J. Math. Soc. Japan
Vol. 27, No. 4, 1975

Weighted norm inequalities for singular integrals

By Makoto KANEKO and Shigeki YANO

(Received Dec. 12, 1974)

§ 1. Introduction.

Recently many results concerning the weighted norm inequalities for various types of integral transforms have been proved. The most fundamental result in this direction is the one due to Muckenhoupt [7], which establishes the weighted norm inequalities for the Hardy-Littlewood maximal functions in $\boldsymbol{R}^n$:

$$f^*(x)=\sup\frac{1}{|Q_x|}\int_{Q_x}|f(y)|dy\,, \tag{1.1}$$

where $f\in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$ and Q_x ranges over all cubes with sides parallel to axes and centered at x. His result is:

THEOREM A. *Let $1<p<\infty$, and ω is non-negative and belongs to $L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$. The inequality*

$$\int_{\boldsymbol{R}^n}[f^*(x)]^p\omega(x)dx\leqq C_p\int_{\boldsymbol{R}^n}|f(x)|^p\omega(x)dx \tag{1.2}$$

is valid for all $f\in L^p(\omega(x)dx)$, if and only if ω satisfies the condition

$$\sup_Q\Big(\frac{1}{|Q|}\int_Q\omega(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q\omega(x)^{-1/(p-1)}dx\Big)^{p-1}<\infty\,, \tag{A_p}$$

where the supremum is taken over all cubes Q.

The weak type inequality

$$m_\omega(\{x\in\boldsymbol{R}^n:f^*(x)>\lambda\})\leqq\frac{C}{\lambda}\int_{\boldsymbol{R}^n}|f(x)|\omega(x)dx \tag{1.3}$$

(here $m_\omega(A)=\int_A\omega(x)dx$ for measurable $A\subset\boldsymbol{R}^n$) is valid for all $f\in L^1(\omega(x)dx)$, if and only if ω satisfies the condition

$$\omega^*(x)\leqq C\omega(x)\qquad a.\,e. \tag{A_1}$$

Shortly after the proof of Theorem A, Hunt, Muckenhoupt, and Wheeden [6] proved the same result for Hilbert transforms in $\boldsymbol{R}^1$ in place of the maximal function.

More recently, extending the latter result to the singular integral operators in $\boldsymbol{R}^n$, Coifman and Fefferman [2] have proved

THEOREM B. *Let $T: f \to K*f$ in $\boldsymbol{R}^n$, be a singular integral operator with a convolution kernel satisfying the conditions*

(a) $$\|\hat{K}\|_\infty \leqq C,$$

(b) $$|K(x)| \leqq \frac{C}{|x|^n},$$

(c) $$|K(x)-K(x-y)| \leqq \frac{C|y|}{|x|^{n+1}} \quad \text{for} \quad |y| < \frac{|x|}{2}.$$

Suppose that the weight function ω satisfies the condition (A_p), $1<p<\infty$, then

$$\int_{\boldsymbol{R}^n} |Tf(x)|^p \omega(x) dx \leqq C_p \int_{\boldsymbol{R}^n} |f(x)|^p \omega(x) dx. \tag{1.4}$$

To prove Theorem B, they introduced the "maximal operator"

$$T^*f(x) = \sup_{Q_x \subset Q'_x} \left| \int_{Q'_x \setminus Q_x} K(x-y) f(y) dy \right|, \tag{1.5}$$

where Q_x and Q'_x range over all cubes centered at x with $Q'_x \supset Q_x$, and showed that the inequality (1.4) is an easy consequence of the "distribution function inequality"

$$m_\omega(\{T^*f > 2\lambda,\ f^* \leqq \gamma\lambda\}) \leqq C\gamma^\delta m_\omega(\{T^*f > \lambda\}), \tag{1.6}$$

where the weight function ω satisfies the condition
(A_∞) There exist positive constants C, $\delta>0$ so that given any cube Q and any measurable subset $E \subset Q$

$$\frac{m_\omega(E)}{m_\omega(Q)} \leqq C\left(\frac{|E|}{|Q|}\right)^\delta.$$

From the inequality (1.6), it follows that

$$\int_{\boldsymbol{R}^n} (T^*f(x))^p \omega(x) dx = C_p \int_{\boldsymbol{R}^n} (f^*(x))^p \omega(x) dx.$$

Combining this and (see [8])

LEMMA 1. (A_p), $1 \leqq p < \infty$, *implies* (A_∞),

they proved Theorem B.

On the other hand, Coifman [1] stated earlier, only with a sketch of proof for the special case of Hilbert transform that, if the convolution kernel K satisfies the conditions

(α) $K(x) = \Omega(x)/|x|^n$, where Ω is homogeneous of degree zero,

(β) Ω satisfies a Lipschitz condition of positive order α, $0 < \alpha \leqq 1$, on the unit sphere S^{n-1},

(γ) $\int_{S^{n-1}} \Omega(x') dx' = 0,$

and ω satisfies the condition (A_∞), then for any $\lambda>0$ and any $\beta>1$, $\gamma>0$ with $\beta-1>C\gamma$

$$m_\omega(\{T^*f>\beta\lambda,\ f^*\leqq\gamma\lambda\})\leqq Ce^{-\frac{c(\beta-1)}{\gamma}}m_\omega(\{T^*f>\lambda\}). \tag{1.7}$$

It is easily seen that the conditions (α), (β) and (γ) imply the condition (a), (b) and

$$|K(x)-K(x-y)|\leqq\frac{C|y|^\alpha}{|x|^{n+\alpha}}\qquad\text{for}\quad |y|<\frac{|x|}{2} \tag{c$'$}$$

(see, for example, Zygmund [11]). The idea of Coifman and Fefferman [2] applies also to the kernel satisfying the conditions (a), (b), and (c$'$) or more general

$$|K(x)-K(x-y)|\leqq\frac{C\theta(|y|/|x|)}{|x|^n}\qquad\text{for}\quad |y|<\frac{|x|}{2}, \tag{c$''$}$$

where θ is an increasing function such that $\theta(2t)\leqq C\theta(t)$ $(t>0)$ and

$$\int_0^1\frac{\theta(t)}{t}\,dt<\infty.$$

The purpose of this note is to give proofs of the distribution function inequalities and the weighted norm inequalities for singular integral with kernels satisfying the general conditions (a), (b) and (c$'$) or (c$''$). In Section 3 we shall prove the distribution function inequalities, and for this end it is convenient to state the weighted norm inequalities and distribution function inequalities for the Marcinkiewicz integrals, so we shall prove them in Section 2. In Section 4 we shall derive the weighted norm inequalities from the results in Section 3, and also give a converse result.

§ 2. Marcinkiewicz integrals.

Let P be a closed set in $\boldsymbol{R}^n$ and $\delta(x)=\delta_P(x)$ denote the distance of the point $x\in\boldsymbol{R}^n$ from P. Let α be any positive number and f be a measurable function on $\boldsymbol{R}^n$. We shall call the integral

$$Mf(x)=M_{P,\alpha}f(x)=\int_{\boldsymbol{R}^n}\frac{\delta^\alpha(y)f(y)}{|x-y|^{n+\alpha}+\delta^{n+\alpha}(y)}\,dy \tag{2.1}$$

to be the Marcinkiewicz integral of f with respect to P and α.

Zygmund [12] proved

THEOREM C. *If $f\in L^p(\boldsymbol{R}^n)$, $1\leqq p<\infty$, then*

$$\Big(\int_{\boldsymbol{R}^n}|Mf(x)|^p dx\Big)^{1/p}\leqq pC\Big(\int_{\boldsymbol{R}^n}|f(x)|^p dx\Big)^{1/p}; \tag{2.2}$$

and if f is bounded and supported in a set of finite measure E, then

$$\int_E \exp(c|Mf(x)|/\|f\|_\infty)dx \leqq C|E|. \tag{2.3}$$

The second named author [**10**] has complemented this result by proving:

THEOREM D. *If $\complement P$ is of finite measure, then, for any $f \in L^\infty(\boldsymbol{R}^n)$, Mf is of* BMO *and*

$$\|Mf\|_* = \sup_Q \frac{1}{|Q|}\int_Q |Mf(x)-(Mf)_Q|dx \leqq C\|f\|_\infty, \tag{2.4}$$

where $g_Q = |Q|^{-1}\int_Q g(x)dx$ and the supremum is taken over all cubes Q.

In this section we shall prove the weighted version of above results. To this end we state a lemma of which we shall make repeated use in sequel.

LEMMA 2. *Suppose that the least decreasing radial majorant of φ is integrable in $\boldsymbol{R}^n$, then*

$$\sup_{\delta>0}|f*\varphi_\delta(x)| \leqq Cf^*(x) \qquad \text{for} \quad f \in L^1_{\mathrm{loc}}(\boldsymbol{R}^n),$$

where $\varphi_\delta(x) = \delta^{-n}\varphi(x/\delta)$ and C is the integral of the majorant.

This is well known (for example, see [**9**; p. 62]).

THEOREM 1. *Suppose Mf is defined by* (2.1) *and the weight function ω satisfies the condition (A_p).*

1°. *If $1 \leqq p < \infty$, then*

$$\int_{\boldsymbol{R}^n}(Mf(x))^p\omega(x)dx \leqq C_p\int_{\boldsymbol{R}^n}|f(x)|^p\omega(x)dx; \tag{2.5}$$

2°. *If $\complement P$ is of finite measure and ω satisfies (A_∞), then*

$$\|f\|_{*,\omega} = \sup_Q \frac{1}{m_\omega(Q)}\int_Q |Mf(x)-(Mf)_{Q,\omega}|\omega(x)dx = C\|f\|_{\infty,\omega} \tag{2.6}$$

where $g_{Q,\omega} = \dfrac{1}{m_\omega(Q)}\int_Q g(y)\omega(y)dy$ and $\|f\|_{\infty,\omega}$ is the essential supremum of f with respect to the measure $\omega(x)dx$.

PROOF. 1°. Let $1<p<\infty$. Then as was pointed out by Folelli [**4**] and is easily verified, the dual space of $L^p(\omega dx)$ is isometrically anti-isomorphic to $L^{p'}(\omega^{-1/(p-1)}dx)$, $1/p+1/p'=1$, with the duality given by $\int_{\boldsymbol{R}^n} f(x)g(x)dx$, where $f \in L^p(\omega dx)$ and $g \in L^{p'}(\omega^{-1/(p-1)}dx)$, so that we know

$$\|f\|_{p,\omega} = \left(\int_{\boldsymbol{R}^n}|f(x)|^p\omega(x)dx\right)^{1/p} = \sup\left|\int_{\boldsymbol{R}^n} f(x)g(x)dx\right|, \tag{2.7}$$

where the supremum is taken over all $g \in L^{p'}(\omega^{-1/(p-1)}dx)$ with $\|g\|_{p',\omega^{-1/(p-1)}} \leqq 1$.

Now

$$\left|\int_{\boldsymbol{R}^n}(Mf(x))g(x)dx\right| \leqq \int_{\boldsymbol{R}^n}|f(y)|\left(\delta^\alpha(y)\int_{\boldsymbol{R}^n}\frac{|g(x)|dx}{|x-y|^{n+\alpha}+\delta^{n+\alpha}(y)}\right)dy.$$

Taking $\varphi(x)=1/(1+|x|^{n+\alpha})$, we can use Lemma 2 and obtain

$$\delta^{\alpha}(y)\int_{R^n}\frac{|g(x)|}{|x-y|^{n+\alpha}+\delta^{n+\alpha}(y)}dx\leqq Cg^*(y). \tag{2.8}$$

Thus (2.8) and the Hölder inequality give

$$\begin{aligned}\left|\int_{R^n}(Mf(x))g(x)dx\right| &\leqq C\int_{R^n}|f(y)|g^*(y)dy\\ &\leqq C\|f\|_{p,\omega}\|g^*\|_{p',\omega^{-1/(p-1)}}.\end{aligned}$$

The condition (A_p) for ω implies the $(A_{p'})$ for $\omega^{-1/(p-1)}$, so that Theorem A yields

$$\|g^*\|_{p',\omega^{-1/(p-1)}}\leqq C_p\|g\|_{p',\omega^{-1/(p-1)}}.$$

Altogether, we obtain the inequality (2.5) for $1<p<\infty$.

In case $p=1$, the proof is more simple: reasoning as above, we obtain

$$\begin{aligned}\int_{R^n}|Mf(x)|\omega(x)dx &\leqq \int_{R^n}|f(y)|\left(\delta^{\delta}(y)\int_{R^n}\frac{\omega(x)dx}{|x-y|^{n+\alpha}+\delta^{n+\alpha}(y)}\right)dy\\ &\leqq C\int_{R^n}|f(y)|\omega^*(y)dy\leqq C\int_{R^n}|f(y)|\omega(y)dy,\end{aligned}$$

which proves (2.5) for $p=1$.

2°. Let Q be a cube, and Q^* be its double. We estimate Mf in Q writing $Mf=Mf_1+Mf_2$, where $f=f_1+f_2$ and $f_1=f\chi_{Q^*}$ and $f_2=f\chi_{\complement Q^*}$. It can be easily proved that

$$\frac{1}{m_\omega(Q)}\int_Q|Mf_1(x)|\omega(x)dx\leqq C\|f_1\|_{\infty,\omega}$$

(see the proof of Theorem 5, part 3°). On the other hand, it is known [10] that $|Mf_2(x)-a_Q|\leqq C\|f\|_{\infty,\omega}$, where

$$a_Q=\int_{\complement Q^*}\frac{f(y)\delta^{\alpha}(y)}{|x_0-y|^{n+\alpha}+\delta^{n+\alpha}(y)}dy$$

and x_0 is the center of Q, so that we have

$$\frac{1}{m_\omega(Q)}\int_Q|Mf_2(x)-a_Q|\omega(x)dx\leqq C\|f\|_{\infty,\omega};$$

altogether we obtain (2.6), and the proof is completed.

For the later use, we state a lemma concerning the distribution function inequality for Marcinkiewicz integrals.

LEMMA 3. *Let P, α and the distance function δ be the same as in the beginning of this section and χ be the characteristic function of the set $\complement P$, and ω satisfy (A_∞), then for any cube Q with $|Q|\geqq a|\complement P|$, we have*

$$m_{\omega}(\{x\in Q: M\chi(x)>\lambda\})\leqq C(a)e^{-c\lambda}m_{\omega}(Q) \tag{2.9}$$

for $\lambda>0$.

PROOF. This is an easy consequence of Theorem 1, (2.6), but we shall give a simple proof based on a result of Zygmund of Theorem C.

For any $\eta>0$, if follows from (2.2) that

$$\int_{Q}\exp(\eta M\chi(x))dx\leqq|Q|+\sum_{p=1}^{\infty}\frac{\eta^{p}}{p!}\int_{Q}(M\chi(x))^{p}dx$$

$$\leqq|Q|+\sum_{p=1}^{\infty}\frac{\eta^{p}}{p!}C^{p}p^{p}\int_{R^{n}}\chi(x)dx\leqq|Q|+\sum_{p=1}^{\infty}\frac{(\eta Cp)^{p}}{p!}|\complement P|$$

$$\leqq\left\{1+\frac{1}{a}\sum_{p=1}^{\infty}\frac{(\eta Cp)^{p}}{p!}\right\}|Q|$$

if $|Q|\geqq a|\complement P|$. Since the last series converges for $\eta=\frac{1}{2Ce}$, we obtain

$$\int_{Q}\exp(cM\chi)dx\leqq C(a)|Q|,$$

and (2.9) follows easily.

§3. Distribution function inequalities for singular integrals.

THEOREM 2. *Let $T: f\rightarrow K*f$ be a singular integral operator in $\boldsymbol{R}^{n}$ with a convolution kernel K satisfying the conditions* (a), (b) *and* (c′) *in the introduction, and the maximal operator T^{*} be defined by* (1.5). *Suppose that the weight function ω satisfies (A_{∞}) and $a>0$ is given. Then there exist positive constants C, c, $C(a)$ such that, for any β, $\gamma>0$ with $\beta>C\gamma$, the inequality*

$$m_{\omega}(\{x\in Q: T^{*}f(x)>\beta\lambda,\ f^{*}(x)\leqq\gamma\lambda\})\leqq C(a)e^{-\frac{c\beta}{\gamma}}m_{\omega}(Q) \tag{3.1}$$

is valid for any $f\in L^{1}(\boldsymbol{R}^{n})$ and for any cube Q with

$$|Q|\geqq\frac{a}{\gamma\lambda}\|f\|_{1}=\frac{a}{\gamma\lambda}\int_{R^{n}}|f(x)|dx. \tag{3.2}$$

PROOF. The set $P=P_{\lambda}=\{x\in\boldsymbol{R}^{n};\ f^{*}(x)\leqq\gamma\lambda\}$ is closed. For the pair f, $\gamma\lambda$ and the open set $\Omega_{\lambda}=\complement P_{\lambda}$, combining the Calderón-Zygmund lemma and Whitney lemma ([**9**; p. 16]), we get the decomposition of the set Ω_{λ} into the union of non-overlapping cubes $\{Q_{j}\}$ in such a way that

(i) $\quad|\Omega_{\lambda}|\leqq\frac{C}{\gamma\lambda}\|f\|_{1}$,

(ii) $\quad\frac{1}{|Q_{j}|}\int_{Q_{j}}|f(y)|dy\leqq C\gamma\lambda$,

(iii) $|f(x)|\leqq\gamma\lambda$ a. e. in P

and

(iv) $2\operatorname{diam} Q_j \leqq \operatorname{dist}(P, Q_j) \leqq 9 \operatorname{diam} Q_j$.

And also we decompose f into the sum of the good part g and the bad part b, where

$$g(x)=\begin{cases} f(x) & \text{for } x\in P, \\ \dfrac{1}{|Q_j|}\displaystyle\int_{Q_j} f(y)dy & \text{for } x\in Q_j,\ j=1, 2, \cdots, \end{cases} \tag{3.3}$$

then

(v) $|g(x)|\leqq C\gamma\lambda$ a. e. and $\|g\|_1\leqq\|f\|_1$,

(vi) $b(x)=0$ in P, and $\int_{Q_j} b(y)dy=0$ and $\|b\|_1\leqq 2\|f\|_1$.

Now, since $T^*f\leqq T^*g+T^*b$, it is sufficient to prove

$$m_\omega\left(\left\{x\in Q : T^*g(x)>\frac{\beta\lambda}{2}\right\}\right)\leqq C(a)e^{-\frac{c\beta}{\gamma}}m_\omega(Q) \tag{3.4}$$

and

$$m_\omega\left(\left\{x\in Q\cap P_\lambda : T^*b(x)>\frac{\beta\lambda}{2}\right\}\right)\leqq C(a)e^{-\frac{c\beta}{\gamma}}m_\omega(Q) \tag{3.5}$$

for any cube Q with (3.2).

To prove (3.4), let Q^* be the double of Q and write $g=g_1+g_2$, where $g_1=g\chi_{Q^*}$ and $g_2=g\chi_{\complement Q^*}$. Then $|g_1(x)|\leqq|g(x)|\leqq C\gamma\lambda$ a. e. by (ii), (iii) and (3.3), and g_1 is supported in the cube Q^*. By a well known fact that

$$\|T^*g_1\|_p\leqq pC\|g_1\|_p \qquad \text{for} \quad p\geqq 2$$

(see Stein [**9**; p. 48]), we obtain for $\eta>0$

$$\begin{aligned}\int_{Q^*}\exp(\eta T^*g_1)dx &\leqq 2\int_{Q^*}\cosh(\eta T^*g_1)dx \\ &= 2\left\{|Q^*|+\sum_{p=1}^{\infty}\frac{\eta^{2p}}{(2p)!}\int_{Q^*}(T^*g_1)^{2p}dx\right\} \\ &\leqq 2\left\{|Q^*|+\sum_{p=1}^{\infty}\frac{(C\eta\gamma\lambda)^{2p}(2p)^{2p}}{(2p)!}|Q^*|\right\}.\end{aligned}$$

Since the last series converges for $C\eta\gamma\lambda e=1/2$, it follows that for this value of η

$$\int_Q \exp(cT^*g_1/\gamma\lambda)dx\leqq C|Q|. \tag{3.6}$$

Next, observing that $|x-y|\geqq c|Q|^{1/n}$ for $x\in Q$ and $y\in\complement Q^*$, the Hölder inequality and (v) yield

$$\left|\int_{Q'_x \setminus Q_x} K(x-y)g_2(y)dy\right| \leqq C\int_{|y-x|\geqq c|Q|^{1/n}} |K(x-y)|\,|g(y)|dy$$

$$\leqq C\left\{\int_{|y-x|\geqq c|Q|^{1/n}} \frac{dy}{|x-y|^{2n}}\right\}^{1/2} \|g\|_2 \leqq C|Q|^{-1/2}\|g\|_2$$

$$\leqq C|Q|^{-1/2}(\gamma\lambda\|f\|_1)^{1/2} = C\left(\frac{\gamma\lambda\|f\|_1}{|Q|}\right)^{1/2} \qquad \text{for} \quad x\in Q\,,$$

so that, if $|Q| \geqq \frac{a}{\gamma\lambda}\|f\|_1$, we obtain

$$T^*g_2(x) \leqq C(a)\gamma\lambda \qquad \text{for} \quad x\in Q\,. \tag{3.7}$$

Altogether by (3.6) and (3.7)

$$\int_Q \exp\left(\frac{c}{\gamma\lambda}T^*g\right)dx \leqq \int_Q \exp\left(\frac{c}{\gamma\lambda}T^*g_1\right)\cdot \exp\left(\frac{c}{\gamma\lambda}T^*g_2\right)dx$$

$$\leqq C(a)|Q|,$$

so that

$$\left|\left\{x\in Q : T^*g(x) > \frac{\beta\lambda}{2}\right\}\right| \leqq C(a)e^{-\frac{c\beta}{\gamma}}|Q|\,.$$

Then the estimate (3.4) follows from the condition (A_∞) for ω.

To prove (3.5), we observe that for $x\in P_\lambda$

$$T^*b(x) \leqq \sum_j \int_{Q_j} |K(x-y)-K(x-y_j)|\,|b(y)|dy + Cb^*(x) \tag{3.8}$$

where y_j is the center of Q_j; this fact is proved in [**9**; pp. 43-44] for a modified set P_λ^* of P_λ. However, the same proof works for P_λ itself if we use the Whitney decomposition of Ω_λ which satisfies the condition (iv).

Now we know that, for $y\in Q_j$ and $x\in P_\lambda$, $2|y-y_j| \leqq |x-y|$ and $|x-y|$ is comparable to $|x-y_j|$, so that the condition (c′) yields

$$\int_{Q_j} |K(x-y)-K(x-y_j)|\,|b(y)|dy \leqq \frac{C(\operatorname{diam} Q_j)^\alpha}{|x-y_j|^{n+\alpha}}\int_{Q_j}|b(y)|dy\,.$$

Since $\int_{Q_j}|b(y)|dy \leqq \int_{Q_j}|f(y)|dy + C\gamma\lambda\int_{Q_j}dy$, $\int_{Q_j}|b(y)|dy \leqq C\gamma\lambda|Q_j|$. From this, observing that $|x-y|^{n+\alpha}$ is comparable to $|x-y|^{n+\alpha}+\delta^{n+\alpha}(y)$ for $x\in P_\lambda$ and for all y, we obtain

$$\int_Q |K(x-y)-K(x-y_j)|\,|b(y)|dy \leqq C\gamma\lambda\int_{Q_j}\frac{\delta^\alpha(y)}{|x-y|^{n+\alpha}+\delta^{n+\alpha}(y)}dy \quad \text{for } x\in P_\lambda\,.$$

Finally, by (3.8),

$$T^*b(x) \leqq C\gamma\lambda \int_{R^n} \frac{\delta^\alpha(y)}{|x-y|^{n+\alpha}+\delta^{n+\alpha}(y)}\, dy + C\gamma\lambda$$
$$\leqq C\gamma\lambda\{M\chi(x)+1\} \qquad \text{for} \quad x \in P_\lambda ,$$

where χ is the characteristic function of the set $\complement P_\lambda$ and $M\chi$ denotes the Marcinkiewicz integral of χ with respect to P_λ and α. Hence, if $|Q| \geqq \frac{a}{\gamma\lambda}\|f\|_1$, then $|Q| \geqq \frac{a}{C}|\complement P|$ by (ii), so that we obtain by Lemma 3 that

$$\left|\left\{x \in Q \cap P_\lambda : T^*b(x) > \frac{\beta\lambda}{2}\right\}\right| = \left|\left\{x \in Q \cap P_\lambda : C\gamma\lambda(M\chi+1) > \frac{\beta\lambda}{2}\right\}\right|$$
$$\leqq \left|\left\{x \in Q : M\chi(x) > \frac{\beta}{2C\gamma} - 1\right\}\right| = C(a)e^{-\frac{c\beta}{\gamma}}\,|Q| ,$$

from this inequality, the estimate (3.5) follows from the (A_∞) property of ω.

REMARK. The estimate (3.1) is the best possible one in the sense that the restriction (3.2) cannot be omitted and the number β/γ in the exponent of the right hand side can not be replaced by a larger one; this fact can easily be seen in the case $n=1$ from the example in which $f=\chi_{[-1,1]}$ and $T^*f(x)= \left|\log \frac{|x+1|}{|x-1|}\right|$.

From Theorem 2, we can deduce the following "distribution function inequality" for the maximal singular integral.

THEOREM 3. *Suppose that T^* is a maximal singular integral operator with a convolution kernel K satisfying the conditions* (a), (b) *and* (c′) *in the introduction and that the weight function ω satisfies the condition* (A_∞). *Then for any $f \in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$ and β, $\gamma>0$ with $\beta>1$, $\beta-1>C\gamma$ and for any $\lambda>0$, we obtain*

(3.9)
$$m_\omega(\{x \in \boldsymbol{R}^n : T^*f(x) > \beta\lambda,\ f^*(x) \leqq \gamma\lambda\})$$
$$\leqq Ce^{-\frac{c(\beta-1)}{\gamma}} m_\omega(\{x \in \boldsymbol{R}^n : T^*f(x) > \lambda\}) .$$

PROOF. The proof is accomplished by refining slightly the idea of Coifman [**1**] and Coifman and Fefferman [**2**]. The open set $\Omega_\lambda = \{x \in \boldsymbol{R}^n : T^*f(x) > \lambda\}$ breaks up as a union of nonoverlapping cubes $\{Q_j\}$ satisfying the condition

$$2 \operatorname{diam} Q_j \leqq \operatorname{dist}(\complement \Omega_\lambda, Q_j) \leqq 9 \operatorname{diam} Q_j .$$

Then there exist points $x_j \in \complement \Omega_\lambda$ such that $\operatorname{dist}(x_j, Q_j) \leqq 9 \operatorname{diam} Q_j$. Let $\bar{Q}_j$ be the cube centered at x_j with diameter $21\sqrt{n}$ times as large as that of Q_j. Then $\bar{Q}_j \supset Q_j^*$, where Q_j^* is the cube concentric with Q_j whose diameter is $2\sqrt{n}$ times as large as that of Q_j.

To prove (3.9) we may assume that $f^*(\xi_j) \leqq \gamma\lambda$ for at least one point $\xi_j \in Q_j$.

We write $f=f_1+f_2$, where $f_1=f\chi_{\bar{Q}_j}$ and $f_2=f\chi_{\complement\bar{Q}_j}$. Since $\xi_j\in Q_j\subset\bar{Q}_j$, it follows that

$$\frac{1}{|\bar{Q}_j|}\int_{\bar{Q}_j}|f_1(y)|dy=\frac{1}{|\bar{Q}_j|}\int_{\bar{Q}_j}|f(y)|dy\leqq f^*(\xi_j)\leqq\gamma\lambda, \tag{3.10}$$

so that we have $|Q_j|\geqq\frac{c}{\gamma\lambda}\|f_1\|_1$. Thus, Theorem 2 yields

$$\begin{aligned}&m_\omega(\{x\in Q_j: T^*f_1>\beta'\lambda,\ f^*\leqq\gamma\lambda\})\\ &\qquad\leqq m_\omega(\{x\in Q_j: T^*f_1>\beta'\lambda,\ f_1^*\leqq\gamma\lambda\})\leqq Ce^{-\frac{c\beta'}{\gamma}}m_\omega(Q_j),\end{aligned}\tag{3.11}$$

where β' will be determined soon after.

Next we shall prove that

$$T^*f_2(x)\leqq(C\gamma+1)\lambda \qquad \text{for}\quad x\in Q_j. \tag{3.12}$$

We fix cubes Q'_x and Q_x centered at x with $Q'_x\supset Q_x$, and let Q'_{x_j} and Q_{x_j} be the cubes of same size as Q'_x and Q_x, respectively, centered at x_j. Then we write

$$\begin{aligned}\left|\int_{Q'_x\setminus Q_x}K(x-y)f_2(y)dy\right|&\leqq\left|\int_{Q'_{x_j}\setminus Q_{x_j}}K(x-y)f_2(y)dy\right|\\ &\quad+\int_{(Q'_{x_j}\setminus Q_{x_j})\Delta(Q'_x\setminus Q_x)}|K(x-y)||f_2(y)|dy\end{aligned}$$

(where Δ denotes the symmetric difference)

$$\begin{aligned}&\leqq\left|\int_{Q'_{x_j}\setminus Q_{x_j}}K(x_j-y)f_2(y)dy\right|+\int_{Q'_{x_j}\setminus Q_{x_j}}|K(x_j-y)-K(x-y)||f_2(y)|dy\\ &\qquad+\int_{(Q'_{x_j}\setminus Q_{x_j})\Delta(Q'_x\setminus Q_x)}|K(x-y)||f_2(y)|dy\\ &=A_1+A_2+A_3.\end{aligned}$$

Now

$$A_1=\left|\int_{(Q'_{x_j}\setminus Q_{x_j})\cap\complement\bar{Q}_j}K(x_j-y)f(y)dy\right|$$

and the set $(Q'_{x_j}\setminus Q_{x_j})\cap\complement\bar{Q}_j$ is empty, or equal to $Q'_{x_j}\setminus\bar{Q}_j$, or $Q'_{x_j}\setminus Q_{x_j}$ according to $\bar{Q}_j\supset Q'_{x_j}$, or $Q_{x_j}\subset\bar{Q}_j\subset Q'_{x_j}$, or $\bar{Q}_j\subset Q_{x_j}$, so that $A_1\leqq T^*f(x_j)\leqq\lambda$, since $x_j\notin\Omega_\lambda$.

Next $A_2\leqq\int_{\complement\bar{Q}_j}|K(x_j-y)-K(x-y)||f(y)|dy$ and the construction of $\bar{Q}_j$ shows that $|x_j-x|<\frac{|x_j-y|}{2}$ for $x\in Q_j$ and $y\in\complement\bar{Q}_j$ and also $|x_j-y|$ is comparable to $|\xi-y|$ for any $\xi\in Q_j$ and $y\in\complement\bar{Q}_j$, so that the condition (c′) and the same reasoning used to obtain (2.6) in the proof of Theorem 2 gives that, for $x\in Q_j$,

$$(3.13)\qquad A_2 \leqq C|x_j-x|^{\alpha}\int_{\complement Q_j}\frac{|f(y)|}{|x_j-y|^{n+\alpha}}dy$$
$$\leqq C\,d_j^{\alpha}\int_{|y-\xi|\geqq cd_j}\frac{|f(y)|}{|\xi-y|^{n+\alpha}}dy\leqq Cf^*(\xi),\qquad \text{for any } \xi\in Q_j\,,$$

where $d_j=\operatorname{diam} Q_j$. In particular, since $f^*(\xi_j)\leqq\gamma\lambda$, we know that $A_2\leqq C\gamma\lambda$.

Finally, a simple geometric observation shows that A_3 is majorized by a sum of at most four integrals of the form

$$C\int_{\rho\leqq|y|\leqq C\rho}\frac{|f(\xi-y)|}{|y|^n}dy\qquad \text{for any}\quad \xi\in Q_j\,,$$

where $\rho>0$, and this integral majorized by $Cf^*(\xi)$ in virtue of Lemma 2, so that $A_3\leqq C\gamma\lambda$.

Altogether we obtain

$$\left|\int_{Q'_x\setminus Q_x}K(x-y)f_2(y)dy\right|\leqq(C\gamma+1)\lambda\,.$$

Taking supremum for Q'_x and Q_x, we have proved the estimate (3.12).

Taking $\beta'=\beta-1-C\gamma$ in (3.11) and then combining (3.11) and (3.12) we have by Theorem 2 that

$$m_\omega(\{x\in Q_j: T^*f>\beta\lambda,\ f^*\leqq\gamma\lambda\})$$
$$\leqq m_\omega(\{x\in Q_j: T^*f_1+T^*f_2>\beta'\lambda+(1+C\gamma)\lambda,\ f^*\leqq\gamma\lambda\})$$
$$\leqq m_\omega(\{x\in Q_j: T^*f_1>\beta'\lambda,\ f^*\leqq\gamma\lambda\})\leqq C\,e^{-\frac{c(\beta-1)}{\gamma}}m_\omega(Q_j)\,.$$

Adding in j yields the inequality (3.9) and the proof of Theorem 3 is completed.

THEOREM 4. *Suppose that T^* be a maximal singular integral operator with a convolution kernel K satisfying the conditions* (a), (b) *and* (c$''$) *in the introduction and the weight function ω satisfies the condition* (A_∞) *and positive numbers β, γ with $\beta-1>C\gamma$ are given, then for any $f\in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$ and any $\lambda>0$*

$$(3.14)\qquad m_\omega(\{x\in\boldsymbol{R}^n: T^*f(x)>\beta\lambda,\ f^*(x)\leqq\gamma\lambda\})$$
$$\leqq C\Big(\frac{\gamma}{\beta-1-C\gamma}\Big)^{\delta}m_\omega(\{x\in\boldsymbol{R}^n: T^*f(x)>\lambda\})\,.$$

PROOF. The proof of Theorem 3 applies equally up to the inequailty (3.11). Then the weak type inequality

$$|\{x\in\boldsymbol{R}^n: T^*f(x)>\lambda\}|\leqq\frac{C}{\lambda}\int_{\boldsymbol{R}^n}|f(x)|dx$$

yields

$$(3.15)\qquad |\{x\in\boldsymbol{R}^n: T^*f_1(x)>\beta'\lambda\}|\leqq\frac{C}{\beta'\lambda}\int_{\boldsymbol{R}^n}|f_1(x)|dx\leqq\frac{C\gamma}{\beta'}\,.$$

Next we prove (3.12) in this case, and its proof reduces to estimate A_1, A_2 and A_3 as in the proof of Theorem 3. The estimates for A_1 and A_3 are not changed because they based on the fact that $f^*(\xi_i)\leqq\gamma\lambda$ and on the condition (b) only. Now the reasoning leading to (3.13) shows

$$\begin{aligned}A_2&\leqq C\int_{\complement\bar{Q}_j}\frac{1}{|x_j-y|^n}\theta\Big(\frac{|x_j-x|}{|x_j-y|}\Big)|f(y)|dy\\&\leqq C\int_{\complement\bar{Q}_j}\frac{1}{|\xi_j-y|^n}\theta\Big(\frac{Cd_j}{|\xi_j-y|^n}\Big)|f(y)|dy\\&\leqq C\int_{|y|\geqq cd_j}\frac{1}{|y|^n}\theta\Big(\frac{Cd_j}{|y|}\Big)|f(\xi_j-y)|dy\leqq Cf^*(\xi_j)\leqq C\gamma\lambda\end{aligned}$$

for $x\in Q_j$, because $\varphi(x)=\theta(1)$ for $|x|\leqq 1$ and $\varphi(x)=\theta(1/|x|)/|x|^n$ for $|x|>1$ satisfies the condition of Lemma 2 if θ satisfies the condition (c″). Altogether we obtain

$$T^*f_2(x)\leqq\lambda+C\gamma\lambda\qquad\text{for}\quad x\in Q_j. \tag{3.12$_{\text{bis}}$}$$

Combining (3.15) and (3.12$_{\text{bis}}$), the same argument as in the proof of Theorem 2 yields

$$|\{x\in Q_j: T^*f(x)>\beta\lambda\}|\leqq\frac{C\gamma}{\beta-1-C\gamma}|Q_j|,$$

which implies (3.14) by the condition (A_∞) for ω, and the proof of Theorem 4 is completed.

We remark that if $K(x)=\Omega(x)/|x|^n$ is any Calderón-Zygmund kernel, i.e. where Ω is homogeneous of degree zero, satisfies a Dini condition and mean-value zero on the unit sphere, then K satisfies the conditions (a), (b) and (c″) and Theorem 4 applies to this kernel.

§4. Weighted norm inequalities for singular integrals.

In this section we shall prove the weighted norm inequalities for singular integral operators. Moreover, we prove the weighted norm inequalities of weak type in $L^1(\omega(x)dx)$ with ω satisfying the condition (A_1) as well as the BMO properties for singular integrals in $L^\infty(\omega(x)dx)$ with ω satisfying (A_∞).

For this end, we define the modified singular integral $\tilde{T}f$:

$$\tilde{T}f(x)=\lim_{|Q|\to 0}\int_{\boldsymbol{R}^n}[K_Q(x-y)-K_{Q_0}(-y)]f(y)dy \tag{4.1}$$

for $f\in L^\infty(\omega(x)dx)$, where $K_Q=K\cdot\chi_{\complement Q}$ for any cube Q centered at the origin and Q_0 is the unit cube.

Besides, we need some more lemmas.

LEMMA 4. *Let $1<p<\infty$ and ω satisfy the condition (A_p), then*

$$\int_{R^n} \frac{\omega(x)}{1+|x|^{np}} dx < \infty .$$

This is due to Hunt, Muckenhoupt and Wheeden [5; Lemma 1] for the case $n=1$; the proof in case $n>1$ is similar.

LEMMA 5. *Any weight function satisfying (A_∞) already satisfies (A_p) for some $p<\infty$.*

This is due to Muckenhoupt [8]; see also Coifman and Fefferman [2].

LEMMA 6. *If ω satisfies (A_p), $1\leqq p<\infty$, then*

$$\frac{m_\omega(E)}{m_\omega(Q)} \geqq c\left(\frac{|E|}{|Q|}\right)^p$$

for any cube Q and any measurable subset $E\subset Q$.

This is due to Muckenhoupt [8], Gundy and Wheeden [4]. A more general result can be found in Coifman and Fefferman [2].

THEOREM 5. *Suppope that $1\leqq p\leqq\infty$, the weight function ω satisfies the condition (A_p) and the convolution kernel K satisfies the conditions* (a), (b) *and* (c″) *in the introduction.*

1°. *If $1<p<\infty$, then*

$$\int_{R^n}(T^*f(x))^p\omega(x)dx \leqq C_p\int_{R^n}|f(x)|^p\omega(x)dx\,; \tag{4.2}$$

2°. *If $p=1$, then for any $\lambda>0$*

$$m_\omega(\{x\in \boldsymbol{R}^n : T^*f(x)>\lambda\}) \leqq \frac{C}{\lambda}\int_{R^n}|f(x)|\omega(x)dx\,; \tag{4.3}$$

3°. *If $p=\infty$, then for any cube Q*

$$\frac{1}{m_\omega(Q)}\int_Q|\tilde{T}f(x)-(\tilde{T}f)_{Q,\omega}|\omega(x)dx \leqq C\|f\|_{\infty,\omega}\,, \tag{4.4}$$

where $(g)_{Q,\omega}=\dfrac{1}{m_\omega(Q)}\displaystyle\int g(x)\omega(x)dx.$

PEOOF. 1°. Let $1<p<\infty$, then the distribution function inequality (3.14) of Theorem 4 with $\beta=2$ yields that

$$\begin{aligned}\int_{R^n}(T^*f)^p\omega dx &\leqq C\int_0^\infty \lambda^{p-1}m_\omega(\{T^*f>2\lambda\})d\lambda\\ &\leqq C\int_0^\infty\lambda^{p-1}m_\omega(\{f^*>\gamma\lambda\})d\lambda+C\gamma^\delta\int_0^\infty\lambda^{p-1}m_\omega(\{T^*f>\lambda\})d\lambda\\ &=C(\gamma)\int_{R^n}(f^*)^p\omega dx+C\gamma^\delta\int_{R^n}(T^*f)^p\omega dx\,.\end{aligned}$$

Taking γ so small that $C\gamma^\delta\leqq 1/2$ and making use of Theorem A, we obtain

$$2\int_{R^n}(T^*f)^p\omega dx \leqq C\int_{R^n}|f|^p\omega dx + \int_{R^n}(T^*f)^p\omega dx.$$

Now, if f is infinitely differentiable and of compact support, then it is easily seen that T^*f is bounded and $O(1/|x|^n)$ as $|x|\to\infty$, so that we know $\int_{R^n}(T^*f)^p\omega dx<\infty$ by Lemma 4, and we obtain (4.2). Since such functions are dense in $L^p(\omega dx)$, standard arguments show that (4.2) holds for $f\in L^p(\omega dx)$.

2°. Next, let $f\in L^1(\omega dx)$ and $P_\lambda=\{x\in R^n: f^*(x)\leqq\lambda\}$. Then by Theorem A, denoting $\Omega_\lambda=\complement P_\lambda$,

$$m_\omega(\Omega_\lambda)\leqq\frac{C}{\lambda}\int_{R^n}|f(x)|\omega(x)dx. \tag{4.5}$$

Again, we decompose the open set Ω_λ into a non-overlapping union of cubes $\{Q_j\}$ satisfying the conditions (i)~(iv) listed in the proof of Theorem 2 with $\gamma=1$, and also we write $f=g+b$ as there.

Since $|g(x)|\leqq C\lambda$ a.e., $\int_{R^n}|g|^2\omega dx\leqq C\lambda\int_{R^n}|f|\omega dx$, the part 1° of this theorem shows that

$$\int_{R^n}(T^*g)^2\omega dx\leqq C\int_{R^n}|g|^2\omega dx\leqq C\lambda\int_{R^n}|f|\omega dx.$$

Since (A_1) implies (A_p) for any $p>1$, it follows that

$$m_\omega\Big(\Big\{T^*g>\frac{\lambda}{2}\Big\}\Big)\leqq\frac{C}{\lambda}\int_{R^n}|f|\omega dx. \tag{4.6}$$

On the other hand, an analogous argument which leads to (3.8) in the proof of Theorem 2 shows that

$$T^*b(x)\leqq\sum_j\int_{Q_j}|x-y|^{-n}\theta\Big(\frac{|y-\xi|}{|x-y|}\Big)|b(y)|dy+C'b^*(x) \tag{4.7}$$

for $x\in P_\lambda$ and for any $\xi\in Q_j$. Let us denote the series on the right hand side of (4.7) by $\Sigma(x)$ and integrate it over the set P_λ, then we know by the condition (c″) for K and the condition (A_1) for ω that

$$\begin{aligned}\int_{P_\lambda}\Sigma(x)\omega(x)dx &\leqq C\sum_j\int_{Q_j}|b(y)|\Big\{\int_{\complement Q_j}|x-y|^n\theta\Big(\frac{|y-\xi|}{|x-y|}\Big)\omega(x)dx\Big\}dy \\ &\leqq C\sum_j\int_{Q_j}|b(y)|\int_{|x|>cd_j}|x|^{-n}\theta\Big(\frac{d_j}{|x|}\Big)\omega(y-x)dx\Big\}dy \\ &\qquad (d_j=\operatorname{diam}Q_j) \\ &\leqq C\sum_j\int_{Q_j}|b(y)|\omega^*(y)dy\leqq C\int_{R^n}|b(y)|\omega(y)dy.\end{aligned} \tag{4.8}$$

The condition (A_1) implies that, for any cube Q,

$$\frac{1}{|Q|}\int_Q \omega(y)dy \leqq C \operatorname*{ess\,inf}_{x\in Q} \omega(x). \tag{4.9}$$

Now, for $x\in Q_j$, we have

$$g(x)=\frac{1}{|Q_j|}\int_{Q_j}|f(y)|dy \leqq \frac{1}{|Q_j|\operatorname*{ess\,inf}_{x\in Q_j}\omega(x)}\int_{Q_j}|f(y)|\omega(y)dy,$$

consequently, by (4.9),

$$\int_{Q_j}|g(x)|\omega(x)dx \leqq \frac{m_\omega(Q_j)}{|Q_j|\operatorname{ess\,inf}\omega(x)}\int_{Q_j}|f(y)|\omega(y)dy$$

$$\leqq C\int_{Q_j}|f(y)|\omega(y)dy,$$

and this implies

$$\int_{R^n}|b(y)|\omega(y)dy \leqq C\int_{R^n}|f(y)|\omega(y)dy.$$

Altogether by (4.8) and the last inequality, we obtain

$$\int_{P_\lambda}\Sigma(x)\omega(x)dx \leqq C\int_{R^n}|f(x)|\omega(x)dx,$$

and it follows that

$$m_\omega(\{x\in P_\lambda : \Sigma(x)>\lambda\}) \leqq \frac{C}{\lambda}\int_{R^n}|f(x)|\omega(x)dx. \tag{4.10}$$

Since $b^*(x)\leqq C''\lambda$ for $x\in P_\lambda$, (4.7), (4.10) show that

$$m_\omega(\{x\in P_\lambda : T^*b(x)>(1+C'C'')\lambda\})$$

$$\leqq m_\omega(\{x\in P_\lambda : \Sigma(x)+C'b^*(x)>\lambda+C'C''\lambda\})$$

$$\leqq m_\omega(\{x\in P_\lambda : \Sigma(x)>\lambda\}) \leqq \frac{C}{\lambda}\int_{R^n}|f(x)|\omega(x)dx,$$

and this implies

$$m_\omega\Big(\Big\{x\in P_\lambda : T^*b(x)>\frac{\lambda}{2}\Big\}\Big) \leqq \frac{C}{\lambda}\int_{R^n}|f(x)|\omega(x)dx. \tag{4.11}$$

Combining (4.5), (4.6) and (4.11) yields

$$m_\omega(\{x\in \boldsymbol{R}^n : T^*b(x)>\lambda\})$$

$$\leqq m_\omega\Big(\Big\{x\in \boldsymbol{R}^n : T^*g(x)>\frac{\lambda}{2}\Big\}\Big)+m_\omega\Big(\Big\{x\in P_\lambda : T^*b(x)>\frac{\lambda}{2}\Big\}\Big)+m_\omega(\Omega_\lambda)$$

$$\leqq \frac{C}{\lambda}\int_{R^n}|f(x)|\omega(x)dx,$$

and we have proved (4.3).

3°. Finally we shall prove (4.4). Let ω satsfies the condition (A_∞) and $f\in L^\infty(\omega(x)dx)$. Since by Lemmas 5 and 6, the measures $\omega(x)dx$ and dx are absolutely continuous with respect to each other, $L^\infty(\omega(x)dx)=L^\infty(\boldsymbol{R}^n)$, so that $\tilde{T}f$ is well defined. Fix any cube Q and let x_0 be its center and $\tilde{Q}$ be the cube centered at x_0, with diameter $2\sqrt{n}$ times as large as that of Q. We write $f=g+h$, where $g=f\chi_{\tilde{Q}}$, $h=f\chi_{\complement\tilde{Q}}$.

Then

$$
\begin{aligned}
\tilde{T}f(x)&=Tg(x)+\int_{\boldsymbol{R}^n}[K(x-y)-K(x_0-y)]h(y)dy\\
&\quad+\Big\{\int_{\boldsymbol{R}^n}[K(x_0-y)-K_{Q_0}(-y)]h(y)dy-\int_{\boldsymbol{R}^n}K_{Q_0}(-y)g(y)dy\Big\} \qquad (4.12)\\
&=Tg(x)+I(x)+C_Q\,.
\end{aligned}
$$

Since, by Lemma 5, ω satisfies (A_p) for some $1<p<\infty$, and $g\in L^p(\omega(x)dx)$ for this p and $\int_{\boldsymbol{R}^n}|g|^p dx\leqq m_\omega(Q)\|f\|_{\infty,\omega}^p$, we obtain by (4.2) and Lemma 6 that

$$
\begin{aligned}
&\frac{1}{m_\omega(Q)}\int_Q|Tg|\omega dx\leqq\Big\{\frac{1}{m_\omega(Q)}\int_Q|Tg|^p\omega dx\Big\}^{1/p} \qquad (4.13)\\
&\leqq\frac{C}{m_\omega(Q)^{1/p}}\Big\{\int_{\boldsymbol{R}^n}|g|^p\omega dx\Big\}^{1/p}\leqq C\Big\{\frac{m_\omega(\tilde{Q})}{m_\omega(Q)}\Big\}^{1/p}\|f\|_{\infty,\omega}\leqq C\|f\|_{\infty,\omega}\,.
\end{aligned}
$$

On the other hand, the condition (c″) yields

$$
\begin{aligned}
|I(x)|&\leqq\|h\|_\infty\int_{|y-x_0|>2|x-x_0|}|K(x_0-y+x-x_0)-K(x_0-y)|\,dy\\
&\leqq C\|h\|_\infty\leqq C\|f\|_{\infty,\omega}\,,
\end{aligned}
$$

so that

$$
\frac{1}{m_\omega(Q)}\int_Q|I(x)|\,\omega(x)dx\leqq C\|f\|_{\infty,\omega}\,. \qquad (4.14)
$$

From (4.12), (4.13) and (4.14), we obtain

$$
\frac{1}{m_\omega(Q)}\int_Q|\tilde{T}f(x)-C_Q|\,\omega(x)dx\leqq C\|f\|_{\infty,\omega}\,,
$$

and this implies (4.4). The proof of Theorem 5 is completed.

If we define

$$
\|f\|_{*,\omega}=\sup_Q\frac{1}{m_\omega(Q)}\int_Q|f(x)-(f)_{Q,\omega}|\,\omega(x)dx
$$

and $(\mathrm{BMO})_\omega=\{f\colon\|f\|_{*,\omega}<\infty\}$, then we can prove that, if ω satisfies (A_p) and $f\in(\mathrm{BMO})_\omega$,

$$
\int_{\boldsymbol{R}^n}\frac{|f(x)-(f)_{Q_0,\omega}|}{(1+|x|)^{pn+\varepsilon}}\,\omega(x)dx\leqq C\,m_\omega(Q_0)\|f\|_{*,\omega} \qquad (4.15)
$$

for any $\varepsilon>0$, where Q_0 is the unit cube in $\boldsymbol{R}^n$, by a similar computation as in Fefferman and Stein [3].

We can prove a converse of Theorem 5 for singular integrals with convolution kernels satisfying the condition (α), (β) and (γ) in the introduction. More generally we have:

THEOREM 6. *Suppose the convolution kernel K is of the form $\Omega(x)/|x|^n$, where Ω is real-valued, homogeneous of degree zero, continuous and does not vanish identcially on the unit sphere S^{n-1}, and the singular integrals $Tf=K*f$ and $\check{T}f=\check{K}*f$ are defined almost everywhere, where $\check{K}(x)=K(-x)$.*

Let $1\leqq p\leqq\infty$, and ω be non-negative and $\omega\in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$. If both of the inequalities

$$m_\omega(\{x\in\boldsymbol{R}^n: |Tf(x)|>\lambda\})\leqq\frac{C}{\lambda^p}\int_{\boldsymbol{R}^n}|f(x)|^p\omega(x)dx \tag{4.16}$$

and

$$m_\omega(\{x\in\boldsymbol{R}^n: |\check{T}f(x)|>\lambda\})\leqq\frac{C}{\lambda^p}\int_{\boldsymbol{R}^n}|f(x)|^p\omega(x)dx \tag{4.17}$$

hold for any $f\in L^p(\omega(x)dx)$ and $\lambda>0$, then ω satisfies the condition (A_p) or $\omega(x)=0$ a.e.

PROOF. If we define $K_1=(K+\check{K})/2$, $K_2=(K-\check{K})/2$, then K_1 and K_2 satisfy the assumption of Theorem 6, so that we may assume that K is odd or even.

Since Ω is continuous on S^{n-1}, there exists a point $\sigma_0\in S^{n-1}$ and a neighborhood U_0 of σ_0 in S^{n-1} such that

$$\Omega \text{ is constant sign on } U_0, \tag{4.18}$$

$$|\Omega(\sigma)|>\frac{1}{2}|\Omega(\sigma_0)|=m>0 \qquad \text{for} \quad \sigma\in U_0. \tag{4.19}$$

Let a cube Q be given, then by a simple geometrical observation, we can find another cube Q' which is congruent to Q and has the following property:

$$\frac{x-y}{|x-y|}\in U_0 \quad \text{and} \quad |x-y|\leqq C\operatorname{diam}Q \quad \text{for all } x\in Q' \text{ and all } y\in Q. \tag{4.20}$$

First, let $1<p<\infty$. Take a function $f\geqq 0$ supported in Q. Then (4.19) and (4.20) show that for any $x\in Q'$

$$|Tf(x)|=\int_Q\frac{|\Omega(x-y)|}{|x-y|^n}f(y)dy\geqq C\frac{1}{|Q|}\int_Q f(y)dy,$$

so that

$$Q'\subset\left\{x\in\boldsymbol{R}^n: |Tf(x)|>\frac{c}{|Q|}\int_Q f\,dy\right\}$$

and thus (4.16) yields

$$m_\omega(Q')\cdot\left(\frac{1}{|Q|}\int_Q f\,dy\right)^p\leqq C\int_Q f^p\omega dx. \tag{4.21}$$

Taking $f=1$, we obtain

$$m_{\omega}(Q')=\int_{Q'}\omega(x)dx\leqq C\int_{Q}\omega(x)dx. \tag{4.22}$$

Starting from $\check{T}f$ and Q', and reversing the above argument we can interchange Q and Q' in (4.21) and obtain

$$\int_{Q}\omega(x)dx\leqq C\int_{Q'}\omega(x)dx.$$

If $\omega(x)>0$ a.e., taking $f=\omega^{-1/(p-1)}$ in (4.21) shows that ω satisfies (A_p), and if $\omega(x)=0$ on a set of positive measure, taking f to be the characteristic function of this set shows that $\omega(x)=0$ a.e.

Next, we treat the case $p=1$. Since $\omega\in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$, $\operatorname*{ess\,inf}_{x\in Q}\omega(x)<\infty$. For any $\varepsilon>0$, we set $E=\{x\in\boldsymbol{R}^n : \omega(x)<\operatorname*{ess\,inf}_{Q}\omega+\varepsilon\}$, then $|E|>0$. Let $f(x)=1$ on E and $f(x)=0$ outside E. Then $f\in L^1(\omega dx)$ and $|Tf(x)|>c|Q|^{-1}|E|$ for all $x\in Q'$ by (4.18), (4.19) and (4.20), so that

$$Q'\subset\{x\in\boldsymbol{R}^n : |Tf(x)|>c|Q|^{-1}|E|\}.$$

Thus, if (4.16) holds, it follows that

$$\begin{aligned}m_{\omega}(Q')&\leqq C|Q||E|^{-1}\int_{E}f(x)\omega(x)dx\\&\leqq C|Q|(\operatorname*{ess\,inf}_{Q}\omega+\varepsilon),\end{aligned}$$

and, since ε is arbitrary, we obtain

$$m_{\omega}(Q')\leqq C|Q|\operatorname*{ess\,inf}_{Q}\omega. \tag{4.23}$$

The relation (4.21) holds for $p=1$ and the argument following it shows that $m_{\omega}(Q')\approx m_{\omega}(Q)$, so that we know that

$$\frac{1}{|Q|}\int_{Q}\omega(x)dx\leqq C\operatorname*{ess\,inf}_{Q}\omega$$

for any cube Q. This inequality is equivalent to the fact that ω satisfies (A_1), and the proof of Theorem 6 is completed.

At last, we shall give an example which shows that the assumption (4.17) for $\check{K}$ cannot be omitted for the validity of Theorem 6.

Let E_j $(j=1,\cdots,4)$ be the j-th quadrant in $\boldsymbol{R}^2$. We take such a kernel $K(x)=\Omega(x)/|x|^2$ as $\operatorname{supp}\Omega\subset S^1\cap E_1$ and such a weight function ω as

$$\omega(x)=\begin{cases}1 & (x\in E_3),\\ 0 & (x\notin E_3).\end{cases}$$

Given any $f \in L^p(\boldsymbol{R}^2)$, we set $g = f\chi_{E_3}$, then $Tf(x) = Tg(x)$ for $x \in E_3$. Since $g \in L^p(\omega dx)$,

$$m_\omega(\{|Tf(x)| > \lambda\}) = |\{|Tf(x)| > \lambda\} \cap E_3|$$

$$= |\{|Tg(x)| > \lambda\} \cap E_3| = |\{|Tg(x)| > \lambda\}|$$

$$\leqq (C\lambda^{-1}\int_{\boldsymbol{R}^2} |g(x)|^p dx)^{1/p} = (C\lambda^{-1}\int_{\boldsymbol{R}^2} |f(x)|^p \omega(x) dx)^{1/p}.$$

However, it is obvious that ω does not satisfy (A_p).

References

[1] R. R. Coifman, Distribution function inequalities for singular integrals, Proc. Nat. Acad. Sci. U. S. A., **69** (1972), 2838–2839.
[2] R. R. Coifman and C. Fefferman, Weighted norm inequalities for maximal functions and singular integrals, Studia Math., to appear.
[3] R. Fefferman and E. M. Stein, H^p spaces of several variables, Acta Math., **129** (1972), 137–193.
[4] F. Forelli, The M. Riesz theorem on conjugate functions, Trans. Amer. Math. Soc., **106** (1968), 369–390.
[5] R. F. Gundy and R. L. Wheeden, Weighted integral inequalities for the non-tangential maximal functions, Lusin area integral and Walsh-Paley series, Studia Math., **49** (1974), 107–124.
[6] R. A. Hunt, B. Muckenhoupt and R. Wheeden, Weighted norm inequalities for the conjugate function and Hilbert transform, Trans. Amer. Math. Soc., **176** (1973), 227–251.
[7] B. Muckenhoupt, Weighted norm inequalities for Hardy maximal function, Trans. Amer. Math. Soc., **165** (1972), 207–226.
[8] B. Muckenhoupt, The equivalence of two conditions for weight functions, Studia Math., **49** (1974), 101–106.
[9] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton, 1970.
[10] S, Yano, On Marcinkiewicz integrals, Tôhoku Math. J., to appear.
[11] A. Zygmund, On singular integrals, Rend. Mat., **16** (1957), 468–505.
[12] A. Zygmund, On certain lemmas of Carleson and Marcinkiewicz, J. Approximation Theory, **2** (1969), 249–257.

Makoto KANEKO
College of General Education
Tohoku University
Kawauchi
Sendai, Japan

Shigeki YANO
Department of Mathematics
Tokyo Metropolitan University
Fukazawa, Setagaya-ku
Tokyo, Japan