\alpha)}=\tau_\beta{}^\mu\tau_\alpha{}^\lambda\gamma_\delta{}^\omega(0)\gamma_\gamma{}^\nu(0)\{H_{\mu\omega}{}^i(0)H_{\lambda\nu}{}^i(0)-H_{\lambda\omega}{}^i(0)H_{\mu\nu}{}^i(0)\}\,,$$

$$Q^{yx}_{(\beta)(\alpha)}=\tau_\beta{}^\mu\tau_\alpha{}^\lambda\gamma_\gamma{}^\omega(0)\gamma_\gamma{}^\nu(0)\{H_{\mu\omega}{}^i(0)H_{\lambda\nu}{}^h(0)-H_{\lambda\omega}{}^i(0)H_{\mu\nu}{}^h(0)\}\,e_y{}^ie_x{}^h\,,$$

hence

$$\begin{aligned}P^{\delta\gamma}_{(\beta)(\alpha)}&=-\tau_\beta{}^\mu\tau_\alpha{}^\lambda\gamma_\delta{}^\omega(0)\gamma_\gamma{}^\nu(0)K_{\omega\nu\mu\lambda}(0)\,,\\ Q^{yx}_{(\beta)(\alpha)}&=\tau_\beta{}^\mu\tau_\alpha{}^\lambda\{H_\mu{}^{\sigma i}(0)H_{\lambda\sigma}{}^h(0)-H_\lambda{}^{\sigma i}(0)H_{\mu\sigma}{}^h(0)\}\,e_y{}^ie_x{}^h\,.\end{aligned} \tag{5.7}$$

From (2.9) and (5.7) we see that a necessary and sufficient condition for $K_{\Gamma,i}(\sigma)$ to vanish at every point and in every direction is that

$$\tau_\beta{}^\mu\tau_\alpha{}^\lambda\gamma_\delta{}^\omega\gamma_\gamma{}^\nu K_{\omega\nu\mu\lambda}=0$$

and

$$\tau_\beta{}^\mu\tau_\alpha{}^\lambda(H_\mu{}^{\sigma i}H_{\lambda\sigma}{}^h-H_\lambda{}^{\sigma i}H_{\mu\sigma}{}^h)C_y{}^iC_x{}^h=0$$

hold for every value of $\alpha, \beta, \gamma, \delta=1, \cdots, m$ and $x, y=m+1, \cdots, n$. Thus we get as the necessary and sufficient condition

(5.8) $$K_{\omega\nu\mu\lambda}=0\,,$$

(5.9) $$H_\mu{}^{\sigma i}H_{\lambda\sigma}{}^h=H_\mu{}^{\sigma h}H_{\lambda\sigma}{}^i\,.$$

Thus we have proved the following lemma.

LEMMA 5.1. *Let M be an m-dimensional manifold and i an immersion $M\to E^n$. A necessary and sufficient condition for the sectional curvature $K_{\Gamma,i}(\sigma)$ of the Gauss map $\Gamma_i: M\to(G(m, n-m), \tilde{g})$ to vanish totally is that (iM, g_i) be a flat Riemannian manifold and the second fundamental form satisfy* (5.9).

Let us study some property of the second fundamental form satisfying (5.9). We fix a point $p\in M$ and choose local coordinates so that $g_{\mu\lambda}=\delta_{\mu\lambda}$ holds at p. Let A^i be the square matrix $(H_{\mu\lambda}{}^i)$ of order m. Then (5.9) is equivalent to $A^iA^h=A^hA^i$. Hence choosing local coordinates again we can transform all matices into diagonal ones simultaneously and get

$$H_{\mu\lambda}{}^h=H_\mu{}^h\delta_{\mu\lambda}\,.$$

On the other hand from the equation of Gauss $K_{\nu\mu\lambda\kappa}=H_{\nu\kappa}{}^hH_{\mu\lambda}{}^h-H_{\mu\kappa}{}^hH_{\nu\lambda}{}^h$ we obtain

$$H_\nu{}^hH_\mu{}^h(\delta_{\nu\kappa}\delta_{\mu\lambda}-\delta_{\mu\kappa}\delta_{\nu\lambda})=0 \qquad \text{(not summed for } \mu,\ \nu)$$

in view of (5.8). Putting $\nu=\kappa\neq\mu=\lambda$ we get

$$H_\nu{}^hH_\mu{}^h=0 \qquad \nu\neq\mu\,.$$

But we have

$$G_{\mu\lambda}=H_\mu{}^{\sigma h}H_{\lambda\sigma}{}^h=H_\mu{}^hH_\lambda{}^h\delta_{\mu\lambda}\,.$$

As we assume $\det(G_{\mu\lambda})\neq 0$, we get

$$H_\nu{}^hH_\nu{}^h>0 \qquad \text{for each } \nu$$

hence $H_1{}^h, \cdots, H_m{}^h$ are m linearly independent vectors of E^n. But these vectors are normal vectors to iM at ip. Consequently there can be no more than $n-m$ linearly independent vectors. This proves the following theorem.

THEOREM 5.2. *Let M be an m-dimensional manifold and i an immersion $M\to E^n$ and assume that the Gauss map $\Gamma_i: M\to(G(m, n-m), \tilde{g})$ is regular. If $n<2m$, the sectional curvature of the Gauss map $K_{\Gamma,i}$ cannot totally vanish. If $n\geqq 2m$, $K_{\Gamma,i}$ vanishes if and only if (iM, g_i) is flat and* (5.9) *holds.*

§ 6. Some examples.

1°. An immersion i of a torus T^2 into E^4 given by

$$x^1=r_1\cos u\,, \qquad x^2=r_1\sin u\,, \qquad x^3=r_2\cos v\,, \qquad x^4=r_2\sin v\,.$$

For this immersion we have

$$g_{11}=(r_1)^2, \qquad g_{12}=0, \qquad g_{22}=(r_2)^2, \qquad G_{\mu\lambda}=\delta_{\mu\lambda}.$$

The Gauss map Γ_i is not homothetic if $r_1\neq r_2$. As (iT^2, g_i) is flat and satisfies (5.9), the sectional curvature $K_{\Gamma,i}(\sigma)$ totally vanishes. As (4.5) is satisfied, (iT^2, g_i) is a *GCS*. If $r_1=r_2=r$, we get $G_{\mu\lambda}=r^{-2}g_{\mu\lambda}$ and the mean curvature vector satisfies

$$H^h=-\frac{1}{2r^2}x^h,$$

$$-\nabla_\mu\nabla^\mu H^h=c^2H^h, \qquad c^2=r^{-2},$$

hence by T. Takahashi's theorem i is a minimal immersion into a hypersphere of E^4 [4]. Moreover in this case we have a pseudo-umbilical submanifold.

2°. An immersion i of a torus T^2 into E^4 given by

$$x^1=\cos u\cos v, \qquad x^2=\cos u\sin v,$$

$$x^3=\sin u\cos v, \qquad x^4=\sin u\sin v.$$

For this immersion we have

$$g_{\mu\lambda}=\delta_{\mu\lambda}, \qquad G_{\mu\lambda}=2g_{\mu\lambda}$$

hence Γ_i is homothetic. The sectional curvature $K_{\Gamma,i}(\sigma)$ totally vanishes. (iT^2, g_i) is also a *GCS* and pseudo-umbilical. The mean curvature vector satisfies

$$H^h=-x^h, \qquad \nabla_\mu\nabla^\mu x^h=-2x^h$$

hence i is a minimal immersion into a hypersphere of E^4 [4].

3°. The Veronese surface in E^5, given by

$$x^1=\frac{\sqrt{3}}{2}\sin 2u\sin v, \qquad x^2=\frac{\sqrt{3}}{2}\sin 2u\cos v,$$

$$x^3=\frac{\sqrt{3}}{2}\sin^2 u\sin 2v, \qquad x^4=\frac{\sqrt{3}}{2}\sin^2 u\cos 2v,$$

$$x^5=\frac{1}{2}(1-3\cos^2 u),$$

$$0\leqq u\leqq\frac{\pi}{2}, \qquad 0\leqq v\leqq 2\pi.$$

It is well-known that the Veronese surface is a minimal submanifold of a hypersphere in E^5. For this homothetic immersion $i: S^2\to E^5$ we have

$$g_{11}=3, \quad g_{12}=0, \quad g_{22}=3\sin^2 u.$$

Γ_i is homothetic, $G_{\mu\lambda}=(5/3)g_{\mu\lambda}$, but the sectional curvature $K_{\Gamma,i}(\sigma)$ does not totally vanish since this submanifold has positive constant curvature. But this submanifold is a *GCS*. The mean curvature vector H^h satisfies $H^h=-x^h$ and it is easy to verify directly that the functions x^h on $S^2(\sqrt{3})$ are eigenfunctions of the Laplacian belonging to the eigenvalue λ_2 of multiplicity 5.

4°. $S^{m_1}(r_1)\times S^{m_2}(r_2)$ in $E^{m_1+m_2+2}$ such that $S^{m_1}(r_1)$ is in $E_1^{m_1+1}$ and $S^{m_2}(r_2)$ in $E_2^{m_2+1}$, $E_1^{m_1+1}\perp E_2^{m_2+1}$.

This submanifold is a *GCS*, but the sectional curvature $K_{\Gamma,i}(\sigma)$ does not totally vanish if $m_1>1$ or $m_2>1$. Γ_i is homothetic only when $r_1=r_2$.

5°. An immersion i of a torus T^3 into E^6 given by

$$x^1=\cos u, \quad x^2=\sin u, \quad x^3=\cos v, \quad x^4=\sin v,$$
$$x^5=\cos w, \quad x^6=\sin w.$$

This is also a minimal immersion into a hypersphere of E^6 [**4**]. For this immersion $G_{\mu\lambda}=g_{\mu\lambda}=\delta_{\mu\lambda}$, hence Γ_i is isometric. $K_{\Gamma,i}(\sigma)$ totally vanishes and (iT^3, g_i) is a *GCS* and a pseudo-umbilical submanifold.

6°. An immersion i of a torus T^3 into E^8 given by

$$x^1=\cos u\cos v\cos w, \quad x^2=\cos u\cos v\sin w,$$
$$x^3=\cos u\sin v\cos w, \quad x^4=\cos u\sin v\sin w,$$
$$x^5=\sin u\cos v\cos w, \quad x^6=\sin u\cos v\sin w,$$
$$x^7=\sin u\sin v\cos w, \quad x^8=\sin u\sin v\sin w.$$

This immersion is also a minimal immersion into a hypersphere of E^8 [**4**]. For this immersion $G_{\mu\lambda}=3g_{\mu\lambda}$, $g_{\mu\lambda}=\delta_{\mu\lambda}$, hence Γ_i is homothetic. (iT^3, g_i) is a *GCS*. But $K_{\Gamma,i}(\sigma)$ does not totally vanish.

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Yosio MUTŌ
2262-150, Tomioka-cho
Kanazawa-ku, Yokohama 236
Japan