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On finite multiplicative subgroups of simple algebras of degree 2

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Amitsur ([1]) determined all finite multiplicative subgroups of division algebras. We try to determine, more generally, all finite multiplicative subgroups of simple algebras of fixed degree. In [5] we characterized p-groups contained in the full matrix algebras $M_n(\Delta)$ of fixed degree n, where Δ is a division algebra of characteristic 0. In this paper we will study multiplicative subgroups of $M_2(\Delta)$.

In § 2 we will determine all finite nilpotent subgroups of $M_2(\Delta)$, and in § 3 all finite subgroups of $M_2(\Delta)$ with abelian Sylow 2-groups. Finally, in § 4, we will give some additional remarks.

§ 1. Preliminaries.

All division algebras considered in this paper are of characteristic 0. As usual $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{H}$ denote respectively the rational number field, the real number field, the complex number field, and the quaternion algebra over $\boldsymbol{R}$.

Let Δ be a division algebra. We denote by $M_n(\Delta)$ the full matrix algebra of degree n over Δ. By a subgroup of $M_n(\Delta)$ we mean a finite multiplicative subgroup of $M_n(\Delta)$. Further let K be a field contained in the center of Δ and let G be a subgroup of $M_n(\Delta)$. We define $V_K(G)=\{\sum \alpha_i g_i \mid \alpha_i \in K,\ g_i \in G\}$. Then $V_K(G)$ is a K-subalgebra of $M_n(\Delta)$ and there is a natural epimorphism $KG \to V_K(G)$. Hence $V_K(G)$ is a semi-simple K-subalgebra of $M_n(\Delta)$.

Let m, r be relatively prime integers, and put $s=(r-1, m)$, $t=m/s$; $n=$the minimal positive integer satisfying $r^n \equiv 1 \bmod m$. Denote by $G_{m,r}$ the group generated by two elements a, b with the relations; $a^m=1$, $b^n=a^t$ and $bab^{-1}=a^r$. Let ζ_m be a fixed primitive m-th root of unity and let $\sigma=\sigma_r$ be the automorphism of $\boldsymbol{Q}(\zeta_m)$ determined by the mapping $\zeta_m \to \zeta_m{}^r$. Let $\{\alpha_{\sigma i,\sigma j}\}$ be the factor set of $\langle\sigma\rangle$ in $\boldsymbol{Q}(\zeta_m)$ defined by

$$\alpha_{\sigma i,\sigma j}=\begin{cases} 1 & \text{when} \quad i+j<n \\ \zeta_s=\zeta_m{}^t & \text{when} \quad i+j \geqq n\,, \end{cases}$$

and denote by $\Lambda_{m,r}$ the crossed product of $\boldsymbol{Q}(\zeta_m)$ and $\langle\sigma\rangle$ by $\{\alpha_{\sigma i,\sigma j}\}$.

Here we recall the results in Amitsur [**1**].

1.1 ([**1**]). *Let G be a finite group and let Δ be a division algebra. If $G \subseteqq \Delta$, then G is one of the following types:*

(1) *All Sylow subgroups of G are cyclic.*

(2) *The odd Sylow subgroups of G are cyclic and the even Sylow subgroup of G is a generalized quaternion group of order $2^{\alpha+1}$, $\alpha \geqq 2$.*

1.2 ([**1**]). *A group G is of type (1) in (1, 1) if and only if $G \cong G_{m,r}$ for some relatively prime integers m, r with $(n, t)=1$. A group G of type (1) or (2) in (1, 1) is metacyclic if and only if $G \cong G_{m,r}$ for some relatively prime integers m, r.*

1.3 ([**1**]). *A group $G_{m,r}$ can be embedded in a division algebra if and only if $\Lambda_{m,r}$ is a division algebra; then we have $V_{\boldsymbol{Q}}(G_{m,r}) \cong \Lambda_{m,r}$ and the isomorphism is obtained by the correspondence $a \leftrightarrow \zeta_m$, $b \leftrightarrow \sigma_r$.*

The group $G_{2m,-1}$ are called the binary dihedral groups. We define $T^* = \langle a, b, c \mid a^4=1,\ a^2=b^2,\ aba^{-1}=b^{-1},\ cac^{-1}=b,\ cbc^{-1}=ab,\ c^3=1\rangle$, $O^* = \langle a, b, c \mid a^8=1,\ a^4=b^2=c^3,\ cba=1\rangle$ and $I^*=SL(2, 5)$.

1.4 ([**1**]). *The finite subgroups of the quaternion algebra $\boldsymbol{H}$ are the cyclic group of any order, the binary dihedral group of order $4m$, the groups T^*, O^* and I^*.*

We remark that the group $G_{4,-1}$ means the ordinary quaternion group of order 8 and that the crossed product $\Lambda_{4,-1}$ means the ordinary quaternion algebra over $\boldsymbol{Q}$. The splitting fields for $\Lambda_{4,-1}$ can be determined by the following:

1.5 ([**3**]). *Let K be an algebraic number field. Then K is a splitting field for $\Lambda_{4,-1}$ if and only if K is totally imaginary and the local degrees of K at all primes of K extending the rational prime (2) are even. In particular, if $4 \mid n$, then $\boldsymbol{Q}(\zeta_n)$ is a splitting field for $\Lambda_{4,-1}$, and, if $n=2m$, m odd, then $\boldsymbol{Q}(\zeta_n)=\boldsymbol{Q}(\zeta_m)$ is a splitting field for $\Lambda_{4,-1}$ if and only if the order of 2 (mod m) is even.*

Next we recall the results in [**5**].

Let $P_0=\langle g\rangle$ be a cyclic group of order p. Let G, G' be finite groups and let $G'_1, G'_2, \cdots, G'_p$ be the copies of G'. We call G a simple (1-fold) p-extension of G' if G is an extension of $G'_1 \times G'_2 \times \cdots \times G'_p$ by P_0 such that $G_1'^{\sigma_g}=G'_2, \cdots, G_{p-1}'^{\sigma_g}=G'_p$, $G_p'^{\sigma_g}=G'_1$ where σ_g denotes the automorphism of $G'_1 \times G'_2 \times \cdots \times G'_p$ corresponding to g. More generally, a finite group G is called an n-fold p-extension of a finite group G' if there exist finite groups $G_0=G', G_1, \cdots, G_{n-1}, G_n=G$ such that, for each $0 \leqq i \leqq n-1$, G_{i+1} is a simple p-extension of G_i. Now we put

$$T_p^{(0)} = \begin{cases} \{\text{all cyclic } p\text{-groups}\} & \text{when } p \neq 2 \\ \{\text{all generalized quaternion 2-groups}\} & \text{when } p=2, \end{cases}$$

and $\tilde{T}_p^{(0)} = \{\text{all cyclic } p\text{-groups}\}$ for any prime p. An element of $T_p^{(0)}$ (resp. $\tilde{T}_p^{(0)}$)

is called a p-group of 0-type (resp. $\tilde{0}$-type). A finite p-group P is said to be of n-type (resp. $\tilde{n}$-type) if P is an n-fold p-extension of a p-group of 0-type (resp. $\tilde{0}$-type). We denote by $T_p^{(n)}$ (resp. $\tilde{T}_p^{(n)}$) the set of all p-groups of n-type (resp. $\tilde{n}$-type).

1.6 ([**5**]). *Let n be a fixed positive integer and let P be a finite p-group. Then the following conditions are equivalent:*

(1) *P is a subgroup of $M_n(\boldsymbol{H})$ (resp. $M_n(\boldsymbol{C})$).*

(2) *P is a subgroup of $M_n(\Delta)$ (resp. $M_n(K)$) for a division algebra Δ (resp. commutative field K).*

(3) *There exist non-negative integers t, $m_0, \cdots, m_t$ with $\sum_{i=0}^{t} p^i m_i \leqq n$ and $P_i^{(1)}, P_i^{(2)}, \cdots, P_i^{(m_i)} \in T_p^{(i)}$ (resp. $\tilde{T}_p^{(i)}$), $0 \leqq i \leqq t$, such that $P \subseteqq \prod_{i=0}^{t} \prod_{j=1}^{m_i} P_i^{(j)}$.*

1.7 ([**5**]). *Let P be a finite non-abelian p-group, Δ a division algebra and K a field contained in the center of Δ. Assume that $P \subseteqq M_n(\Delta)$ and $V_K(P) = M_n(\Delta)$.*

(1) *If P is a 2-group which is not of type 0 and Δ is non-commutative, then there exists a subgroup P_0 of P of index 2 such that $V_K(P_0) \cong M_{n/2}(\Delta) \oplus M_{n/2}(\Delta)$.*

(2) *If Δ is commutative, then we have $V_{\bar{K}}(P) = M_n(\bar{K})$ and there exists a subgroup P_0 of P of index p such that*

$$V_{\bar{K}}(P_0) \cong \overbrace{M_{n/p}(\bar{K}) \oplus \cdots \oplus M_{n/p}(\bar{K})}^{p},$$

where $\bar{K}$ is the algebraic closure of K.

(In [**5**], we proved (1.7) for $K = \boldsymbol{Q}$. But that proof holds good for any field K contained in the center of Δ.)

§ 2. Nilpotent groups.

We begin with

LEMMA 2.1. *Let Δ be a division algebra (of characteristic 0) and let K be a field contained in the center of Δ. Let G be a finite subgroup of $M_2(\Delta)$. Then we have $V_K(G) \cong \Delta_1$, $M_2(\Delta_2)$ or $\Delta_3 \oplus \Delta_4$ where Δ_i, $1 \leqq i \leqq 4$, are division algebras.*

PROOF. This is evident, because $V_K(G)$ is semi-simple.

Here we give the following basic lemmas.

LEMMA 2.2. *Let Δ be a division algebra and K be a subfield of the center of Δ. Let H, J be finite groups and G_σ, $\sigma \in H$, be normal subgroup of J. Let G be an extension of J by H. Assume the following conditions;*

(1) *$\bigcap_{\sigma \in H} G_\sigma = 1$ and $G_\sigma \neq G_\tau$ for any $\sigma \neq \tau$ in H.*

(2) *Let $\{u_\tau\}_{\tau \in H}$ be a set of representatives of H in G. Then $G_\sigma{}^{u_\tau} = G_{\sigma\tau}$ for*

any σ, $\tau \in H$.

(3) $J/G_1 \subseteq M_n(\Delta)$ *and* $V_K(J/G_1)=M_n(\Delta)$.

Then we have $G \subseteq M_{nh}(\Delta)$ *and* $V_K(G)=M_{nh}(\Delta)$, *where* h *is the order of* H.

PROOF. Let V be an irreducible $M_n(\Delta)$-module. Then V can be regarded as a $K[J/G_1]$-module because $V_K(J/G_1)=M_n(\Delta)$. Let ϕ ; $J \to J/G_1$ be the natural homomorphism. Then we may further regard V as a KJ-module through ϕ. Now we have $V^G=KG \otimes_{KJ} V=\sum_{\sigma \in H} \oplus u_\sigma^{-1}V$. Since $\operatorname{Ker} u_\sigma^{-1}V=\{g \in J \mid gu_\sigma^{-1}v=u_\sigma^{-1}v$ for all $v \in V\}=\{g \in J \mid gv=v$ for all $v \in V\}^{u_\sigma}=\operatorname{Ker} V^{u_\sigma}=G_1{}^{u_\sigma}=G_\sigma$ by our assumptions (2), (3), we have $u_\sigma^{-1}V \cong u_\tau^{-1}V$ as KJ-module for any $\sigma \neq \tau$. Therefore $\operatorname{Hom}_{KG}(V^G, V^G) \cong \operatorname{Hom}_{KJ}(u_\sigma^{-1}V, V^G) \cong \operatorname{Hom}_{KJ}(u_\sigma^{-1}V, u_\sigma^{-1}V) \cong \Delta^{op}$. Because $\dim_K V^G =h \dim_K V$, the simple component of KG corresponding to V^G is $M_{nh}(\Delta)$. As is easily seen, V^G is G-faithful. Hence $G \subseteq M_{nh}(\Delta)$ and $V_K(G)=M_{nh}(\Delta)$.

LEMMA 2.3. *Let* G *be a finite group and let* N *be a normal subgroup of* G. *Let* Δ *be a division algebra and let* K *be a field contained in the center of* Δ. *Assume that* $G \subseteq M_2(\Delta)$ *and* $V_K(G)=M_2(\Delta)$. *Let* V *be an irreducible* $M_2(\Delta)$-*(so,* KG*-) module. Further let* W *be an irreducible* KN*-submodule of* V *and let* U *be the sum of all* KN*-submodules of* V *which are isomorphic to* W. *Define* $H=\{g \in G \mid gU=U\}$. *Then the number* m *of all isomorphism classes of irreducible* KN*-submodules of* V *is* 1 *or* 2. *In the case where* $m=2$, *we have* $[G:H]=2$ *and* $V_K(H) \cong \Delta \oplus \Delta$. *Moreover* H *has normal subgroups* H_1, H_2 *satisfying* $H_1 \cap H_2=\{1\}$, $H_1{}^g=H_2$ *and* $H/H_1 \cong \rho(H)$, *where* $\{1, g\}$ *is a set of representative of* G/H *in* G *and* ρ *is the projection of* $V_K(H)$ *on the first component of* $\Delta \oplus \Delta$. *In the case where* $m=1$, *we have* $V_K(N) \cong M_2(\Delta')$ *or* Δ' *for a division algebra* Δ', *and, especially, if* $|N|$ *is odd, then* $V_K(N)$ *is a division algebra.*

PROOF. By Clifford's theorem (e. g. [2]) U is irreducible as a KH-module, $V=U^G$ and $m=[G:H]$. Let $M_r(\Delta')$ be the simple component of KH corresponding to U. Let $V=U^G=U \oplus U_1 \oplus \cdots \oplus U_k$ be a decomposition of V into irreducible KH-modules. By the assumption on U, $U \not\cong U_i$ as KN-module for all $1 \leqq i \leqq k$. Therefore $U \not\cong U_i$ as KH-module, so we have $\Delta^{op} \cong \operatorname{Hom}_{KG}(V, V) \cong \operatorname{Hom}_{KH}(U, U) \cong \Delta'^{op}$. Since $2 \dim_K \Delta=\dim_K V=m \dim_K U=mr \dim_K \Delta'=mr \dim_K \Delta$, we have $mr=2$ and so $m \leqq 2$.

We now assume that $m=2$. Then $r=1$ and so $V_K(H) \cong \Delta \oplus \Delta$. Since $[G:H]=2$, H is a normal subgroup of G. Let $G=H \cup gH$ be the decomposition of G into cosets of H. Then $V=U \oplus g^{-1}U$ and $U \not\cong g^{-1}U$ as KH-modules. And we may assume that U (resp. gU) is the irreducible $V_K(H)$-module corresponding to the first (resp. second) component of $\Delta \oplus \Delta$. Put $H_1=\operatorname{Ker} \rho$ and $H_2=H_1{}^g=(\operatorname{Ker} \rho)^g$. Because $V=U \oplus g^{-1}U$ is H-faithful, $1=\operatorname{Ker}(U \oplus g^{-1}U)=\{h \in H \mid hu+hg^{-1}u=u+g^{-1}u$ for all $u \in U\}=\{h \in H \mid hu=u$ for all $u \in U\} \cap \{h \in H \mid hg^{-1}u=g^{-1}u$ for all $u \in U\}=\operatorname{Ker} \rho \cap (\operatorname{Ker} \rho)^g=H_1 \cap H_2$. Furthermore since $V_K(H)=\Delta \oplus \Delta$, we have $H_i \neq \{1\}$ and so $H_1 \neq H_2$.

Assume $m=1$. Then it is easily shown that $V_K(N)\cong M_2(\Delta')$ or Δ' for a division algebra Δ'. Let $\bar{K}$ be the algebraic closure of K. Then we have $\bar{K}\otimes_K M_2(\Delta')\cong M_2(\bar{K}\otimes_K\Delta')\cong \overbrace{M_{2t}(\bar{K})\oplus\cdots\oplus M_{2t}(\bar{K})}^{s}$ for some integers s and t. Therefore, if $V_K(N)\cong M_2(\Delta')$, then we have $2t\,|\,|N|$.

Now we will determine all finite nilpotent subgroups of simple algebras $M_2(\Delta)$ where Δ are division algebras.

LEMMA 2.4. *Let Δ be a division algebra and let K be a field contained in the center of Δ. Let G be a finite nilpotent subgroup of $M_2(\Delta)$. For each prime $p\,|\,|G|$ let S_p be the Sylow p-subgroup of G, and let $|S_2|=2^l$ and $|\prod_{p\neq 2}S_p|=m$. Assume that $V_K(G)$ is simple. Then $\prod_{p\neq 2}S_p$ is cyclic, $V_K(\prod_{p\neq 2}S_p)\cong K(\zeta_m)$, $V_K(S_2)$ is simple and $V_{L(\zeta_m)}(S_2)=V_K(G)$, where L is the center of $V_K(S_2)$. Further assume that $V_K(S_2)$ is a division algebra. Then one of the following conditions is satisfied:*

(1) *S_2 is a cyclic group and $V_K(G)\cong K(\zeta_{2^l m})$.*

(2) *S_2 is a generalized quaternion group and $V_K(G)\cong K(\zeta_{2^{l-1}}+\zeta_{2^{l-1}}^{-1}, \zeta_m)\otimes_{\boldsymbol{Q}}\Lambda_{4,-1}$.*

PROOF. By (1.6) $\prod_{p\neq 2}S_p$ is abelian. Then $V_K(\prod_{p\neq 2}S_p)$ is contained in the center of $V_K(G)$, and therefore $V_K(\prod_{p\neq 2}S_p)$ is a field. Hence $\prod_{p\neq 2}S_p$ is cyclic and $V_K(\prod_{p\neq 2}S_p)\cong K(\zeta_m)$. Further we easily see that the center of $V_K(S_2)$ is contained in the center of $V_K(G)$. Therefore $V_K(S_2)$ is simple and $V_{L(\zeta_m)}(S_2)\subseteqq V_K(G)$. On the other hand it clearly holds that $V_K(G)\subseteqq V_{L(\zeta_m)}(S_2)$. Hence $V_{L(\zeta_m)}(S_2)=V_K(G)$. Now assume that $V_K(S_2)$ is a division algebra. Then by (1.1) S_2 is a cyclic group or a generalized quaternion group. If S_2 is cyclic, then we have $V_K(S_2)\cong K(\zeta_{2^l})$, and so $V_K(G)\cong K(\zeta_{2^l m})$. On the other hand, if S_2 is a generalized quaternion group, i.e., if $S_2=G_{2^{l-1},-1}$, then we have by (1.3) $V_{\boldsymbol{Q}}(S_2)=\Lambda_{2^{l-1},-1}$. The center of $\Lambda_{2^{l-1},-1}$ is $\boldsymbol{Q}(\zeta_{2^{l-1}}+\zeta_{2^{l-1}}^{-1})$, so $V_K(S_2)\cong K\otimes_{\boldsymbol{Q}(\zeta_{2^{l-1}}+\zeta_{2^{l-1}}^{-1})}V_{\boldsymbol{Q}}(S_2)\cong K(\zeta_{2^{l-1}}+\zeta_{2^{l-1}}^{-1})\otimes_{\boldsymbol{Q}}\Lambda_{4,-1}$. Therefore we get $V_K(G)\cong K(\zeta_{2^{l-1}}+\zeta_{2^{l-1}}^{-1}, \zeta_m)\otimes_{\boldsymbol{Q}}\Lambda_{4,-1}$. This completes the proof of the lemma.

We now give

THEOREM 2.5. *Let G be a finite nilpotent group and for each prime $p\,|\,|G|$ let S_p be the Sylow p-subgroup of G. Let $|S_2|=2^s$ and $|\prod_{p\neq 2}S_p|=m$. Let Δ be a division algebra and K a field contained in the center of Δ. Assume that G can be embedded in $M_2(\Delta)$ in the form that $V_K(G)=M_2(\Delta)$. Then G satisfies the following conditions* (a) *and* (b).

(a) *S_2 has a subgroup S of index 2 and S has two normal subgroups T_1, T_2 $(\neq\{1\})$ of S such that $T_1\cap T_2=\{1\}$ and $T_1{}^g=T_2$, where $\{1, g\}$ is a set of representatives of S_2/S in S_2.*

(b) *S/T_1 and $\prod_{p\neq 2} S_p$ satisfy one of the following conditions:*

(1) *S/T_1 and $\prod_{p\neq 2} S_p$ are cyclic groups.*

(2) *S/T_1 is a quaternion group of order 2^l, $\prod_{p\neq 2} S_p$ is a cyclic group and $K(\zeta_{2^{l-1}}+\zeta_{2^{l-1}}^{-1}, \zeta_m)\otimes_{\boldsymbol{Q}}\Lambda_{4,-1}$ is a division algebra.*

Conversely, assume that G satisfies the condition (a). *Let $|S/T_1|=2^l$. Furthermore if G satisfies the condition* (1) *in* (b), *then $G\subseteqq M_2(K(\zeta_{2^l m}))$ and $V_K(G)=M_2(K(\zeta_{2^l m}))$. If G satisfies the condition* (2) *in* (b), *then $G\subseteqq M_2(K(\zeta_{2^{l-1}}+\zeta_{2^{l-1}}^{-1}, \zeta_m)\otimes_{\boldsymbol{Q}}\Lambda_{4,-1})$ and $V_K(G)=M_2(K(\zeta_{2^{l-1}}+\zeta_{2^{l-1}}^{-1}, \zeta_m)\otimes_{\boldsymbol{Q}}\Lambda_{4,-1})$.*

PROOF. By (2.4) $\prod_{p\neq 2} S_p$ is cyclic. First assume that $V_K(S_2)$ is a division algebra. Because $V_K(G)=M_2(\Delta)$, again by (2.4) S_2 is a generalized quaternion group and $V_K(S_2)=K(\zeta_{2^{s-1}}+\zeta_{2^{s-1}}^{-1})\otimes_{\boldsymbol{Q}}\Lambda_{4,-1}$. Therefore $S_2\subseteqq K(\zeta_{2^{s-1}}+\zeta_{2^{s-1}}^{-1})\otimes_{\boldsymbol{Q}}\Lambda_{4,-1}\subseteqq \bar{K}\otimes_{\boldsymbol{Q}}\Lambda_{4,-1}=M_2(\bar{K})$ and $V_{\bar{K}}(S_2)=M_2(\bar{K})$, where $\bar{K}$ is the algebraic closure of K. By (1.7) there exists a subgroup S of S_2 of index 2 such that $V_{\bar{K}}(S)=\bar{K}\oplus\bar{K}$. Hence by (2.3) S has normal subgroups T_1, T_2 satisfying the condition (a) such that S/T_1 is the subgroup of $\bar{K}$. So G satisfies the conditions (a) and (1) in (b). Next assume that $V_K(S_2)\cong M_2(\Delta')$ for a division algebra Δ'. If Δ' is commutative, then, by the same reason as above, G satisfies the conditions (a) and (1) in (b). On the other hand, if Δ' is non-commutative, then S_2 is not of type 0. Therefore by (1.7) there exists a subgroup S of S_2 of index 2 such that $V_K(S)=\Delta'\oplus\Delta'$. Then by virtue of (2.3) S has normal subgroups T_1, T_2 satisfying (a), S/T_1 is a subgroup of Δ' and $V_K(S/T_1)=\Delta'$. It follows from (2.4) that S/T_1 is a generalized quaternion group and $\Delta'=V_K(S/T_1)\cong K(\zeta_{2^{l-1}}+\zeta_{2^{l-1}}^{-1})\otimes_{\boldsymbol{Q}}\Lambda_{4,-1}$. Therefore again by (2.4), $M_2(\Delta)=V_K(G)=K(\zeta_{2^{l-1}}+\zeta_{2^{l-1}}^{-1}, \zeta_m)\otimes_{\boldsymbol{Q}}M_2(\Lambda_{4,-1})$. Hence G satisfies the conditions (2) in (b).

Finally we prove the converse. If G satisfies the condition (a), then G is an extension of $S\times\prod_{p\neq 2} S_p$ by S_2/S and G satisfies the conditions (1), (2) in (2.2). So, if $S/T_1\times\prod_{p\neq 2} S_p$ is a subgroup of a division algebra Δ and $V_K(S/T_1\times\prod_{p\neq 2} S_p)=\Delta$, then we have by (2.2) $G\subseteqq M_2(\Delta)$ and $V_K(G)=M_2(\Delta)$. Therefore it remains only to prove that $S/T_1\times\prod_{p\neq 2} S_p$ satisfies the above condition. First assume that G satisfies the condition (1) in (b). Since $S/T_1\times\prod_{p\neq 2} S_p$ is a cyclic group of order $2^l m$, $S/T_1\times\prod_{p\neq 2} S_p$ can be embedded in $K(\zeta_{2^l m})$ in the form that $V_K(S/T_1\times\prod_{p\neq 2} S_p)=K(\zeta_{2^l m})$. If G satisfies the condition (2) in (b), we have by (1.3) S/T_1 is a subgroup of $\Lambda_{2^{l-1},-1}=\boldsymbol{Q}(\zeta_{2^{l-1}}+\zeta_{2^{l-1}}^{-1})\otimes_{\boldsymbol{Q}}\Lambda_{4,-1}$ such that $V_{\boldsymbol{Q}}(S/T_1)=\Lambda_{2^{l-1},-1}$. Then $S/T_1\times\prod_{p\neq 2} S_p$ can be embedded in $K(\zeta_{2^{l-1}}+\zeta_{2^{l-1}}^{-1}, \zeta_m)\otimes_{\boldsymbol{Q}}\Lambda_{4,-1}$ and $V_K(S/T_1\times\prod_{p\neq 2} S_p)=K(\zeta_{2^{l-1}}+\zeta_{2^{l-1}}^{-1}, \zeta_m)\otimes_{\boldsymbol{Q}}\Lambda_{4,-1}$. Thus the proof of the theorem is completed.

COROLLARY 2.6. *Let G be a finite nilpotent group and for each prime*

$p \mid |G|$, let S_p be the Sylow subgroup of G. Assume that G can be embedded in $M_2(\Delta)$ for a division algebra Δ in the form that $V_{\boldsymbol{Q}}(G)=M_2(\Delta)$. Then G satisfies the following condition (a) *and one of the following conditions* (b-1)$\sim$(b-3).

(a) *S_2 has a subgroup S of index 2 with normal subgroups T_1, T_2 $(\neq\{1\})$ such that $T_1 \cap T_2=\{1\}$ and $T_1^g=T_2$, where $\{1, g\}$ is a set of representatives of S_2/S in S_2.*

(b-1) *S/T_1 and $\prod_{p\neq 2} S_p$ are cyclic groups.*

(b-2) *S/T_1 is a quaternion group of order 8, $\prod_{p\neq 2} S_p$ is a cyclic group and the order of 2 $(\bmod m)$ is odd.*

(b-3) *S/T_1 is a generalized quaternion group of order >8 and $\prod_{p\neq 2} S_p=\{1\}$.*

Conversely, assume that G satisfies the condition (a). *Let $|\prod_{p\neq 2} S_p|=m$, $|S/T_1|=2^l$. Furthermore if G satisfies the condition* (b-1), *then $G \subseteqq M_2(\boldsymbol{Q}(\zeta_{2^l m}))$ and $V_{\boldsymbol{Q}}(G)=M_2(\boldsymbol{Q}(\zeta_{2^l m}))$. If G satisfies the condition* (b-2), *then $G \subseteqq M_2(\boldsymbol{Q}(\zeta_m)\otimes_{\boldsymbol{Q}} \Lambda_{4,-1})$ and $V_{\boldsymbol{Q}}(G)=M_2(\boldsymbol{Q}(\zeta_m)\otimes_{\boldsymbol{Q}} \Lambda_{4,-1})$. And if G satisfies* (b-3), *then $G \subseteqq M_2(\Lambda_{2^{l-1},-1})$ and $V_{\boldsymbol{Q}}(G)=M_2(\Lambda_{2^{l-1},-1})$.*

PROOF. We may only check this when $\boldsymbol{Q}(\zeta_{2^{l-1}}+\zeta_{2^{l-1}}^{-1}, \zeta_m)\otimes_{\boldsymbol{Q}} \Lambda_{4,-1}$ is a division algebra. Let $|S/T_1|=2^l$. According to (1.5), if $l=3$, then the order of 2 $(\bmod m)$ is odd, if $l>3$, then $m=1$.

We conclude this section with the following

COROLLARY 2.7. *Let G be a finite nilpotent group. Then the following conditions are equivalent;*

(1) *G can be embedded in $M_2(\boldsymbol{H})$ in the form that $V_{\boldsymbol{R}}(G)=M_2(\boldsymbol{H})$.*

(2) *G is a 2-group. And G has a subgroup S of index 2 with normal subgroups T_1, T_2 $(\neq\{1\})$ such that $T_1 \cap T_2=\{1\}$ and $T_1^g=T_2$, where $\{1, g\}$ is a set of representatives of G/S in G, and S/T_1 is a generalized quaternion group.*

PROOF. Assume that G satisfies the condition (1). For each prime $p \mid |G|$ let S_p be the Sylow p-subgroup of G. Then by (2.5) $\prod_{p\neq 2} S_p$ is cyclic, so $V_{\boldsymbol{R}}(\prod_{p\neq 2} S_p)$ is a field contained in the center of $V_{\boldsymbol{R}}(G)=M_2(\boldsymbol{H})$. Therefore $V_{\boldsymbol{R}}(\prod_{p\neq 2} S_p)=\boldsymbol{R}$ and $\prod_{p\neq 2} S_p=\{1\}$ i. e., $G=S_2$. It follows from (1.8) and (2.2) that G satisfies the condition (2).

§ 3. Groups with abelian Sylow 2-subgroups.

In this section we will study subgroups of $M_2(\Delta)$ with abelian Sylow 2-subgroups.

Let G be a group and let H be a subgroup of G. As usual $N_G(H)$, $C_G(H)$, $Z(G)$ denote respectively the normalizer of H in G, the centralizer of H in G, and the center of G.

LEMMA 3.1. *Let G be a finite group which can be embedded in $M_2(\Delta)$ for* *a division algebra Δ. Let p be the minimal prime divisor of $|G|$.*

(1) *If p is odd, then G has a normal p-complement.*

(2) *If $p=2$ and the Sylow 2-subgroup of G is abelian, then G has a normal 2-complement.*

PROOF. If p is odd, then by (1.6) the Sylow p-subgroup of G is abelian. Therefore in both cases the Sylow p-subgroup of G is abelian. Let P be a Sylow p-subgroup of G and put $N=N_G(P)$. Now it suffices by Burnside's theorem ([4], (20.13)) to prove that $P\subseteqq Z(N)$. By (2.1) we have $V_Q(N)\cong\Delta_1$, $M_2(\Delta_2)$ or $\Delta_3\oplus\Delta_4$ for some division algebras Δ_i. If $V_Q(N)\cong\Delta_1$, then P is cyclic, and therefore, by ([4], (20.14)), we have $P\subseteqq Z(N)$. If $V_Q(N)\cong M_2(\Delta_2)$, then, by the proof of (2.3), $2\,|\,|N|$ and so $p=2$. Because $2\nmid[N:P]$, it follows from (2.3) that $V_Q(P)$ is a division algebra. Then P is cyclic. Therefore again by ([4], (20.14)) we have $P\subseteqq Z(N)$. Assume that $V_Q(N)\cong\Delta_3\oplus\Delta_4$ and let ρ_i be the projection of $V_Q(N)$ on Δ_i, $i=3, 4$. Since $\rho_i(P)\subseteqq\Delta_i$, $\rho_i(P)$ is cyclic, and so $\rho_i(P)\subseteqq Z(\rho_i(N))$. Hence $\rho_3(P)\times\rho_4(P)\subseteqq Z(\rho_3(N)\times\rho_4(N))$. Thus we get $P\subseteqq Z(N)$, and this completes the proof of the lemma.

As a direct consequence of (3.1) we get

PROPOSITION 3.2. *Let G be a finite group with abelian Sylow 2-subgroups. Assume that $G\subseteqq M_2(\Delta)$ for a division algebra Δ. Then G is solvable.*

We now give, as a main result in this section,

THEOREM 3.3. *Let G be a finite group with abelian Sylow 2-subgroups. Let Δ be a division algebra and K a field contained in the center of Δ. Assume that G can be embedded in $M_2(\Delta)$ in the form that $V_K(G)=M_2(\Delta)$. Then G satisfies one of the following conditions* (a), (b);

(a) *G has a subgroup G_0 of index 2. Put $G/G_0=\{G_0, gG_0\}$. Then there exist normal subgroups T_1, T_2 $(\neq\{1\})$ of G_0 and two integers m, r such that $T_1\cap T_2=\{1\}$, $T_1{}^g=T_2$, $G_0/T_1\cong G_{m,r}$ and $K\otimes_Z\Lambda_{m,r}\cong\Delta$, where Z is the center of $\Lambda_{m,r}$.*

(b) *There exist a positive integer s, an odd number m and a group homomorphism σ from G to $\mathrm{Gal}(K(\zeta_{2^s m})/K(\zeta_{2^s}))$, which satisfy the following conditions;*

(1) *$\operatorname{Ker}\sigma$ can be embedded in $K(\zeta_{2^s m})$ in the form $V_K(\operatorname{Ker}\sigma)=K(\zeta_{2^s m})$.*

(2) *Put $G/\operatorname{Ker}\sigma=\{g_1\operatorname{Ker}\sigma, \cdots, g_k\operatorname{Ker}\sigma\}$ and $\alpha_{\sigma(g_r),\sigma(g_s)}=g_t^{-1}g_rg_s$ for $g_rg_s\operatorname{Ker}\sigma=g_t\operatorname{Ker}\sigma$. Then the crossed product $(K(\zeta_{2^s m}), G/\operatorname{Ker}\sigma, \{\alpha_{\sigma(g_r),\sigma(g_s)}\})\cong M_2(\Delta)$.*

Conversely, if G satisfies the condition (a) *or* (b), *then G can be embedded in $M_2(\Delta)$ in the form that $V_K(G)=M_2(\Delta)$.*

PROOF. Let V be an irreducible $M_2(\Delta)$-module. Then we may regard V as a KG-module. Denote by G_1 the normal 2-complement of G. So, it follows from (2.3) that the number m of all isomorphism classes of irreducible KG_1-

submodules of V is 1 or 2. In the case where $m=2$, again by (2.3) there exists a subgroup G_0 of G of index 2 with normal subgroups T_1, T_2 $(\neq\{1\})$ such that $T_1\cap T_2=\{1\}$, $T_1^g=T_2$, $G_0/T_1\subseteqq\Delta$ and $V_K(G_0/T_1)=\Delta$, where $\{1, g\}$ is a set of representatives of G/G_0 in G. Since any Sylow subgroup of G_0/T_1 is abelian, it follows from (1.1), (1.2) and (1.3) that $G_0/T_1\cong G_{m,r}$ for some integers m, r and $\Delta\cong K\otimes_Z\Lambda_{m,r}$. Conversely if G satisfies the condition (a), then by (2.2) $G\subseteqq M_2(K\otimes_Z\Lambda_{m,r})$ and $V_K(G)=M_2(K\otimes_Z\Lambda_{m,r})$.

In the case where $m=1$, because $|G_1|$ is odd, it follows from (2.3) that $V_K(G_1)$ is a division algebra. Therefore by (1.2) we have that $G_1\cong G_{m,r}$ for some relatively prime integers m, r and that $V_K(G_1)\cong K\otimes_Z\Lambda_{m,r}$, where Z is the center of $\Lambda_{m,r}$. We recall the notation of $G_{m,r}$. $G_{m,r}=\langle a, b \mid a^m=1,\ b^n=a^t$ and $bab^{-1}=a^r\rangle$, where $s=(r-1, m)$, $t=m/s$; $n=$the minimal positive integer satisfying $r^n\equiv 1 \bmod m$. Let S_2 be a Sylow 2-subgroup of G. And put $S_2'=S_2\cap C_G(\langle a\rangle)$. Since $G=S_2G_{m,r}$ and $(|S_2|, |G_{m,r}|)=1$, we have $C_G(\langle a\rangle)=\langle a\rangle\times S_2'$. So, the fact that $\langle a\rangle\triangleleft G$ implies $S_2'\triangleleft G$. Therefore $C_G(S_2')$ contains S_2 and $G_{m,r}$ and we have $Z(G)\supseteqq S_2'$. Hence $V_K(S_2')$ is contained in the center of $M_2(\Delta)=V_K(G)$. So, if we put $|S_2'|=2^s$ and $S_2'=\langle c\rangle$, then we have $V_K(S_2')\cong K(\zeta_2 s)$, $V_K(C_G(\langle a\rangle))\cong K(\zeta_2 s_m)$ and the isomorphism is obtained by the correspondence $a\leftrightarrow\zeta_m$, $c\leftrightarrow\zeta_2 s$. Denote by ϕ the above isomorphism from $V_K(C_G(\langle a\rangle))$ to $K(\zeta_2 s_m)$. For $g\in G$, we construct an automorphism $\sigma(g)$ of $K(\zeta_2 s_m)$, by mapping $\zeta_2 s\to\zeta_2 s$ and $\zeta_m\to\zeta_m{}^r$, where $a^g=a^r$. Since $K(\zeta_2 s)=V_K(S_2')$ is contained in the center of $V_K(G)$, $\sigma(g)$ is an element of $\mathrm{Gal}(K(\zeta_2 s_m)/K(\zeta_2 s))$, so σ is a group homomorphism from G to $\mathrm{Gal}(K(\zeta_2 s_m)/K(\zeta_2 s))$ and $\mathrm{Ker}\,\sigma=C_G(\langle a\rangle)=\langle a\rangle\times\langle c\rangle$. We recall that $\Lambda=(K(\zeta_2 s_m), G/\mathrm{Ker}\,\sigma, \{\alpha_{\sigma(g_r),\sigma(g_s)}\})$ is a simple algebra with the following structure;

$\Lambda=u_{\sigma(g_1)}K(\zeta_2 s_m)\oplus\cdots\oplus u_{\sigma(g_k)}K(\zeta_2 s_m)$ as $K(\zeta_2 s_m)$-space; $\alpha u_{\sigma(g_i)}=u_{\sigma(g_i)}\alpha^{\sigma(g_i)}$ for α in $K(\zeta_2 s_m)$ and $u_{\sigma(g_r)}u_{\sigma(g_s)}=u_{\sigma(g_r)\sigma(g_s)}\alpha_{\sigma(g_r),\sigma(g_s)}$. In the above notations the mapping $\sum f_i u_{\sigma(g_i)}\to\sum\phi^{-1}(f_i)g_i$ determines a homomorphism from Λ onto $V_K(G)$, where $f_i\in K(\zeta_2 s_m)$. Since Λ is simple and $V_K(G)\neq 0$, this is an isomorphism. Therefore $\Lambda=M_2(\Delta)$. Conversely, if G satisfies the condition (b), then the factor set $\{\alpha_{\sigma(g_r),\sigma(g_s)}\}$ defines an extension of $\mathrm{Ker}\,\sigma$ by $G/\mathrm{Ker}\,\sigma$, which is isomorphic to G. Hence G can be embedded in $M_2(\Delta)$ in the form $V_K(G)=M_2(\Delta)$.

COROLLARY 3.4. *Let G be a finite group with abelian Sylow 2-groups. Then the following conditions are equivalent;*

(1) *G can be embedded in $M_2(\boldsymbol{H})$ in the form $V_{\boldsymbol{R}}(G)=M_2(\boldsymbol{H})$.*

(2) *G has a subgroup G_0 of index 2 with normal subgroups T_1, T_2 $(\neq\{1\})$ such that $T_1\cap T_2=\{1\}$, $T_1^g=T_2$ and $G_0/T_1\cong G_{2m,-1}$ for some integer m, where $\{1, g\}$ is a set of representatives of G/G_0 in G.*

PROOF. Since by (1.3) $G_{2m,-1}$ can be embedded in $\Lambda_{2m,-1}$ in the form $V_{\boldsymbol{Q}}(G_{2m,-1})=\Lambda_{2m,-1}$, $G_{2m,-1}$ can be embedded in $\Lambda_{2m,-1}\otimes_{\boldsymbol{Q}(\zeta_{2m}+\zeta_{2m}^{-1})}\boldsymbol{R}=\boldsymbol{H}$ in the form $V_{\boldsymbol{R}}(G_{2m,-1})=\boldsymbol{H}$. Therefore if G satisfies the condition (2), then it follows from (2.2) that G can be embedded in $M_2(\boldsymbol{H})$ in the form $V_{\boldsymbol{R}}(G)=M_2(\boldsymbol{H})$.

Assume that G satisfies the conditions (1). So, G satisfies one of the conditions (a) and (b) for $K=\boldsymbol{R}$ in (3.3). Since $|\mathrm{Gal}(\boldsymbol{R}(\zeta_2 s_m)/\boldsymbol{R}(\zeta_2 s))|\leqq 2$, we have $\dim_{\boldsymbol{R}}(\boldsymbol{R}(\zeta_2 s_m),\ G/\mathrm{Ker}\,\sigma,\ \{\alpha_{\sigma(gr),\sigma(gs)}\})\leqq 4$. On the other hand $\dim_{\boldsymbol{R}} M_2(\boldsymbol{H})=16$, and it implies that G satisfies the conditions (a). Because $\boldsymbol{R}\otimes_Z \Lambda_{m,r}=\boldsymbol{H}$, $G_{m,r}$ is a subgroup of $\boldsymbol{H}$. Hence it follows from (1.4) that $G_{m,r}$ is the binary dihedral group of order $4l$. This completes the proof of the corollary.

§ 4. Additional results.

LEMMA 4.1. *Let Δ be a division algebra. Let P be a 2-subgroup of $M_2(\Delta)$ and N a normal subgroup of P. Then any elementary abelian subgroup of P/N has order $\leqq 2^4$.*

PROOF. By (1.6) P is a subgroup of $P_1\times P_2$ for some $P_1, P_2\in T_2^{(0)}$, or a subgroup of $\tilde{P}$ for some $\tilde{P}\in T_2^{(1)}$. Since P_i is a cyclic group or a generalized quaternion group, there exists a generalized quaternion group P_3 which contains P_i, $i=1, 2$. It follows from the definition of the 2-group of 1-type that for some $\tilde{P}\in T_2^{(1)}$, $P\subseteqq P_1\times P_2\subseteqq P_3\times P_3\subseteqq \tilde{P}$. Therefore P is a subgroup of a 2-group of 1-type $\tilde{P}$. So, there exist generalized quaternion groups of order 2^{n+1}, $P'=\langle x, y \mid x^{2^n}=1,\ y^2=x^{2^{n-1}},\ y^{-1}xy=x^{-1}\rangle$, and $P''=\langle s, t \mid s^{2^n}=1,\ t^2=s^{2^{n-1}},\ t^{-1}st=s^{-1}\rangle$ such that $P'\times P''$ is a subgroup of $\tilde{P}$ of index 2 and for some $g\in\tilde{P}-(P'\times P'')$ $x^g=s$, $y^g=t$.

Let Q/N be an elementary abelian subgroup of P/N. Since $N\supseteqq [Q, Q]$, we only need to prove $\mathrm{rank}(Q/[Q, Q])\leqq 4$. Let $Q_0=(P'\times P'')\cap Q$. Then $Q/Q_0\subseteqq \tilde{P}/P'\times P''$, so we have $|Q/Q_0|\leqq 2$. Also $Q_0/Q_0\cap\langle x, s\rangle\subseteqq P'\times P''/\langle x, s\rangle$ implies $|Q_0/Q_0\cap\langle x, s\rangle|=1$, 2 or 4. In the case where $|Q_0/Q_0\cap\langle x, s\rangle|\leqq 2$ or $Q=Q_0$, Q is generated by at most 4 elements, for $Q_0\cap\langle x, s\rangle$ is generated by at most 2 elements. It means $\mathrm{rank}(Q/[Q, Q])\leqq 4$.

Assume that $|Q_0/Q_0\cap\langle x, s\rangle|=4$ and $Q\neq Q_0$. Since $P'^h=P''$ for any $h\in Q-Q_0$, by changing s, t, g into x^h, y^h, h respectively, if it is necessary, we may assume that $g\in Q-Q_0$. Because $|P'\times P''/\langle x, s\rangle|=4$, $Q_0/Q_0\cap\langle x, s\rangle\cong P'\times P''/\langle x, s\rangle$, and this means $Q_0\ni yx^i s^j$ for some integers i, j. Using the fact that $s^g=x^{g^2}\in\langle x\rangle$, we have $g^{-1}(yx^i s^j)g(yx^i s^j)^{-1}=tyx^m s^n$ for some integers m, n. Let ρ be the natural homomorphism from Q onto $Q/[Q, Q]$. Then $Q/[Q, Q]$ is generated by $\rho(g)$, $\rho(yx^i s^j)$ and $\rho(Q_0\cap\langle x, s\rangle)$. Therefore $\mathrm{rank}(Q/[Q, Q])\leqq 4$.

PROPOSITION 4.2. *Let G be a solvable subgroup of $M_2(\Delta)$. Let $\pi=\{2, 3, 5, 7\}$. Then G has a normal Hall π'-subgroup.*

PROOF. Let $G=H_0\supseteqq H_1\supseteqq \cdots \supseteqq H_r=\{1\}$ be a chain of normal subgroups of G such that H_i/H_{i+1} is a non-trivial elementary abelian group for each $0\leqq i\leqq r-1$. We shall prove this proposition by induction on $|G|$. Since $G=H_0\neq H_1$,

H_1 has a normal Hall π'-subgroup N. If H_0/H_1 is an elementary p-group for some $p\in\pi$, then our proof is done. Therefore we may assume that $p\notin\pi$. Let D be a $2'$-group of G. By (3.1) D has a normal Hall π'-subgroup D'. Let P be a Sylow p-subgroup of D'. Then P is a Sylow p-subgroup of G. We shall prove that $D'N=PN$. Let Q be a Sylow q-group of D' for any $q\neq p$ and Q' a Sylow q-group of N. Since Q and Q' are Sylow q-groups of G, there exists an element g of G such that $Q=Q'^g$. So $N\lhd G$ means $Q=Q'^g\subseteqq N$ and $D'N\subseteqq PN$. Moreover, it is easily seen that $D'N\supseteqq PN$. Hence $D'N=PN$.

Since H_1 contains a normal Hall π'-subgroup, we may assume that H_1/H_2 is a q-group for some $q\in\pi$. If we can prove that $PH_2\lhd G$ and G/PH_2 is a non-trivial q-group, then by the induction hypothesis PH_2 has a normal Hall π'-subgroup N', implying G has a normal Hall π'-subgroup N'. Therefore we only need to prove that $PH_2\lhd G$ and G/PH_2 is a non-trivial q-group. In the case where $q=2$, H_1/H_2 is an elementary abelian 2-group of order $\leqq 2^4$ by (4.1). It implies $\mathrm{Aut}(H_1/H_2)\subseteqq GL(4, 2)$, and so $|\mathrm{Aut}(H_1/H_2)| \mid 2^6\cdot 3^2\cdot 5\cdot 7$. Since $p\nmid |\mathrm{Aut}(H_1/H_2)|$ and $PH_2/H_2/C_{PH_2/H_2}(H_1/H_2)\subseteqq \mathrm{Aut}(H_1/H_2)$, we have $PH_2/H_2=C_{PH_2/H_2}(H_1/H_2)$. On the other hand $PH_1/H_2=G/H_2$, which implies that $G/H_2\rhd PH_2/H_2$ and G/PH_2 is a non-trivial 2-group. In the case where $q\in\{3, 5, 7\}$, $H_0/H_2=DH_2/H_2\rhd D'H_2/H_2=PH_2/H_2$ means $H_0\rhd PH_2$ and H_0/PH_2 is a non-trivial q-group. This completes the proof of the proposition.

Finally we give a remark on nilpotent subgroups of $M_n(K)$ over an algebraically closed field K of characteristic 0.

In case $n=1$, a group N is a subgroup of K if and only if N is cyclic. We assume $n>1$. Suppose that we can determine the nilpotent subgroups of $M_r(K)$ for $r<n$. Let N be a nilpotent subgroup of $M_n(K)$. If $V_K(N)\neq M_n(K)$, then $V_K(N)=M_{r_1}(K)\oplus\cdots\oplus M_{r_t}(K)$ for some integers $r_1,\cdots,r_t$ such that $\sum_{i=1}^{t} r_i\leqq n$ and $r_i<n$. By our assumption, we can determine the subgroup of $M_{r_i}(K)$, $i=1,\cdots,t$ and we can determine N as a subgroup of a direct product of such groups. Conversely if N_i is a nilpotent subgroup of $M_{r_i}(K)$, then $N_1\times\cdots\times N_t$ is a subgroup of $M_n(K)$. Assume that $V_K(N)=M_n(K)$. In this case N is not abelian, and let S_p be a non-abelian Sylow p-subgroup of N. Since $V_K(S_p)$ is a semi-simple subalgebra of $V_K(N)=M_n(K)$, by the Schur's commutation theorem $V_K(S_p)\cong\prod_{i=1}^{r} M_{n_i}^{m_i}(K)$ and the commutant of $V_K(S_p)$ is isomorphic to $\prod_{i=1}^{r} M_{m_i}^{n_i}(K)$, where $\sum_{i=1}^{r} n_i m_i=n$ and

$$M_{n_i}^{m_i}(K)=\left\{\begin{pmatrix} A & & 0\\ & \ddots & \\ 0 & & A\end{pmatrix}\in M_{n_i m_i}(K) \mid A\in M_{n_i}(K)\right\}.$$

Since S_p is not abelian, we have $n_i>1$ for at least one $1\leqq i\leqq r$. Hence

$V_K(O_{p'}(N)) \neq M_n(K)$, so such groups can be determined by the assumption. On the other hand by (1.6) we can determine S_p. Hence the nilpotent subgroups of $M_n(K)$ can be determined inductively.

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