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Structure of a single pseudo-differential equation in a real domain

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We investigate the micro-local structure of a single pseudo-differential equation in a real domain under the assumption that their characteristic variety has singularities of normal crossing type. (Precise conditions are given in the below.)

We note that the micro-local structure of a single pseudo-differential equation of this type has been completely investigated in a complex domain by Kashiwara, Kawai and Oshima [2]. (See Theorem 1 in the below.)

The most interesting phenomenon peculiar to the problem in a real domain is that new invariant (the function h appearing in Theorem 2) appears.

Firstly we recall the theorem which clarifies the structure in a complex domain of a single pseudo-differential equation $\mathcal{M}=\mathcal{P}/\mathcal{J}$ whose characteristic variety V has the singularity of normal crossing type. Precise conditions on $\mathcal{M}$ and V are the following:

(1) The symbol ideal J of $\mathcal{J}$ is reduced.

(2) V has the form $V_1\cup V_2$, where V_1 and V_2 are regular submanifolds of a $(2n-1)$-dimensional complex contact manifold (X^C, ω) and cross transversally.

(3) The canonical 1-form ω restricted to $V_1\cap V_2$ never vanishes.

Then a suitable "quantized" contact transformation will bring micro-locally the generator P of $\mathcal{J}$ to $z_1D_1+Q(z', D_{z'})$. Here z' and $D_{z'}$ denote $(z_2, \cdots, z_n)$ and $\left(\frac{\partial}{\partial z_2}, \cdots, \frac{\partial}{\partial z_n}\right)$ respectively and $Q(z', D_{z'})$ is a pseudo-differential operator of order at most zero. (Theorem 3 of Kashiwara, Kawai and Oshima [2]). Moreover $\kappa=\sigma_0(Q)/\{\zeta_1, z_1\}|_{V_1\cap V_2}$ is invariant under contact transformation. Then using this invariant κ we have the following theorem.

THEOREM 1. *Assume conditions* (1)$\sim$(3). *Further assume that*

$$(d\kappa\wedge\omega)|_{V_1\cap V_2}\neq 0\,. \tag{4}$$

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Then a suitable "quantized" contact transformation brings micro-locally the generator $P(z, D)$ of $\mathcal{J}$ to $z_1D_1+z_2$.

Thus we have seen the micro-local structure of $\mathcal{M}$ satisfying conditions (1)$\sim$(4) in a complex domain. So the most important step in the investigation of the structure of the microfunction solutions of $\mathcal{M}$ is to know the micro-local structure of $\mathcal{M}$ in a real domain. Hence in this note we concentrate ourselves to the study of real contact geometry related to the interaction of V_1 and V_2. As for the investigation of the structure of cohomology groups having microfunction solution sheaves of $\mathcal{M}$ as coefficients, we refer the reader to Kashiwara, Kawai and Oshima [2], [3].

The first case that we are concerned with is the following.

THEOREM 2. *Let $V_j=\{f_j=0\}$ $(j=1, 2)$ be regular hypersurfaces of a complex contact manifold (X^C, ω), complexification of a purely imaginary contact manifold (X, ω). Assume following conditions* (5) *and* (6).

(5) $$\{f_j, f_k^c\}\neq 0\ (j, k=1, 2) \quad \text{and} \quad \{f_j, f_k\}\neq 0\ (j\neq k)\,.$$

For the definiteness' sake we assume that $\{f_1, f_1^c\}>0$.

(6) $$V_1\cap V_1^c=V_2\cap V_2^c\,.$$

Then locally we may take f_1 and f_2 so that they satisfy

(7) $$\{f_1, f_1^c\}=1\,.$$

(8) $f_2=f_1-hf_2^c$, *where h is real valued on X and satisfies $\{f_1, h\}=\{f_1^c, h\}=0$.*

Here $f^c(x, i\eta)$ denotes $\overline{f(\bar{x}, i\bar{\eta})}$ using the canonical coordinate system $(x, i\eta)$ on X.

REMARK. Lemma 2.3.3 in Chapter III of Sato, Kawai and Kashiwara [4] says that we can choose a canonical coordinate system $(x, i\eta)$ on X so that f_1 has the form

$$\frac{\eta_1}{\sqrt{i\eta_n}}+x_1\sqrt{i\eta_n}$$

near $(x, i\eta)=(0; i(0, \cdots, 0, 1))$. Then condition (8) asserts that f_2 has the form

$$\frac{\eta_1}{\sqrt{i\eta_n}}+hx_1\sqrt{i\eta_n}$$

with $h=h\left(x_n-\frac{x_1\eta_1}{2\eta_n}, x_2, \cdots, x_n, \eta_2, \cdots, \eta_n\right)$.

PROOF OF THEOREM 2. All the problems in the below are considered in a neighborhood U of x_0 in $V_1\cap V_1^c=V_2\cap V_2^c\subset X^C$. Firstly we note that we can find a holomorphic function k so that $f_2=f_1-kf_1^c$ with $k\neq 0$ because of conditions (5) and (6). If we define θ_1 so that it satisfies

$$\exp(2i\theta_1)=\frac{k}{\sqrt{kk^c}}$$

and replace f_1 by $\exp(i\theta_1)f_1$, then the corresponding k, denoted by k_1 is real valued on $V_2\cap V_2^c$. Moreover, by solving the differential equation $\{f_2, k_2\}=0$ with initial data k_1 on $V_2\cap V_2^c$ and replacing k_1 by k_2, we may assume from the beginning that f_2 has the form $f_1-kf_1^c$ where $k|_{V_1\cap V_2}$ is real and k is constant along any bicharacteristics of V_2. In fact, it is sufficient to replace f_1 by $e^{i\theta_2}f_1$ with θ_2 satisfying $\exp(2i\theta_2)=\frac{k_1}{k_2}$.

On the other hand the condition that $\{f_1, f_1^c\}\neq 0$ allows us to find φ which is real valued on X and satisfies $\{\varphi f_1, \varphi f_1^c\}=1$. (Lemma 2.3.3 in Chapter III of Sato, Kawai and Kashiwara [4]). Therefore we may assume from the beginning that $\{f_1, f_1^c\}=1$.

Now consider the function $k-k^c$. Since $\{f_2, f_2^c\}\neq 0$, we can find holomorphic functions a and b so that $k-k^c=af_2+bf_2^c$. Then clearly we have $k-k^c=\frac{a-b^c}{2}f_2+\frac{b-a^c}{2}f_2^c$, since $k-k^c=-b^cf_2-a^cf_2^c$.

Let us define h_0 by $k-\frac{a-b^c}{2}f_2$. The function h_0 thus defined clearly coincides with k on $V_2=\{f_2=0\}$ and is real valued on X. It is also clear that $\{f_2, h_0\}|_{V_2}=0$ because $\{f_2, k\}|_{V_2}=0$.

Now consider the following first order differential equation (9).

$$\begin{cases}\{f_1^c, h\}=0\\ h|_{V_2}=h_0|_{V_2}.\end{cases}\tag{9}$$

Since $\{f_1^c, f_2\}\neq 0$, this constitutes a non-characteristic Cauchy problem for h. Therefore equation (9) admits a unique solution h. By the initial condition given in (9) asserts that $V_2=\{f_1-hf_1^c=0\}$ holds.

Now we want to show that the function h thus defined satisfies condition (8).

In order to see this, we first note the following

$$\{f_1^c, \{f_1, h\}\}=0\,.\tag{10}$$

In fact, the Jacobi identity implies that

$$\{f_1^c, \{f_1, h\}\}=\{f_1, \{h, f_1^c\}\}+\{h, \{f_1^c, f_1\}\}\,.\tag{11}$$

On the other hand equation (9) implies $\{h, f_1^c\}=0$ and the choice of f_1 explained before asserts $\{f_1^c, f_1\}=-1$. Therefore (11) implies (10).

Moreover we can easily verify that $\{f_1, h\}|_{V_2}=0$ holds. In fact, $\{f_1, h\}|_{V_2}=\{f_1-hf_1^c, h\}|_{V_2}+\{hf_1^c, h\}|_{V_2}=0$ holds, because $\{f_2, h\}|_{V_2}=\{f_2, h_0\}|_{V_2}=0$ and because $V_2=\{f_1-hf_1^c=0\}$.

Then, using again the fact that $\{f_1^c, f_2\}\neq 0$, we can conclude by the uniqueness of the solution of the non-characteristic Cauchy problem that $\{f_1, h\}=0$.

Lastly we show that $h=h^c$. As we have proved in the above, $\{f_1, h\}=\{f_1^c, h\}=0$. Therefore $\{f_1, h-h^c\}=\{f_1^c, h-h^c\}=0$ holds. On the other hand $(h-h^c)|_{V_2\cap V_2^c}=(h_0-h_0^c)|_{V_2\cap V_2^c}=0$. Moreover, as we see later in Lemma 3,

$$\det\begin{pmatrix}\{f_1, f_2\} & \{f_1, f_2^c\}\\ \{f_1^c, f_2\} & \{f_1^c, f_2^c\}\end{pmatrix}\neq 0 \tag{12}$$

holds.

Therefore the submanifold $V_2\cap V_2^c$ is non-characteristic with respect to the system of equations $\{f_1, u\}=\{f_1^c, u\}=0$. This fact implies that $h=h^c$ holds identically because of the uniqueness of solutions for non-characteristic Cauchy problem. This ends the proof of Theorem 2 except for the proof of the relation (12).

LEMMA 3. *Assumptions* (5) *and* (6) *imply* (12).

PROOF. Firstly note that assumptions (5) and (6) allow us to assume that f_1 takes the form $\eta_1-ix_1\eta_n$ and that f_2 takes the form $\eta_1-ix_1\varphi\eta_n$ with $\varphi=\varphi^c$ near $(x, i\eta)=(0, i(0, \cdots, 0, 1))$.

In order to see this, we use the inhomogeneous coordinate system (x, p), i.e., $p_j=-\eta_j/\eta_n$ $(j=1, \cdots, n-1)$. Assumptions (5) and (6) then imply that f_2 has the form

$$(p_1+\psi_1x_1)\pm i(\psi_2p_1+\theta x_1)$$

where $\theta(0)\neq 0$ and ψ_1, ψ_2 and θ are real valued on X. Multiplying f_2 by $(1\pm i\psi_1/\theta)$, we may assume from the beginning that $\psi_1=0$.

Now we try to find the required φ by multiplying f_1 and f_2 by $(1\pm i\alpha_1)$ and $(1\pm i\alpha_2)$. It is readily verified that α_2/α_1 can be taken to be φ if $\alpha_1(1-\psi_2\alpha_2)=\alpha_2\theta$ and $\alpha_1\theta=\psi_2+\alpha_2$ hold for α_1 and α_2 which are real valued on X. Direct calculations will show that it suffices to take

$$\alpha_2=\frac{1}{\theta}(\psi_2+\alpha_1)$$

and

$$\alpha_1=\frac{1-\psi_2^2-\theta^2+\sqrt{(\psi_2^2+\theta^2-1)^2+4\psi_2^2}}{2\psi_2}.$$

Note that $\theta(0)^2-1\neq 0$ if $\psi_2(0)=0$ by assumption (5). So α_1, hence α_2, is always well-defined and holomorphic. Clearly α_1 and α_2 are real valued on X. Thus we have verified that f_2 may be chosen to be $\eta_1-ix_1\varphi\eta_n$ with $\varphi=\varphi^c$.

Now the direct calculations show that

$$\det\begin{pmatrix}\{f_1, f_2\} & \{f_1, f_2^c\} \\ \{f_1^c, f_2\} & \{f_1^c, f_2^c\}\end{pmatrix}$$
$$=|\{f_1, f_2\}|^2-|\{f_1^c, f_2\}|$$
$$=|(1-\varphi)\eta_n|^2-|(1+\varphi)\eta_n|^2+O(|x_1|)\,.$$

Since φ is real valued on X, this shows that (12) holds near $(x, i\eta)=(0, i(0, \cdots, 0, 1))$.

This ends the proof of Lemma 3 and, at the same time, completes the proof of Theorem 2.

The case treated by Theorem 2 is, so to speak, the case of crossing of two characteristic varieties of Lewy-Mizohata type. The second case we treat in the following Theorem 4 is the case where a characteristic variety of Lewy-Mizohata type and that of de Rham type cross.

Precise statement is the following.

THEOREM 4. *Let V_1 and V_2 be regular hypersurfaces in $X^{\mathbf{C}}$. Assume that V_1 and V_2 intersect transversally and that $\omega|_{V_1\cap V_2}\neq 0$. Assume further that $V_1=\{f_1=0\}$ is real and that $V_2=\{f_2=0\}$ is of Lewy-Mizohata type, that is, $f_1=f_1^c$ and $\{f_2, f_2^c\}\neq 0$. Then we can find a suitable canonical coordinate system on X so that $V_1=\{x_1=0\}$ and $V_2=\{\eta_1\pm ix_1\eta_2=0\}$ near $(x; i\eta)=(0; i(0, 1, 0, \cdots, 0))$. Here the sign in the defining function of V_2 is chosen according to that of $\{f_2, f_2^c\}$.*

PROOF. Under the assumptions of the theorem the real codimension of $V_2\cap X$ is 1 in $V_1\cap X$. This implies the existence of h_1 and h_2 which satisfy the following:

$$h_1=h_1^c \quad\text{and}\quad h_2=h_2^c \tag{13}$$

$$V_1=\{h_1=0\} \quad\text{and}\quad V_2=\{h_1+ih_2=0\}\,. \tag{14}$$

Since V_2 is of Lewy-Mizohata type, we can find $\varphi\neq 0$ so that $\{\varphi h_1, \varphi h_2\}=\pm 1$ with $\varphi=\varphi^c$. Here the sign is that of $\{h_1, h_2\}$. (Lemma 2.3.3 in Chapter III of Sato, Kawai and Kashiwara [**4**]). Then $V_1=\{\varphi h_1=0\}$ and $V_2=\{\varphi h_1+i\varphi h_2=0\}$ with $\{\varphi h_1, \varphi h_2\}=\pm 1$. This immediately implies that a suitable choice of canonical coordinate system of X makes $V_1=\{x_1=0\}$ and $V_2=\{\eta_1\pm ix_1\eta_2=0\}$ near $(x, i\eta)=(0; i(0, 1, 0, \cdots, 0))$.

Before ending this note we mention the solvability of the pseudo-differential equation $P(x, D_x)u=f$ whose characteristic variety V has the form $V_1\cup V_2$ where V_j satisfies the conditions posed in Theorem 4. We assume that the symbol ideal J of $\mathcal{J}=\mathcal{P}P$ is reduced.

In this case the most important point that makes the arguments simpler is the following observation:

The generator $P(x, D_x)$ of $\mathcal{J}$ may be chosen to be of the form

$$x_1(D_1+\alpha x_1 D_2)+\lambda\left(x_2-\frac{\alpha x_1^2}{2}, x_3, \cdots, x_n, D_2, \cdots, D_n\right), \tag{15}$$

where $\alpha=\pm i$ and λ is of order at most 0.

This fact is an obvious consequence of Theorem 1, because the lower order term λ of P may be chosen to satisfy $[\lambda, x_1]=[\lambda, D_1+\alpha x_1 D_2]=0$.

The expression of P in the form (15) will allow one to construct the fundamental solution for P if the principal symbol $\sigma_0(\lambda)$ of λ restricted to $x_1=0$ does not attain integral values, that is, we can assert that the pseudo-differential equation $P(x, D_x)u=f$ is always solvable micro-locally as long as $\sigma_0(\lambda)(x_2, \cdots, x_n, \eta_2, \cdots, \eta_n)$ does not attain integral values. In fact, it is possible to give the meaning to x_{1+}^{λ} as a boundary value of a pseudo-differential operator if λ is of order at most zero and is independent of D_1 (cf. Kashiwara and Kawai [**1**]). This topic will be discussed elsewhere.

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