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On the prolongation of local holomorphic solutions of partial differential equations, III, equations of the Fuchsian type

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§1. Introduction.

In the theory of partial differential equations in the complex n-dimensional space $\boldsymbol{C}^n$, one of interesting problems is the holomorphic continuation of the homogeneous solutions of $P(z, D)u(z)=0$. There obtained several results concerning this problem ([**2**], [**3**], [**5**], [**7**], [**8**] and others). In our preceding paper [**6**], we study the holomorphic continuation over the pluri-harmonic surface and the result obtained is the following: If the surface is defined by $\mathrm{Re}\,\Phi(z)=0$ for some non-degenerate holomorphic function Φ, and the principal part of P, $P_m(z, \mathrm{grad}\,\Phi)$, does not vanish identically on the analytic variety $\{\Phi(z)=0\}$, then every solution of $P(z, D)u(z)=0$ in $\{z \mid \mathrm{Re}\,\Phi(z)>0\}$ becomes holomorphic near the boundary.

In this paper we study the case where $P_m(z, \mathrm{grad}\,\Phi)$ is identically zero on the set $\{\Phi(z)=0\}$ but the function $\Phi(z)$ is not characteristic, that is, $P_m(z, \mathrm{grad}\,\Phi)$ does not vanish identically near the boundary. In such case, $P_m(z, \mathrm{grad}\,\Phi)$ can be divided by Φ^k for some $k \geqq 1$ because the variety $\{\Phi=0\}$ is irreducible and then the notion of *the differential operator of the Fuchsian type* with respect to Φ is naturally introduced (see also M. S. Baouendi and C. Goulaouic [**1**]). The main result in this paper is roughly expressed as follows: If the operator $P(z, D)$ is of the Fuchsian type with respect to $\Phi(z)$ then every homogeneous solution of $P(z, D)u(z)=0$ in $\{z \mid \mathrm{Re}\,\Phi(z)>0\}$ is holomorphic near the boundary of the function of $\log \Phi(z)$ and z.

§2. Partial differential operators of the Fuchsian type.

Let Ω be a domain in the complex n-dimensional space $\boldsymbol{C}^n$ whose boundary $\partial\Omega$ is defined by the level surface of some *pluri-harmonic function*, and $P(z, D)$ be a linear partial differential operator of order m near $\partial\Omega$ with holomorphic coefficients. We denote its principal part by $P_m(z, D)$. Since we study only the local properties of the holomorphic solutions of $P(z, D)u(z)=0$, we may assume without loss of generality that $\Omega=\{z\in U \mid \mathrm{Re}\, z_1>0\}$ for some neighborhood U of 0. In our preceding paper [6], we deal with the case where $P_m(z, N)$ does not vanish identically on the complex hypersurface $z_1=0$ where $N=(1, 0, \cdots, 0)$. Now in this paper we study the case where $P_m(z, N)$ vanishes identically on the set $\{z_1=0\}$ but the function $\Phi(z)=z_1$ is not characteristic, this means that $P_m(z, N)$ does not vanish identically in U. We write $P(z, D) = \sum_{p+|\alpha|\leqq m} a_{(p,\alpha)}(z)(\partial/\partial z_1)^p(\partial/\partial z')^\alpha$ where $z'=(z_2, \cdots, z_n)$ and $\alpha=(\alpha_2, \cdots, \alpha_n)$ is a multi-index, and use the terminology in M. S. Baouendi and C. Goulaouic [1]. We say that a differential monomial which may be written as $c(z_1, z')z_1^l(\partial/\partial z_1)^p(\partial/\partial z')^\alpha$, where $c(0, z')\not\equiv 0$, has the *weight* $p-l$. We now consider the following conditions on the differential operator $P(z, D)$:

(i) the coefficient of $(\partial/\partial z_1)^m$ is $a(z)z_1^k$ with $0\leqq k\leqq m$ and $a(z)\neq 0$ in a neighborhood of 0,

(ii) $P(z, D)$ can be written as the finite sum of differential monomials each of which has the weight at most $m-k$,

(iii) each monomial in the principal part $P_m(z, D)$, except $a(z)z_1^k(\partial/\partial z_1)^m$, has the weight at most $m-k-1$.

DEFINITION. A differential operator satisfying the above conditions (i), (ii) and (iii) is said to be of *the Fuchsian type* (with respect to z_1).

REMARK 1 (see also [1], Remark 1). These conditions (i), (ii) and (iii) are invariant under the coordinate transformation which preserves the hypersurface $z_1=0$. Therefore we may say in general that $P(z, D)$ is of the Fuchsian type with respect to the complex hypersurface S if and only if, in some local coordinates (which associate to S the hyperplane $z_1=0$), the operator $P(z, D)$ satisfies (i), (ii) and (iii).

REMARK 2. Our definition of the Fuchsian operator is weaker than that of M. S. Baouendi and C. Goulaouic [1]. They request the Fuchsian operator to satisfy the condition (iii) for *all differential monomial* in $P(z, D)$ except for the terms $a_{(k,0)}(z)(\partial/\partial z_1)^k$ $(k=0, 1, \cdots, m)$.

Since we deal only with the homogeneous equation $P(z, D)u(z)=0$, we may assume in general that $m=k$ and the coefficient of $(\partial/\partial z_1)^m$ is equal to z_1^m in the above definition. Then the Fuchsian operator is written in the form

$$P(z, D)=\sum_{p+|\alpha|\leqq m} a_{(p,\alpha)}(z)z_1^p\left(\frac{\partial}{\partial z_1}\right)^p\left(\frac{\partial}{\partial z'}\right)^\alpha, \tag{1}$$

and especially its principal part is in the form

$$P_m(z, D)=z_1^m\left(\frac{\partial}{\partial z_1}\right)^m+z_1\sum_{\substack{p+|\alpha|=m\\ p<m}} b_{(p,\alpha)}(z)z_1^p\left(\frac{\partial}{\partial z_1}\right)^p\left(\frac{\partial}{\partial z'}\right)^\alpha, \tag{2}$$

where $a_{(p,\alpha)}(z)$ and $b_{(p,\alpha)}(z)$ are holomorphic in U.

Now we take U as the set

(3) $$U_z(\rho, r)=\{z \mid |z_1|<\rho,\ |z_j|<r \quad j=2, \cdots, n\}$$

for some positive constants ρ and r, and set

(4) $$\Omega_z(\rho, r)=\{z\in U_z(\rho, r) \mid \operatorname{Re} z_1>0\}\ .$$

Here we may assume that there exists a constant M such that

(5) $$|b_{(p,\alpha)}(z)|\leqq M$$

in $U_z(\rho, r)$ for every (p, α) with $p+|\alpha|=m$ and $p<m$. Then we make the holomorphic transformation of coordinates from (z_1, z')-variables to (t, z')-variables as follows:

(6) $$z_1=e^t\ .$$

This change of variables is well-known for the Euler equation in the theory of ordinary differential equations and we have the next relations,

(7) $$z_1^k\Big(\frac{\partial}{\partial z_1}\Big)^k=\frac{\partial}{\partial t}\Big(\frac{\partial}{\partial t}-1\Big)\cdots\Big(\frac{\partial}{\partial t}-k+1\Big) \qquad k=1, 2, \cdots\ .$$

Under this transformation the domain Ω given by (4) is bi-holomorphically mapped into the following domain,

(8) $$\tilde{\Omega}_{(t,z')}(\log\rho, r)=\{(t, z') \mid \operatorname{Re} t<\log\rho,\ |\operatorname{Im} t|<\pi/2,\ |z_j|<r \quad j=2, \cdots, n\}$$

and by the relation (7), the Fuchsian operator $P(z, D)$ given by (1) is transformed to a differential operator $\tilde{P}(t, z'\,;\,\partial/\partial t, \partial/\partial z')$ whose coefficients are holomorphic in

(9) $$\tilde{U}_{(t,z')}(\log\rho, r)=\{(t, z') \mid \operatorname{Re} t<\log\rho,\ |z_j|<r \quad j=2, \cdots, n\}\ .$$

Then the principal part of $\tilde{P}(t, z'\,;\,\partial/\partial t, \partial/\partial z')$ with respect to $(\partial/\partial t, \partial/\partial z')$ is, by (2),

(10) $$\tilde{P}_m\Big(t, z'\,;\,\frac{\partial}{\partial t}, \frac{\partial}{\partial z'}\Big)$$
$$=\Big(\frac{\partial}{\partial t}\Big)^m+e^t\sum_{\substack{p+|\alpha|=m\\ p<m}} b_{(p,\alpha)}(e^t, z')\Big(\frac{\partial}{\partial t}\Big)^p\Big(\frac{\partial}{\partial z'}\Big)^\alpha .$$

Here we should pay the attention to the following two points:

(i) under this transformation, the hyperplane $t=0$ becomes non-characteristic,

(ii) there exists the term e^t in every coefficients of differential monomials $(\partial/\partial t)^p(\partial/\partial z')^\alpha$ $(p+|\alpha|=m,\ p<m)$.

These conditions mean that if $\operatorname{Re} t$ is sufficiently small, then the characteristic hyperplane becomes "almost" parallel to the t-axis (see Lemma 1 in the next section), and this is essential in our theory.

§ 3. Prolongation of local holomorphic solutions.

In this section we study the holomorphic continuation of the homogeneous solutions of the Fuchsian partial differential equation $P(z, D)u(z)=0$.

We define the bilinear inner product $\langle , \rangle$ in $\boldsymbol{C}^n$ by $\langle z, \lambda\rangle = \sum_{j=1}^{n} z_j\lambda_j$ and the norm of z by $|z|^2=\langle z, \bar{z}\rangle$ and set $S^{2n-1}=\{\zeta\in\boldsymbol{C}^n \mid |\zeta|=1\}$. The real hyperplane $H(\zeta, z_0)$ through the point z_0 with the complex normal direction $\zeta\in S^{2n-1}$ is defined by

$$H(\zeta, z_0)=\{z \mid \operatorname{Re}\langle z-z_0, \zeta\rangle=0\} . \tag{11}$$

We also denote this by $H(\zeta)$ when z_0 has no need to be mentioned. The vector $\zeta\in S^{2n-1}$ is said to be characteristic with respect to $P(z, D)$ at z_0 if ζ satisfies the equation $P_m(z_0, \zeta)=0$. For an open set V in $\boldsymbol{C}^n$ and a differential operator $P(z, D)$ in V, we denote by $\operatorname{Car}_P(V)$ the closure in S^{2n-1} of all vectors that are characteristic for some point in V. Then we have the next theorem.

THEOREM 1 (J. M. Bony and P. Schapira [3] Théorème 2.1, see also L. Hörmander [4], Theorem 5.3.3). *Let Ω_1 and Ω_2 be two open convex sets in $\boldsymbol{C}^n$ such that $\Omega_1\subset\Omega_2$ and let $P(z, D)$ be a differential operator in Ω_2. We assume that every hyperplane $H(\zeta)$ with $\zeta\in\operatorname{Car}_P(\Omega_2)$ which intersects Ω_2 also meets Ω_1. Then every $u(z)$ holomorphic in Ω_1 and satisfying the equation $P(z, D)u(z)=0$ becomes holomorphic in Ω_2.*

We remark that the function $u(z)$ extended holomorphically to that on Ω_2 satisfies also the equation $P(z, D)u(z)=0$ in Ω_2 by the theorem of identity.

We now study the vector ζ in $\operatorname{Car}_{\tilde{P}}(\tilde{U}_{(t,z')}(\log\rho, r))$ for the operator $\tilde{P}(t, z' ; \partial/\partial t, \partial/\partial z')$, where $\tilde{U}_{(t,z')}(\log\rho, r)$ is given by (9) and the principal part of $\tilde{P}(t, z' ; \partial/\partial t, \partial/\partial z')$ is given by (10) in the preceding section. Then we have the following lemma.

LEMMA 1. *For any number $C>0$, there exists τ ($\tau<\log\rho$) such that*

$$C|\zeta_1|\leqq|\zeta_2|+\cdots+|\zeta_n|$$

for any $\zeta\in\operatorname{Car}_{\tilde{P}}(\tilde{U}_{(t,z')}(\tau, r))$.

PROOF. Let ζ be any vector that is characteristic with respect to $\tilde{P}$ at some point in $\tilde{U}_{(t,z')}(\tau, r)$. Then by (5) and (10) we have

$$\begin{aligned}|\zeta_1|^m &\leqq e^{\tau}\sum_{\substack{p+|\alpha|=m\\ p<m}}|b_{(p,\alpha)}||\zeta_1^p\zeta'^{\alpha}| \\ &\leqq Me^{\tau}\{(|\zeta_1|+\cdots+|\zeta_n|)^m-|\zeta_1|^m\} .\end{aligned}$$

Therefore

$$\{(1+M^{-1}e^{-\tau})^{1/m}-1\}|\zeta_1|\leqq|\zeta_2|+\cdots+|\zeta_n|,$$

and this inequality completes the proof because the coefficient of $|\zeta_1|$ tends to infinity as $\tau\to-\infty$.

We next study the relation between open convex sets and hyperplanes. Let $\Delta(a, b)$ (a, b real) be an open convex set in the complex plane $\boldsymbol{C}$ with the variable t defined by

$$\Delta(a, b)=\{t \mid \operatorname{Re} t<a,\ |\operatorname{Im} t|<b\}. \tag{12}$$

Using this notation, we have

$$\tilde{\Omega}_{(t,z')}(a, r)=\{(t, z') \mid t\in\Delta(a, \pi/2),\ |z_j|<r \quad j=2, \cdots, n\}.$$

Furthermore we set $\hat{\Omega}_{(t,z')}(\Delta(a, b), r)$ the convex hull of the set $\tilde{\Omega}_{(t,z')}(a, r)$ and the set $\{(t, z') \mid t\in\Delta(a, b),\ z'=0\}$. We remark that $\hat{\Omega}$ obtained above is an open convex set in $\boldsymbol{C}^n$.

LEMMA 2. *Let $\zeta\in S^{2n-1}$ be any vector satisfying the inequality*

$$C|\zeta_1|\leqq|\zeta_2|+\cdots+|\zeta_n|$$

for some constant $C>0$. Then if the hyperplane $H(\zeta)$ intersects $\hat{\Omega}_{(t,z')}(\Delta(a, \pi/2+Cr), r)$, it also meets $\tilde{\Omega}_{(t,z')}(a, r)$.

PROOF. Since $\hat{\Omega}_{(t,z')}(\Delta(a, \pi/2+Cr), r)$ is the convex hull of $\tilde{\Omega}_{(t,z')}(a, r)$ and the set $\{(t, z') \mid t\in\Delta(a, \pi/2+Cr),\ z'=0\}$, the hyperplane $H(\zeta)$ which intersects $\hat{\Omega}$ must also meets $\tilde{\Omega}$ or $\{(t, z') \mid t\in\Delta(a, \pi/2+Cr),\ z'=0\}$. Thus for the poof of this lemma it is sufficient to show that the hyperplane $H(\zeta)$ meets $\tilde{\Omega}$ if there is a point $(\alpha+i\beta, 0, \cdots, 0)$ (α, β real) in $H(\zeta)$ such that $\alpha<a$ and $|\beta|<\pi/2+Cr$.

We now write $\zeta_j=\xi_j+i\eta_j$ ($j=1, \cdots, n$) (ξ_j, η_j real) and $t=x+iy$ (x, y real). Then $H(\zeta)$ is the set of all points $(x+iy, z_2, \cdots, z_n)$ satisfying

$$\xi_1(x-\alpha)-\eta_1(y-\beta)=-\operatorname{Re}\sum_{j=2}^{n}\zeta_j z_j. \tag{13}$$

If we take $x_0=\alpha$ and $|y_0|<\pi/2$ such that $|y_0-\beta|<Cr$, then by the assumption we have

$$|\xi_1(x_0-\alpha)-\eta_1(y_0-\beta)|<C|\zeta_1|r\leqq r(|\zeta_2|+\cdots+|\zeta_n|).$$

On the other hand the right hand side of (13) can take any value whose absolute value is less than $r\,(|\zeta_2|+\cdots+|\zeta_n|)$ at some point $(z_2, \cdots, z_n)$ satisfying $|z_j|<r$ ($j=2, \cdots, n$). Thus there exists a point $(t_0, z_0')=(x_0+iy_0, z_2^{(0)}, \cdots, z_n^{(0)})$ in $H(\zeta)$ which is also contained in $\tilde{\Omega}$. This completes the proof.

Now we have the following main theorem.

THEOREM 2. *Let $\tilde{P}(t, z'; \partial/\partial t, \partial/\partial z')$ be a differential operator in the domain $\tilde{U}_{(t,z')}(\log\rho, r)$ given by* (9) *with the principal part given by* (10). *Then for any positive number C there exists τ ($\tau<\log\rho$) such that every $u(t, z')$ holomorphic* *in* $\tilde{\Omega}_{(t,z')}(\tau, r)$ *and satisfying the equation* $\tilde{P}(t, z'; \partial/\partial t, \partial/\partial z')u(t, z')=0$ *becomes holomorphic in* $\hat{\Omega}_{(t,z')}(\Delta(\tau, \pi/2+Cr), r)$.

PROOF. For a given number C we take τ by Lemma 1 such that

$$C|\zeta_1| \leqq |\zeta_2| + \cdots + |\zeta_n|$$

for any $\zeta \in \mathrm{Car}_{\tilde{P}}(\tilde{U}_{(t,z')}(\tau, r))$. We then apply Theorem 1 with $\Omega_1 = \tilde{\Omega}_{(t,z')}(\tau, r)$ and $\Omega_2 = \hat{\Omega}_{(t,z')}(\Delta(\tau, \pi/2+Cr), r)$ and, using Lemma 2, we get this theorem.

Since for any number ε $(0<\varepsilon<Cr)$ there exists ρ $(0<\rho<\min(\exp\tau, r))$ such that the set $\{(t, z') \mid \mathrm{Re}\, t<\log\rho,\ |\mathrm{Im}\, t|<\pi/2+Cr-\varepsilon,\ |z_j|<\rho\ \ j=2, \cdots, n\}$ is contained in $\hat{\Omega}_{(t,z')}(\Delta(\tau, \pi/2+Cr), r)$, we can now restate the above theorem as follows.

THEOREM 2^{bis}. *Let* $P(z, D)$ *be a differential operator of the Fuchsian type with respect to* z_1 *in a neighborhood* U *of* 0 *in* $\boldsymbol{C}^n$. *Then for any positive number* C *we can choose* $r>0$ *such that every* $u(z)$ *holomorphic in* $\Omega=\{z\in U \mid \mathrm{Re}\, z_1>0\}$ *and satisfying the equation* $P(z, D)u(z)=0$ *becomes holomorphic with respect to the variables* $(\log z_1, z_2, \cdots, z_n)$ *in the following domain*

$$\begin{cases} |z_j|<r & (j=1, 2, \cdots, n) \\ |\arg z_1|<C. \end{cases}$$

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