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Cusps of certain symmetric bounded domains

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Introduction.

Let K be a quadratic extension of a field k whose characteristic is not 2; $K=k(\sqrt{\alpha})$. Let (V, H) be a non-degenerate Hermitian vector space defined over K/k. There exists a functor sending each (V, H) to an alternating vector space (V', A') given by $V'=\mathfrak{R}_{K/k}V$, $A'(x, y)=(H(x, y)-H(y, x))/2\sqrt{\alpha}$. When K/k is a totally imaginary quadratic extension of a totally real number field, the above functor gives rise to a "holomorphic imbedding" sending a symmetric bounded domain of type I, denoted by D^{I}, into a Siegel space, or a symmetric bounded domain of type III, denoted by D^{III}. Furthermore, when we are given a lattice L in V, the above functor induces a homomorphism sending the arithmetic subgroup Γ^{I} of $SU(V, H)$ stabilizing L into the subgroup Γ^{III} of $Sp(V', A')$ stabilizing $\mathfrak{R}_{K/k}L$. Thus we obtain a mapping ρ sending "cusps" of D^{I} with respect to Γ^{I} into "cusps" of D^{III} with respect to Γ^{III}. Each "cusp" of D^{I} with respect to Γ^{I} is, by definition, a Γ^{I}-orbit of rational boundary components of the compactification $\bar{D}^{\mathrm{I}}$; a rational boundary component of $\bar{D}^{\mathrm{I}}$ is, on the other hand, associated to a totally isotropic subspace of V. The totality of the rational boundary components associated to totally isotropic subspaces of dimension s constitutes an $SU(V, H)$-orbit which is decomposed into a finite number of cusps, the totality of the latter being denoted by $\mathcal{C}_s^{\mathrm{I}}(L)$. The mapping ρ sends $\mathcal{C}_s^{\mathrm{I}}(L)$ into $\mathcal{C}_{2s}^{\mathrm{III}}(\mathfrak{R}_{K/k}L)$. When L is an "$\mathfrak{J}$-modular lattice", with $\mathfrak{J}$ assumed to be an ideal in k, the lattice $\mathfrak{R}_{K/k}L$ is maximal in V' and there exists a bijection Φ^{III} sending $\mathcal{C}_{2s}^{\mathrm{III}}(\mathfrak{R}_{K/k}L)$ onto the ideal class group $C(k)$, $(s>0)$. On the other hand, the association of each element of $\mathcal{C}_s^{\mathrm{I}}(L)$ corresponding to a totally isotropic subspace U of V to the ideal class of the lattice $L\cap U$ gives a mapping $\tilde{\Phi}^{\mathrm{I}}: \mathcal{C}_s^{\mathrm{I}}(L)\to C(K)$ which is, under a certain condition, bijective (Theorem 1.8 and its Corollaries, Ch. II). The main result of this note asserts the existence of a surjection $\nu_s: C(K)\to C(k)$ closely connected to the norm $N_{K/k}$, such that the following diagram is commutative (Theorem 3.3, Ch. II):

$$\begin{tikzcd} C_s^{\mathrm{I}}(L) \arrow[r, "\rho"] \arrow[d, "\tilde{\Phi}^{\mathrm{I}}"'] & C_{2s}^{\mathrm{III}}(\mathcal{R}_{K/k}L) \arrow[d, "\Phi^{\mathrm{III}}"] \\ C(K) \arrow[r, "\nu_s"] & C(k), \end{tikzcd} \qquad (s>0).$$

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Chapter I. Boundary components of certain symmetric bounded domains.

In this chapter we shall summarize the facts, which will be utilized as foundation for chapter II, concerning the symmetric bounded domains associated to Hermitian forms and alternating forms. A large part of these facts and their proofs are found, although not always explicitly, in [9] and [10].

§1. Basic notions.

Let k be a field with the characteristic not 2 and let K be a commutative ring with unity containing k as a subring. We assume the existence of a non-zero element w of K such that:

1) $w^2 \in k$,

2) $K = k\cdot 1 + k\cdot w$, (direct sum of k-vector spaces).

An element w' of K satisfies the requirements 1) and 2) above if and only if $w' = aw$, for an element $a \in k^*$.

The ring K then admits a uniquely determined non-trivial involution σ which stabilizes each element of k; namely, for an element $\alpha = a + bw$ of K we have $\alpha^\sigma = a - bw$.

Given an element α of K we set $N(\alpha) = N_{K/k}(\alpha) = \alpha^\sigma \alpha$, $\mathrm{Tr}(\alpha) = \mathrm{Tr}_{K/k}(\alpha) = \alpha^\sigma + \alpha$.

Let (V, H) be the pair of a finite dimensional free K-module V and a non-degenerate σ-sesquilinear form H on V which shall be called a Hermitian form on V. We have by definition

$$H(\alpha u, \beta v) = \alpha^\sigma H(u, v)\beta, \qquad (\alpha, \beta \in K,\ u, v \in V),$$

$$H(u, v) = H(v, u)^\sigma,$$

$$\mathrm{rad}_H V = \{v \in V;\ H(v, u) = 0 \quad \text{for all } u \in V\} = \{0\}.$$

The K-module V is, naturally, a k-vector space of finite dimension admitting a non-degenerate alternating form $A = A_w$ defined by:

$$A_w(u, v) = (H(u, v) - H(v, u))/2w.$$

We have, for $a\in k^*$, $A_{aw}=a^{-1}A_w$. The K-module V is, when regarded as a k-vector space, denoted by $\mathcal{R}_{K/k}V$, or simply by $\mathcal{R}V$, and the alternating form A_w is denoted by $\mathrm{Im}_w H$, or $\mathrm{Im}\,H$. Furthermore, we set:

$$\mathcal{R}_{K/k}(V, H)=\mathcal{R}(V, H)=(\mathcal{R}V, \mathrm{Im}\,H).$$

We denote the group of K-module automorphisms of V by $GL(V)$ and set

$$U(V, H)=\{g\in GL(V)\,;\ H(gu, gv)=H(u, v),\ \text{ for all } u, v\in V\},$$
$$SU(V, H)=U(V, H)\cap SL(V),$$

where, of course, $SL(V)=\{g\in GL(V)\,;\ \det g=1\}$.

The above functor $\mathcal{R}_{K/k}$ then naturally gives rise to a homomorphism

$$\rho:\ U(V, H)\longrightarrow Sp(\mathcal{R}V, \mathrm{Im}\,H),$$

where $Sp(\mathcal{R}V, A)=\{g\in GL(\mathcal{R}V)\,;\ A(gu, gv)=A(u, v),$ for all $u, v\in\mathcal{R}V\}$.

§ 2. The symmetric bounded domain $D^{\mathrm{I}}(V, H)$ and its boundary components.

2.1. Let $K=\boldsymbol{C}=$complex number field, and let $k=\boldsymbol{R}=$real number field. Let V be a vector space over $\boldsymbol{C}$ with the base $\{v_1, \cdots, v_n\}$, supplied with a non-degenerate Hermitian form H given by $H(v_i, v_j)=\varepsilon_i\delta_{ij}$; $\varepsilon_i=1$ for $1\leqq i\leqq t$, $\varepsilon_j=-1$ for $t+1\leqq j\leqq n$. We set $t'=n-t$ and assume that $t\geqq t'$. We put

$$D^{\mathrm{I}}(V, H)=\{U\,;\ U \text{ is a } t'\text{-dimensional subspace of } V \text{ such that } H|_U<0\},$$

which is, when $t'>0$, an open submanifold of a complex Grassmannian manifold, whereas, when $t'=0$, $D^{\mathrm{I}}=D^{\mathrm{I}}(V, H)$ is a point. D^{I} has the "origin" $o^{\mathrm{I}}=U_0=\{v_{t+1}, \cdots, v_n\}_{\boldsymbol{C}}$, (when $t'=0$, $U_0=\{0\}$). The group $SU(V, H)$, which is also denoted by G^{I}, naturally operates transitively on D^{I}; the isotropy subgroup stabilizing o^{I}, denoted by K^{I}, is clearly a maximal compact subgroup of the Lie group G^{I}. We have, therefore, isomorphism of manifolds: $D^{\mathrm{I}}\cong G^{\mathrm{I}}/K^{\mathrm{I}}$, where the latter homogeneous space is well known to be a symmetric bounded domain. A realization of D^{I} as a bounded domain is given in the following manner.

We shall denote the orthogonal projection of V onto its subspace U_0 by p_0. Then by the above definition of D^{I}, p_0 is injective on each element U of D^{I} and hence $p_0: U\to U_0$ is an isomorphism of vector spaces. This implies that there exists uniquely a base $\{u_1, \cdots, u_{t'}\}$ of U such that $p_0(u_j)=v_{t+j}$, $(1\leqq j\leqq t')$. Hence each element U of D^{I} determines uniquely at $t\times t'$ complex matrix $Z(U)=(z_{ij})$ with $u_j=\sum_{i=1}^{t} z_{ij}v_i+v_{t+j}$, $(1\leqq j\leqq t')$. The matrix $Z=Z(U)$ satisfies that

(1) $$1-{}^t\bar{Z}Z>0\,.$$

Conversely, if Z is an arbitrarily given complex $t\times t'$ matrix satisfying (1), then there exists uniquely an element $U\in D^{\mathrm{I}}$ such that $Z=Z(U)$. The space D^{I} is, in this manner, realized as a bounded domain in $\boldsymbol{C}^{t\times t'}$. Each element $g\in G^{\mathrm{I}}$ represented with respect to the base $\{v_1,\cdots,v_n\}$ as $g=\begin{array}{cc} \begin{pmatrix} A & B \\ C & D \end{pmatrix} & \begin{matrix} \}t \\ \}t' \end{matrix} \\ \begin{matrix} \underbrace{}_{t} & \underbrace{}_{t'} \end{matrix} & \end{array}$ operates on D^{I} as follows:

$$g(Z)=(AZ+B)(CZ+D)^{-1}\,,\qquad (Z=Z(U)\,,\ U\in D^{\mathrm{I}})\,.$$

We shall denote the analytic isomorphism sending each element U of D^{I} to $Z(U)$ of $\boldsymbol{C}^{t\times t'}$ by ι^{I}.

Let us denote the Lie algebra of G^{I} by $\mathfrak{g}^{\mathrm{I}}$, which is, with respect to the base $\{v_1,\cdots,v_n\}$, represented as a subalgebra of the total matrix algebra $M(n,\boldsymbol{C})$ consisting of the matrices:

$$S=\begin{array}{cc} \begin{matrix} \overbrace{}^{t} & \overbrace{}^{t'} \end{matrix} & \\ \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} & \begin{matrix} \}t \\ \}t' \end{matrix} \end{array},\ {}^t\bar{X}=-X,\ Z={}^t\bar{Y},\ {}^t\bar{W}=-W,\ \mathrm{tr}\,X+\mathrm{tr}\,W=0\,.$$

The Lie algebra $\mathfrak{g}^{\mathrm{I}}$ admits a Cartan involution θ determined by:

$$\theta:S\longrightarrow -{}^t\bar{S}\,,$$

which gives rise to the Cartan decomposition:

$$\mathfrak{g}^{\mathrm{I}}=\mathfrak{k}^{\mathrm{I}}+\mathfrak{p}^{\mathrm{I}}\,,$$

where $\mathfrak{k}^{\mathrm{I}}=\{S\in\mathfrak{g}^{\mathrm{I}}\,;\,\theta(S)=S\}$, $\mathfrak{p}^{\mathrm{I}}=\{S\in\mathfrak{g}^{\mathrm{I}}\,;\,\theta(S)=-S\}=\left\{\begin{pmatrix} 0 & Z \\ {}^t\bar{Z} & 0 \end{pmatrix}\in\mathfrak{g}^{\mathrm{I}}\right\}$, and $\mathfrak{k}^{\mathrm{I}}$ is the Lie algebra of K^{I}, (cf. Helgason [**3**, Ch. IV]). The subspace $\mathfrak{p}^{\mathrm{I}}$ may be identified with the tangent space of D^{I} at o^{I} and admits a complex structure, (or, according to the terminology used by Helgason in [**3**], a canonical almost complex structure), J^{I} given by:

$$J^{\mathrm{I}}:\begin{pmatrix} 0 & Z \\ {}^t\bar{Z} & 0 \end{pmatrix}\longrightarrow\begin{pmatrix} 0 & \sqrt{-1}Z \\ -\sqrt{-1}\,{}^t\bar{Z} & 0 \end{pmatrix}.$$

(For this paragraph and also for the paragraph 3.1, concerning the domain D^{III}, cf. Satake [**10**].)

2.2. The closure $\bar{D}^{\mathrm{I}}$ of D^{I} in the ambient Grassmannian manifold is given by:

$$\bar{D}^{\mathrm{I}}=\{U\,;\,U\text{ is a }t'\text{-dimensional subspace of }V\text{ such that }H|_U\leqq 0\}\,.$$

The analytic isomorphism ι^{I} is naturally extended to a continuous mapping, again denoted by ι^{I}, sending $\bar{D}^{\mathrm{I}}$ onto the closure of $\iota^{\mathrm{I}}(D^{\mathrm{I}})$ in $\boldsymbol{C}^{t\times t'}$ which consists of $t\times t'$-complex matrices Z such that $1-{}^t\bar{Z}Z\geqq 0$.

Given elements $U_1,\,U_2\in\bar{D}^{\mathrm{I}}$ we shall write $U_1\underset{G^{\mathrm{I}}}{\sim}U_2$ if and only if there exists an element $g\in G^{\mathrm{I}}$ sending U_1 onto U_2. It is obvious that we have:

$$U_1 \underset{G^{\mathrm{I}}}{\sim} U_2 \iff \dim(\mathrm{rad}_H U_1)=\dim(\mathrm{rad}_H U_2).$$

Hence $\bar{D}^{\mathrm{I}}$ is decomposed into disjoint union of G^{I}-orbits:

$$\bar{D}^{\mathrm{I}}=\bigcup_{s=0}^{t'} C_s^{\mathrm{I}}, \qquad C_s^{\mathrm{I}}=\{U\in\bar{D}^{\mathrm{I}};\ \dim(\mathrm{rad}_H U)=s\};$$

the operation of $g=\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in G^{\mathrm{I}}$ on C_s^{I} is given by:

$$g(Z)=(AZ+B)(CZ+D)^{-1}, \qquad (Z\in C_s^{\mathrm{I}}).$$

It is evident that for an arbitrarily given totally isotropic subspace W of V there always exists an element $U\in\bar{D}^{\mathrm{I}}$ with $\mathrm{rad}_H U=W$. We shall set:

$$B_W^{\mathrm{I}}=\{U\in\bar{D}^{\mathrm{I}};\ \mathrm{rad}_H U=W\}.$$

We have, then,

$$C_s^{\mathrm{I}}=\bigcup_{\dim W=s} B_W^{\mathrm{I}}.$$

It is furthermore known that each B_W^{I} is a "boundary component" of $\bar{D}^{\mathrm{I}}$, i.e. it is an arc-wise connected component with respect to complex analytic curves, (cf. Pyatetski-Shapiro, [**9**, Ch. II, § 6, Theorem 1]).

Setting $G_W^{\mathrm{I}}=\{g\in G^{\mathrm{I}};\ gW=W\}$, the space B_W^{I} carries the structure of a G_W^{I}-space.

Generally if W is a totally isotropic subspace of V the quotient space $W^{\perp}/W$ carries a non-degenerate Hermitian form H_W canonically induced by H. We then have a mapping:

$$\varphi_W^{\mathrm{I}}:\ B_W^{\mathrm{I}}\longrightarrow D^{\mathrm{I}}(W^{\perp}/W,\ H_W),$$

determined by $\varphi_W^{\mathrm{I}}(U)=U/\mathrm{rad}_H U=U/W$. It is obvious that φ_W^{I} is bijective. Furthermore φ_W^{I} is an analytic isomorphism from the submanifold B_W^{I} of $\boldsymbol{C}^{t\times t'}$ onto the symmetric bounded domain $D^{\mathrm{I}}(W^{\perp}/W,\ H_W)$.

Particularly, when $W=\{v_1+v_{t+1},\cdots,v_s+v_{t+s}\}_C$, $(0\leqq s\leqq t')$, we have:

$$\iota^{\mathrm{I}}(B_W^{\mathrm{I}})=\left\{\begin{pmatrix} 1_s & 0 \\ 0 & Z\end{pmatrix};\ 1-{}^t\bar{Z}Z>0,\ Z\in M(t-s,\ \boldsymbol{C})\right\},$$

and φ_W^{I} sends the element corresponding to $\begin{pmatrix} 1 & 0 \\ 0 & Z\end{pmatrix}$ to the element of $D^{\mathrm{I}}(W^{\perp}/W,\ H_W)$ associated to Z.

2.3. In this paragraph we let k be a totally real number field and let K be its totally imaginary quadratic extension. We set $\dim_K V=n$ and assume that $\mathrm{ind}(H)=r>0$. ($\mathrm{ind}(H)$ is, by definition, the dimension of a maximal totally isotropic subspace of V, and is equal to the k-rank of algebraic group $SU(V,\ H)$.)

Let $\{\tau_1, \cdots, \tau_d\}$ be the set of mutually distinct isomorphisms sending k into $\boldsymbol{R}$. Then each $(V^{\tau_i})_{\boldsymbol{R}} = V^{\tau_i} \otimes_{k^{\tau_i}} \boldsymbol{R}$ carries a non-degenerate Hermitian form H^{τ_i} naturally induced by H. Since $\mathrm{ind}_{\boldsymbol{R}}(H^{\tau_i}) \geqq r$, $(1 \leqq i \leqq d)$, each $D^{\mathrm{I}}((V^{\tau_i})_{\boldsymbol{R}}, H^{\tau_i})$ is a non-trivial symmetric bounded domain. Let us set

$$D^{\mathrm{I}}(\tilde{V}, \tilde{H}) = \prod_{i=1}^{d} D^{\mathrm{I}}((V^{\tau_i})_{\boldsymbol{R}}, H^{\tau_i}).$$

This is a symmetric domain of type I, (cf. (9), (10)).

Each boundary component of $\bar{D}^{\mathrm{I}}(\tilde{V}, \tilde{H}) = \prod_{i=1}^{d} \bar{D}^{\mathrm{I}}((V^{\tau_i})_{\boldsymbol{R}}, H^{\tau_i})$, B^{I}, may be written as $B^{\mathrm{I}} = \prod_{i=1}^{d} B^{\mathrm{I}}_{W_i}$, where $B^{\mathrm{I}}_{W_i}$ is a boundary component of $\bar{D}^{\mathrm{I}}((V^{\tau_i})_{\boldsymbol{R}}, H^{\tau_i})$ associated to a totally isotropic subspace W_i of $(V^{\tau_i})_{\boldsymbol{R}}$. $B^{\mathrm{I}} = \prod B^{\mathrm{I}}_{W_i}$ is called *rational* if there exists a totally isotropic subspace W of V such that $W_i = (W^{\tau_i})_{\boldsymbol{R}}$, $(1 \leqq i \leqq d)$; in such a case we denote B^{I} by B^{I}_W.

(This definition of "rational boundary components" is in accordance with the definition of "Γ-rational boundary components" for an arithmetic subgroup Γ, given in Borel-Baily, [2].)

The set of rational boundary components of $\bar{D}^{\mathrm{I}}(\tilde{V}, \tilde{H})$ is therefore decomposed into the disjoint union of G^{I}-orbits, $(G^{\mathrm{I}} = SU(V, H))$ each of which shall be denoted by C^{I}_s, where $C^{\mathrm{I}}_s = \bigcup_{\dim W = s} B^{\mathrm{I}}_W$. Furthermore it is easily seen that $B^{\mathrm{I}}_W \cong D^{\mathrm{I}}((W^{\perp}/W)^{\sim}, \tilde{H}_W)$.

§ 3. The symmetric bounded domain $D^{\mathrm{III}}(V, A)$ and its boundary components.

3.1. Let V be a finite dimensional vector space over $\boldsymbol{R}$ supplied with a non-degenerate alternating form A. We shall set:

$$D^{\mathrm{III}}(V, A) = \{I;\ I \text{ is a complex structure on } V \text{ such that the bilinear form } S(x, y) = A(x, Iy),\ (x, y \in V) \text{ is symmetric and positive definite}\}.$$

The group $G^{\mathrm{III}} = Sp(V, A)$ then operates on $D^{\mathrm{III}}(V, A)$ as follows:

$$g : I \longrightarrow gIg^{-1}, \qquad (g \in G^{\mathrm{III}},\ I \in D^{\mathrm{III}} = D^{\mathrm{III}}(V, A)).$$

Let V_C denote the complexification of V. V_C admits the involution:

$$V_C \ni x \longrightarrow \bar{x} \in V_C,$$

such that $V = \{x \in V_C;\ \bar{x} = x\}$. The form A may be naturally extended to a non-degenerate alternating form, again denoted by A, on V_C with $A(\bar{x}, \bar{y}) = \overline{A(x, y)}$. Let us set:

$$F(x, y) = \sqrt{-1} A(\bar{x}, y), \qquad (x, y \in V_C).$$

F is, then, a non-degenerate Hermitian form on V_C.

Given an element $I \in D^{\mathrm{III}}$ we denote its extension to V_C again by I. We then have a decomposition:

$$V_C = U_I + \bar{U}_I,\ U_I = \{x \in V_C\,;\, Ix = -\sqrt{-1}x\},\ \bar{U}_I = \{x \in V_C\,;\, Ix = \sqrt{-1}x\}\,,$$

(cf. [10]). We have furthermore,

$$(1) \qquad A|U_1 = 0\,, \qquad F|U_I < 0\,.$$

It is, on the other hand, easily seen that F is positive definite on $\bar{U}_I$ and therefore U_I belongs to $D^{\mathrm{I}}(V_C, F)$.

When, on the other hand, we are given a subspace U of V_C with $\dim U = \frac{1}{2}\dim V_C$ satisfying the condition (1) above, we obtain a decomposition:

$$V_C = U + \bar{U}\,, \qquad U \cap \bar{U} = \{0\}\,.$$

Accordingly we can determine a complex structure I of V_C by:

$$I|U = -\sqrt{-1}\cdot 1\,, \qquad I|\bar{U} = \sqrt{-1}\cdot 1\,,$$

and by restricting this I on V we obtain an element $I(U)$ of D^{III}. We have $I = I(U_I)$ and $U = U_{I(U)}$. Hence the mapping $I \to U_I$ determines a bijection from D^{III} onto the following closed submanifold of $D^{\mathrm{I}}(V_C, F)$:

$$\boldsymbol{D}^{\mathrm{III}}(V_C, F) = \left\{U \subset V_C\,;\, A|U = 0,\ F|U < 0,\ \dim U = \frac{1}{2}\dim V_C\right\}.$$

The space D^{III} may be in this manner identified with the manifold $\boldsymbol{D}^{\mathrm{III}}$ which is known to be a symmetric bounded domain of type III, (cf. [10]).

The group G^{III} is naturally identified with the following group:

$$\boldsymbol{G}^{\mathrm{III}} = \{g \in U(V_C, F)\,;\, g(\bar{x}) = \overline{g(x)}\,, \qquad \text{for all} \quad x \in V_C\}\,,$$

and we have, for $g \in G^{\mathrm{III}} (= \boldsymbol{G}^{\mathrm{III}})$, the following commutative diagram:

$$\begin{array}{ccc} D^{\mathrm{III}} \ni I & \longrightarrow & gIg^{-1} \in D^{\mathrm{III}} \\ \wr\wr & & \wr\wr \\ \boldsymbol{D}^{\mathrm{III}} \ni U_I & \longrightarrow & gU_I \in \boldsymbol{D}^{\mathrm{III}}\,. \end{array}$$

Let $\{v_1, \cdots, v_n, v_1', \cdots, v_n'\}$ be a base of V with $A(v_i, v_j) = A(v_i', v_j') = 0$, $A(v_i, v_j') = \delta_{ij}$. Let us set

$$e_i = (v_i - \sqrt{-1}\,v_i')/(1 + \sqrt{-1})\,, \qquad (1 \leqq i \leqq n)\,.$$

Then $\{e_1, \cdots, e_n, \bar{e}_1, \cdots, \bar{e}_n\}$ constitutes a base of V_C with $F(e_i, e_j) = \delta_{ij}$, $F(\bar{e}_i, \bar{e}_j) = -\delta_{ij}$, $F(e_i, \bar{e}_j) = 0$. The element $o^{\mathrm{III}} = U_0 = \{\bar{e}_1, \cdots, \bar{e}_n\}_C$ belongs to $\boldsymbol{D}^{\mathrm{III}}$ and the corresponding $I_0 = I(U_0)$ is determined by:

$$I_0(v_i) = v_i'\,, \qquad I_0(v_i') = -v_i\,, \qquad (1 \leqq i \leqq n)\,.$$

The fact that G^{III} operates transitively on D^{III} immediately follows from the observation that every $I\in D^{\mathrm{III}}$ is associated to a base $\{u_1, \cdots, u_n, u_1', \cdots, u_n'\}$ of V with $A(u_i, u_j)=A(u_i', u_j')=0$, $A(u_i, u_j')=\delta_{ij}$; $I(u_i)=u_i'$, $I(u_i')=-u_i$. The isotropy subgroup K^{III} of G^{III} stabilizing o^{III} is easily seen to be a maximal compact subgroup. The symmetric space $D^{\mathrm{III}}=\boldsymbol{D}^{\mathrm{III}}$ is realized as a bounded domain in $\boldsymbol{C}^{n(n+1)/2}$ in the following manner. First, since $\boldsymbol{D}^{\mathrm{III}}$ is contained in $D^{\mathrm{I}}(V_C, F)$, to each $U\in\boldsymbol{D}^{\mathrm{III}}$ there corresponds a complex $n\times n$ matrix $Z(U)=(z_{ij})$ such that $\{z_i=\sum_{j=1}^{n} z_{ij}e_j+\bar{e}_i,\ (1\leqq i\leqq n)\}$ constitutes a base of U. Taking the additional condition "$A|U=0$" into account we have:

$$(2)\qquad {}^tZ=Z, \qquad 1-{}^t\bar{Z}Z>0, \qquad (Z=Z(U)).$$

The mapping $\iota^{\mathrm{III}}: U\to Z(U)$ gives an analytic isomorphism sending $\boldsymbol{D}^{\mathrm{III}}$ onto the bounded domain in $\boldsymbol{C}^{n(n+1)/2}$ consisting of the $n\times n$ complex matrices Z satisfying the condition (2).

With respect to the base $\{e_1, \cdots, e_n, \bar{e}_1, \cdots, \bar{e}_n\}$ the group $\boldsymbol{G}^{\mathrm{III}}$ is written as the set of $2n\times 2n$ complex matrices as follows:

$$\left\{\begin{pmatrix} \overbrace{A}^{n} & \overbrace{B}^{n} \\ \bar{B} & \bar{A}\end{pmatrix}\begin{matrix}\}n\\ \}n\end{matrix};\ {}^t\bar{B}A={}^tA\bar{B},\ {}^tA\bar{A}-{}^t\bar{B}B=1\right\},$$

and for $g=\begin{pmatrix} A & B \\ \bar{B} & \bar{A}\end{pmatrix}\in\boldsymbol{G}^{\mathrm{III}}$ and $U\in\boldsymbol{D}^{\mathrm{III}}$ we have:

$$Z(gU)=(AZ+B)(\bar{B}Z+\bar{A})^{-1}, \qquad (Z=Z(U)).$$

The Lie algebra of $\boldsymbol{G}^{\mathrm{III}}$ is, with respect to the base $\{e_1, \cdots, e_n, \bar{e}_1, \cdots, \bar{e}_n\}$, written as follows:

$$\mathfrak{g}^{\mathrm{III}}=\left\{\begin{pmatrix} \overbrace{X}^{n} & \overbrace{Y}^{n} \\ \bar{Y} & \bar{X}\end{pmatrix}\begin{matrix}\}n\\ \}n\end{matrix};\ {}^t\bar{X}=-X,\ {}^tY=Y\right\},$$

which admits the following Cartan involution:

$$\theta: S\longrightarrow -{}^t\bar{S}, \qquad (S\in\mathfrak{g}^{\mathrm{III}}),$$

according to which we have the Cartan decomposition:

$$\mathfrak{g}^{\mathrm{III}}=\mathfrak{k}^{\mathrm{III}}+\mathfrak{p}^{\mathrm{III}},$$

$$\mathfrak{k}^{\mathrm{III}}=\left\{\begin{pmatrix} X & 0 \\ 0 & \bar{X}\end{pmatrix};\ {}^t\bar{X}=-X\right\}, \qquad \mathfrak{p}^{\mathrm{III}}=\left\{\begin{pmatrix} 0 & Y \\ \bar{Y} & 0\end{pmatrix};\ {}^tY=Y\right\},$$

and the latter space $\mathfrak{p}^{\mathrm{III}}$ is identified with the tangent space of $\boldsymbol{D}^{\mathrm{III}}$ at o^{III} on which acts the complex structure J^{III} defined by

$$J^{\mathrm{III}}: \begin{pmatrix} 0 & Y \\ \bar{Y} & 0\end{pmatrix}\longrightarrow \sqrt{-1}\begin{pmatrix} 0 & Y \\ -\bar{Y} & 0\end{pmatrix}.$$

3.2. $\boldsymbol{D}^{\mathrm{III}}(V_C, F)$ is a submanifold of the Grassmann manifold consisting of the subspaces U of V_C with $\dim U = \frac{1}{2}\dim V_C$. The closure of $\boldsymbol{D}^{\mathrm{III}}$ in the Grassmann manifold is given by:

$$\bar{\boldsymbol{D}}^{\mathrm{III}} = \left\{U\,;\, U \subset V_C,\ \dim U = \frac{1}{2}\dim V_C,\ A|U=0,\ F|U \leqq 0\right\}.$$

The analytic isomorphism ι^{III} is extendible to a continuous mapping, again denoted by ι^{III}, sending $\bar{\boldsymbol{D}}^{\mathrm{III}}$ onto the closure of $\iota^{\mathrm{III}}(\boldsymbol{D}^{\mathrm{III}})$ in $\boldsymbol{C}^{n(n+1)/2}$, $\left(n = \frac{1}{2}\dim V_C\right)$, which is the set of $n \times n$ complex matrices Z such that

$${}^tZ = Z, \qquad 1 - {}^t\bar{Z}Z \geqq 0.$$

Suppose that we are given $U \in \bar{\boldsymbol{D}}^{\mathrm{III}}$. We shall set $\widetilde{W} = \mathrm{rad}_F U$. Then it can be easily verified that the subspace $U/\widetilde{W}$ of $\widetilde{W}^{\perp}/\widetilde{W}$ is an element of $\boldsymbol{D}^{\mathrm{III}}(\widetilde{W}^{\perp}/\widetilde{W}, F_{\widetilde{W}})$. Let us set, furthermore, $W = W(U) = \widetilde{W} \cap V$. Then $A|W=0$ and we obtain in a natural manner a non-degenerate alternating vector space $(W^{\perp}/W, A_W)$ where A_W is induced by A. $\widetilde{W}^{\perp}/\widetilde{W}$ may be, then, regarded as the complexification of $W^{\perp}/W$, and $F_{\widetilde{W}}$ is obtained from A_W in the same manner as in 3.1. Hence we may identify $\boldsymbol{D}^{\mathrm{III}}(\widetilde{W}^{\perp}/\widetilde{W}, F_{\widetilde{W}})$ with $D^{\mathrm{III}}(W^{\perp}/W, A_W)$. We shall denote the element of the latter space corresponding to U/W by $I_{W,U}$.

Let us set:

$$C(V, A) = \{(W, I_W)\,;\, W \subset V,\ A|W=0,\ I_W \in D^{\mathrm{III}}(W^{\perp}/W, A_W)\}.$$

We assert that the mapping:

$$\alpha : \bar{\boldsymbol{D}}^{\mathrm{III}} \ni U \longrightarrow (W(U), I_{W(U),U}) \in C(V, A)$$

is bijective. Clearly, it is enough to show that α is surjective. For the purpose of showing the latter let us note that the operation of $\boldsymbol{G}^{\mathrm{III}} (= G^{\mathrm{III}})$ is naturally extended to the operation on $\bar{\boldsymbol{D}}^{\mathrm{III}}$, "$g : U \to gU$, $g \in \boldsymbol{G}^{\mathrm{III}}$, $U \in \bar{\boldsymbol{D}}^{\mathrm{III}}$"; precisely, we have for $g = \begin{pmatrix} A & B \\ \bar{B} & \bar{A}\end{pmatrix} \in \boldsymbol{G}^{\mathrm{III}}$ and $Z \in \iota^{\mathrm{III}}(\bar{\boldsymbol{D}}^{\mathrm{III}})$, the following:

$$g(Z) = (AZ+B)(\bar{B}Z+\bar{A})^{-1}.$$

Now, suppose that W is a subspace of V with $A|W=0$. Then the subgroup of G^{III} consisting of the elements stabilizing W, denoted by G_W^{III}, obviously operates transitively on the subset of $\bar{\boldsymbol{D}}^{\mathrm{III}}$ consisting of elements U with $W(U) = W$. This fact, together with the fact that totally isotropic subspaces W, W' of V are G^{III}-equivalent (i.e. there exists $g \in G^{\mathrm{III}}$ such that $gW = W'$) if and only if $\dim W = \dim W'$, imply the surjectivity of α.

Given a totally isotropic subspace W of V we shall set

$$B_W^{\mathrm{III}} = \{U\,;\, U \in \bar{\boldsymbol{D}}^{\mathrm{III}},\ W(U) = W\}, \quad C_s^{\mathrm{III}} = \bigcup_{\dim W = s} B_W^{\mathrm{III}}.$$

Then B_W^{III} is a boundary component of $\bar{\boldsymbol{D}}^{\mathrm{III}}$ and the mapping $\varphi_W^{\mathrm{III}} : U \to I_{W,U}$

gives an isomorphism sending B_W^{III} onto $D^{\mathrm{III}}(W^\perp/W, A_W)$. B_W^{III} is, as mentioned above, a G_W^{III}-space, whereas C_s^{III} is the G^{III}-orbit of B_W^{III} in $\bar{\boldsymbol{D}}^{\mathrm{III}}$ with $\dim W=s$. We have $\bar{\boldsymbol{D}}^{\mathrm{III}}=\bigcup_{s=0}^{n} C_s^{\mathrm{III}}$, $(2n=\dim V)$.

3.3. Let k be a totally real number field and let $\{\tau_1, \cdots, \tau_d\}$ be the set of mutually distinct isomorphisms sending k into $\boldsymbol{R}$. Let (V, A) be a non-degenerate alternating vector space over k. Denoting the alternating form on $V_{\boldsymbol{R}}^{\tau_i}$ naturally induced by A by A^{τ_i}, we obtain a symmetric bounded domain $D^{\mathrm{III}}(\tilde{V}, \tilde{A})=\prod_{i=1}^{d} D^{\mathrm{III}}(V_{\boldsymbol{R}}^{\tau_i}, A^{\tau_i})$. A boundary component $\prod_{i=1}^{d} B_{W_i}^{\mathrm{III}}$ of $\bar{\boldsymbol{D}}^{\mathrm{III}}(\tilde{V}, \tilde{A})$ is *rational* when there exists a totally isotropic subspace W of V such that $W_i=W_{\boldsymbol{R}}^{\tau_i}$ for $1\leqq i\leqq d$; we denote such a boundary component by B_W^{III}. The set of rational boundary components is, therefore, decomposed into disjoint union of G^{III}-orbits each of which shall be denoted by $C_s^{\mathrm{III}}=\bigcup_{\dim W=s} B_W^{\mathrm{III}}$. Each B_W^{III} may be identified with $D^{\mathrm{III}}(\widetilde{W^\perp/W}, \tilde{A}_W)$.

§ 4. The holomorphic imbedding $\rho: D^{\mathrm{I}}\to D^{\mathrm{III}}$.

4.1. Let (V, H) be a non-degenerate finite dimensional Hermitian vector space over $\boldsymbol{C}/\boldsymbol{R}$ and let $(V', A')=\mathcal{R}_{\boldsymbol{C}/\boldsymbol{R}}(V, H)$, $(A'=\mathrm{Im}_{\sqrt{-1}}H)$. The homomorphism $\rho: G^{\mathrm{I}}\to G^{\mathrm{III}}=Sp(V', A')$ induces, as follows, a mapping $\rho: D^{\mathrm{I}}(V, H)\to D^{\mathrm{III}}(V', A')$ which sends o^{I} onto o^{III}, (cf. [10]).

Given an element $U\in D^{\mathrm{I}}$ we have the direct sum decomposition:

$$V=U+U^\perp .$$

We then obtain an element $I=\rho(U)\in D^{\mathrm{III}}$ given as follows:

$$I|U=-\sqrt{-1}\cdot 1_U , \qquad I|U^\perp=\sqrt{-1}\cdot 1_{U^\perp} .$$

The mapping ρ is "equivariant", i.e. we have, for $g\in G^{\mathrm{I}}$,

$$\rho(gU)=\rho(g)\rho(U)\rho(g)^{-1} , \qquad (U\in D^{\mathrm{I}}) .$$

The mapping ρ induces a mapping, again denoted by ρ, sending D^{I} into $\boldsymbol{D}^{\mathrm{III}}$. For the purpose of obtaining an explicit description of the latter mapping, we shall first consider the Hermitian vector space (V^*, H^*) naturally determined by (V, H) as follows:

i) V^* is, as a vector space over $\boldsymbol{C}$, identical to V;

ii) $V(=V^*)$ admits an $\boldsymbol{R}$-linear involution $x\to x^*$ satisfying that $(\alpha x)^*=\bar{\alpha}x^*$, $(\alpha\in\boldsymbol{C})$;

iii) $H^*(x^*, y^*)=-H(y, x)$.

Let us set:

$$\hat{V}=V\oplus V^*, \ \hat{H}=H\oplus H^*, \ (\text{i.e. } \hat{H}(x_1+y_1^*, x_2+y_2^*)=H(x_1, x_2)+H^*(y_1^*, y_2^*)) .$$

$\hat{V}$ admits an involution $\hat{V}\ni x=x_1+y_1^*\to x^*=y_1+x_1^*\in \hat{V}$. We set:

$$A(x, y)=-\sqrt{-1}\hat{H}(x^*, y), \qquad (x, y\in \hat{V}).$$

The $\boldsymbol{R}$-subspace of $\hat{V}$, $V_0=\{x\in \hat{V}; x=x^*\}$, is, then, supplied with an alternating form A_0 naturally induced by the above A. It is obvious that the space (V_0, A_0) is isometric to $\mathscr{R}_{\boldsymbol{C}/\boldsymbol{R}}(V, H)=(V', A')$ and that $\hat{V}$ is identifiable to the complexification $V'_{\boldsymbol{C}}$. Suppose, now, that U is an element of $D^{\mathrm{I}}(V, H)$. Then the subspace $U+(U^{\perp})^*$ of $\hat{V}$ belongs to $\boldsymbol{D}^{\mathrm{III}}(V, H)$ and is identifiable to $\rho(U)$.

For the purpose of obtaining an expression of the mapping $\rho: D^{\mathrm{I}}\to\boldsymbol{D}^{\mathrm{III}}$ utilizing coordinate systems, let $\{v_1, \cdots, v_n\}$ be a base of V over $\boldsymbol{C}$ such that $H(v_i, v_j)=\varepsilon_i\delta_{ij}$, where $\varepsilon_i=1$, for $1\leqq i\leqq t$, $\varepsilon_j=-1$ for $t+1\leqq j\leqq n$. Then, by setting $v'_i=\sqrt{-1}v_i$, $(1\leqq i\leqq t)$, $v'_j=-\sqrt{-1}\,v_j$, $(t+1\leqq j\leqq n)$ we obtain a base $\{v_1, \cdots, v_n, v'_1, \cdots, v'_n\}$ of V' over $\boldsymbol{R}$. We may, then, form a base $\{e_1, \cdots, e_n, \bar{e}_1, \cdots, \bar{e}_n\}$ of $V'_{\boldsymbol{C}}$ as in 3.1. Given an element $g\in G^{\mathrm{I}}$, expressed with respect to the base $\{v_1, \cdots, v_n\}$ as:

$$g=\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

we have $\rho(g)\in \boldsymbol{G}^{\mathrm{III}}$, expressed with respect to the base $\{e_1, \cdots, e_n, \bar{e}_1, \cdots, \bar{e}_n\}$ as follows:

$$\rho(g)=\begin{pmatrix} A & 0 & 0 & iB \\ 0 & \bar{D} & i\bar{C} & 0 \\ 0 & \bar{B}/i & \bar{A} & 0 \\ C/i & 0 & 0 & D \end{pmatrix}, \qquad (i=\sqrt{-1}).$$

Hence, we have:

$$\rho:\ D^{\mathrm{I}}\ni Z\longrightarrow i\begin{pmatrix} 0 & Z \\ {}^tZ & 0 \end{pmatrix}\in \boldsymbol{D}^{\mathrm{III}}.$$

It is also clear that we have $d\rho(\mathfrak{k}^{\mathrm{I}})\subset \mathfrak{k}^{\mathrm{III}}$, $d\rho(\mathfrak{p}^{\mathrm{I}})\subset \mathfrak{p}^{\mathrm{III}}$, and hence $\rho: D^{\mathrm{I}}\to\boldsymbol{D}^{\mathrm{III}}$ is a "holomorphic imbedding", i.e. it is a holomorphic isometry such that $\rho(D^{\mathrm{I}})$ is totally geodesic in $\boldsymbol{D}^{\mathrm{III}}$, (cf. [**10**]).

4.2. The holomorphic imbedding $\rho: D^{\mathrm{I}}\to\boldsymbol{D}^{\mathrm{III}}$ may be uniquely extended to a continuous mapping $\rho: \bar{D}^{\mathrm{I}}\to\bar{\boldsymbol{D}}^{\mathrm{III}}$ as follows. First, for an element $U\in\bar{D}^{\mathrm{I}}$ we set $W=\mathrm{rad}_H U$. Then we have the direct sum decomposition $U=W+U'$, where U' may be regarded as a subspace of $W^{\perp}/W$. We then have $U^{\perp}=W+U'^{\perp}$, where $U'^{\perp}$ is identifiable with the complement of U' in $W^{\perp}/W$. Hence, $(U^{\perp})^*=W^*+(U'^{\perp})^*$. We shall set $\rho(U)=U+(U^{\perp})^*=W+W^*+U'+(U'^{\perp})^*$; $W+W^*$ may be identified with $\mathscr{R}(W)_{\boldsymbol{C}}$. Therefore ρ maps B_W^{I} into $B_{W'}^{\mathrm{III}}$, $(W'=\mathscr{R}(W))$, and induces a holomorphic imbedding $\rho_W: D^{\mathrm{I}}(W^{\perp}/W, H_W)\to D^{\mathrm{III}}(W'^{\perp}/W', A'_{W'})$, which coincides to the holomorphic imbedding naturally induced by the functor $\mathscr{R}_{\boldsymbol{C}/\boldsymbol{R}}$. Furthermore, for $g\in G^{\mathrm{I}}$ we have $\rho(gU)=$

$\rho(g)\rho(U)$, $(U\in\bar{D}^{\mathrm{I}})$. Explicitly, we have for $Z\in\bar{D}^{\mathrm{I}}$, $\rho(Z)=\sqrt{-1}\begin{pmatrix}0 & Z\\ {}^{t}Z & 0\end{pmatrix}$.

4.3. Suppose that we are in the same situation as in 2.3, and let $(V', A')=\mathcal{R}_{K/k}(V, H)$. Then we obtain a mapping $\rho=\prod_{i=1}^{d}\rho^{\tau_i}$, sending $D^{\mathrm{I}}(\tilde{V}, \tilde{H})$ into $D^{\mathrm{III}}(\tilde{V}', \tilde{A}')$, where each ρ^{τ_i} is, by definition, the holomorphic imbedding naturally induced by $\mathcal{R}_{C/R}:(V_R^{\tau_i}, H^{\tau_i})\rightarrow(V_R'^{\tau_i}, A'^{\tau_i})$. The mapping ρ is, therefore, a holomorphic imbedding and is extendible to a continuous mapping, again denoted by ρ, sending $\bar{D}^{\mathrm{I}}(\tilde{V}, \tilde{H})$ into $\bar{D}^{\mathrm{III}}(\tilde{V}', \tilde{A}')$ which is equivariant. Furthermore, ρ induces a holomorphic imbedding sending a rational boundary component B_W^{I} of $\bar{D}^{\mathrm{I}}$ into $B_{W'}^{\mathrm{III}}$, $(W'=\mathcal{R}_{K/k}W)$. We have, therefore, $\rho(C_s^{\mathrm{I}})\subset C_{2s}^{\mathrm{III}}$.

Chapter II. Cusps of certain symmetric bounded domains.

§ 1. Cusps of D^{I}.

1.1. Let k be a totally real number field and let K be a totally imaginary quadratic extension of k. Let (V, H) be a non-degenerate Hermitian vector space over K/k with ind $(H)>0$, $G^{\mathrm{I}}=SU(V, H)$, and let $D^{\mathrm{I}}=D^{\mathrm{I}}(\tilde{V}, \tilde{H})$. Suppose that we are given an arithmetic subgroup Γ of G^{I}. A G^{I}-orbit C_s^{I} of $\bar{D}^{\mathrm{I}}$ is, then, decomposed into disjoint union of Γ-orbits, each of which is called a *cusp of level s, (with respect to Γ)*. When W is a totally isotropic subspace of dimension s in V, the G^{I}-orbit C_s^{I} may be identified with the quotient space $G^{\mathrm{I}}/G_W^{\mathrm{I}}$, and the space of cusps of level s is in a bijective correspondence with the double coset space $\Gamma\backslash G^{\mathrm{I}}/G_W^{\mathrm{I}}$. We shall choose for an arithmetic subgroup Γ a subgroup G_L^{I} of G^{I} consisting of elements stabilizing a "lattice" L in V.

1.2. Let us recall some of basic notions and properties of lattices. Let $\mathfrak{o}$, (or $\mathfrak{o}_k$) be a Dedekind domain of characteristic not 2, k its quotient field, and let K be a commutative ring with unity satisfying the requirements 1), 2) of Ch. I, § 1. Let $\mathfrak{O}_K$ denote the ring of $\mathfrak{o}$-integers in K, (when $K=k\times k$, $\mathfrak{O}_K=\mathfrak{o}\times\mathfrak{o}$).

Let V be a free K (resp. k)-module of finite rank. A sub $\mathfrak{O}_K$ (resp. $\mathfrak{o}$)-module L of V is called an $\mathfrak{O}_K$ (resp. $\mathfrak{o}$)-*lattice* in V if L is finitely generated and if it contains a base of V over K (resp. k). Specifically, when $V=K$ (resp. k), an $\mathfrak{O}_K$ (resp. $\mathfrak{o}$)-lattice in V is called an $\mathfrak{O}_K$ (resp. $\mathfrak{o}$)-*ideal* in K (resp. k).

Given a subgroup G of $GL(V)$ and a lattice L in V we set $G_L=\{g\in G; gL=L\}$. Given $\mathfrak{O}_K$ (resp. $\mathfrak{o}$)-submodules M, N of V we shall denote $M\underset{G}{\sim}N$ if and only if there exists $g\in G$ such that $N=gM$, and set $(M)_G=\{N; N$ is an $\mathfrak{O}_K$ (resp. $\mathfrak{o}$)-submodule of V, $N\underset{G}{\sim}M\}$; $(M)_G$ is called the *G-equivalence* *class* of M.

Suppose that we are in the same situation as in 1.1, $\mathfrak{o}$ is the ring of integers in k, L is an $\mathfrak{O}_K$-lattice in V and $\Gamma=G_L^{\mathrm{I}}$. Let W be a totally isotropic subspace of V with dimension s. Let us denote the space of cusps of level s with respect to G_L^{I} by $C_s^{\mathrm{I}}(L)$. Then, there exists a bijective correspondence between $C_s^{\mathrm{I}}(L)$ and $G_L^{\mathrm{I}}\backslash G^{\mathrm{I}}/G_W^{\mathrm{I}}$, and the latter double coset space may be identified with $(W)_{G^{\mathrm{I}}}/G_L^{\mathrm{I}}$. Furthermore the correspondence $G^{\mathrm{I}}\ni g\to g^{-1}\in G^{\mathrm{I}}$ induces the bijection: $G_L^{\mathrm{I}}\backslash G^{\mathrm{I}}/G_W^{\mathrm{I}}\approx G_W^{\mathrm{I}}\backslash G^{\mathrm{I}}/G_L^{\mathrm{I}}$, and the latter space is identifiable with $(L)_{G^{\mathrm{I}}}/G_W^{\mathrm{I}}$.

An $\mathfrak{O}_K$ (resp. $\mathfrak{o}$)-lattice L in V admits the decomposition:

$$L=\mathfrak{A}_1v_1+\cdots+\mathfrak{A}_nv_n\,,\qquad \text{(direct sum)}\,, \tag{1}$$

where $\mathfrak{A}_i$ are $\mathfrak{O}_K$ (resp. $\mathfrak{o}$)-ideals in K (resp. k), and $\{v_1,\cdots,v_n\}$ is a base of V. The ideal class $c(\prod_{i=1}^{n}\mathfrak{A}_i)$ is, then, uniquely determined by L and is denoted by $c_K(L)$ (resp. $c_k(L)$) or simply $c(L)$. We have $L\underset{GL(V)}{\sim}M$ if and only if $c(L)=c(M)$.

Given $\mathfrak{O}_K$ (resp. $\mathfrak{o}$)-lattices L and M in V we set $d_K(L,M)$ (resp. $d_k(L,M)$), or simply $d(L,M)$ to be $\mathfrak{O}_K$ (resp. $\mathfrak{o}$)-ideal generated by $\det(g)$ with g ranging over all the endomorphisms of V sending L into M. When L is decomposed as in (1), and when we have $M=\mathfrak{B}_1u_1+\cdots+\mathfrak{B}_nu_n$, the following formula holds:

$$d(L,M)=(\det(g))\prod_{i=1}^{n}\mathfrak{A}_i^{-1}\mathfrak{B}_i\,, \tag{2}$$

where $g\in GL(V)$ is determined by $g(v_i)=u_i$, $(1\leqq i\leqq n)$. We have, therefore,

$$c(d(L,M))=c(L)^{-1}c(M)\,. \tag{3}$$

1.3. The situation being the same as in 1.2, we let (V,H) be a non-degenerate Hermitian vector space over K/k and let L be an $\mathfrak{O}_K$-lattice in V. We shall set:

$$\mu^{\mathrm{I}}(L)=\mathfrak{O}_K\text{-ideal generated by } H(x)=H(x,x)\,,\qquad \text{for all}\quad x\in L\,,$$

$$\mu_0^{\mathrm{I}}(L)=\mathfrak{O}_K\text{-ideal generated by } H(x,y)\,,\qquad \text{for all}\quad x,y\in L\,,$$

$$L^{\#}=\{x\in V\,;\ H(L,x)\subset\mathfrak{O}_K\}\,.$$

It is known and easily seen that

$$\mathfrak{O}_K\operatorname{Tr}\mu_0^{\mathrm{I}}(L)\subset\mu^{\mathrm{I}}(L)\subset\mu_0^{\mathrm{I}}(L)\subset\mu^{\mathrm{I}}(L)\mathfrak{D}^{-1}\,,$$

where $\mathfrak{D}^{-1}=\{x\in K;\ \operatorname{Tr}(x\mathfrak{O}_K)\subset\mathfrak{o}_k\}$. The lattice L is called ($\mu_0^{\mathrm{I}}(L)$-) *modular* with respect to H if one has $L=\mu_0^{\mathrm{I}}(L)L^{\#}$. L is *normal* with respect to H if $\mu^{\mathrm{I}}(L)=\mu_0^{\mathrm{I}}(L)$.

When we are given a totally isotropic subspace W of V and an $\mathfrak{I}$-modular lattice L in V we have the following "W-decomposition" of L:

$$L=\sum_{i=1}^{s}(\mathfrak{A}_i^{-\sigma}\mathfrak{I}w_i+\mathfrak{A}_i w_i')\oplus L', \qquad \text{(orthogonal sum)}, \tag{4}$$

where $\{w_1, \cdots, w_s\}$ is a base of W, $w_i \in L$, $\mathfrak{A}_i$ are $\mathfrak{O}_K$-ideals such that $\mathfrak{A}_1 \supset \mathfrak{A}_2 \supset \cdots \supset \mathfrak{A}_s$, $H(w_i, w_j')=\delta_{ij}$, $H(w_i', w_j')=m_i\delta_{ij}$, (Proposition 3.3, [**6**]). We have, therefore, the following:

$$L \cap W=\sum_{i=1}^{s}\mathfrak{A}_i^{-\sigma}\mathfrak{I}w_i, \qquad L' \cong L \cap W^{\perp}/L \cap W; \tag{5}$$

when $W^{\perp} \supsetneqq W$, L' is an $\mathfrak{I}$-modular lattice in $W^{\perp}/W$ with respect to H_W.

We shall set, for a given $\mathfrak{I}$-modular lattice L,

$$\Sigma_H(L, W)=H(L')+\mathrm{Tr}\,\mathfrak{I}=\{H(x)+\mathrm{Tr}\,\alpha\,;\ x \in L',\ \alpha \in \mathfrak{I}\},$$
$$\Sigma_H(L)=H(L)+\mathrm{Tr}\,\mathfrak{I}\,;$$

$\Sigma_H(L, W)$ and $\Sigma_H(L)$ are submodules of $\mathfrak{o}_k$, and if k is an algebraic number field of finite degree the index $s_H(L, W)=[\Sigma_H(L) : \Sigma_H(L, W)]$ is finite, (Proposition 4.2, [**6**]). Furthermore it is easily seen that when $\Sigma_H(L)=\Sigma_H(L, W)$ we may set, in the decomposition (4), $m_i=0$, $(1 \leqq i \leqq s)$, ([**6**], 3.7). When, in particular, $\mathfrak{O}_K\,\mathrm{Tr}\,\mu_0^{\mathrm{I}}(L)=\mu^{\mathrm{I}}(L)$, it is obvious that $\Sigma_H(L)=\Sigma_H(L, W)$.

1.4. LEMMA. 1) *If* $\mathfrak{o}_k=N\mathfrak{O}_K+\mathrm{Tr}\,\mathfrak{O}_K$, *then* $\Sigma_H(L)=\mu^{\mathrm{I}}(L) \cap k$, $\Sigma_H(L, W)=\mu^{\mathrm{I}}(L \cap W^{\perp}/L \cap W) \cap k$,

2) *If k is an algebraic number field of finite degree and if* $\mathfrak{o}_k$ *is the ring of integers in k, then* $\mathfrak{o}_k=N\mathfrak{O}_K+\mathrm{Tr}\,\mathfrak{O}_K$.

PROOF. 1) Since $H(x+y)=H(x)+H(y)+\mathrm{Tr}\,(H(x, y))$, $(x, y \in V)$, $\Sigma_H(L)$ is contained in $\mu^{\mathrm{I}}(L) \cap k$. For the purpose of showing that $\Sigma_H(L)=\mu^{\mathrm{I}}(L) \cap k$, it suffices to show that $\Sigma_H(L)$ is an $\mathfrak{o}_k$-ideal, which follows from the fact that every element $a \in \mathfrak{o}_k$ may be, in our case, written as:

$$a=N\alpha+\mathrm{Tr}\,\beta, \qquad (\alpha, \beta \in \mathfrak{O}_K).$$

Indeed, for $x \in L$ and $\xi \in \mathfrak{I}$ we have:

$$\begin{aligned} a(H(x)+\mathrm{Tr}\,\xi) &= H(\alpha x)+\mathrm{Tr}\,\beta \cdot H(x)+a \cdot \mathrm{Tr}\,\xi \\ &= H(\alpha x)+\mathrm{Tr}\,(\beta H(x)+a\xi) \in \Sigma_H(L). \end{aligned}$$

It can be shown in the similar manner that $\Sigma_H(L, W)=\mu^{\mathrm{I}}(L \cap W^{\perp}/L \cap W) \cap k$.

2) When $K=k \times k$ one has $N\mathfrak{O}_K=\mathrm{Tr}\,\mathfrak{O}_K=\mathfrak{o}$. When K is a quadratic extension of k it is known that $N\mathfrak{O}_K+\mathrm{Tr}\,\mathfrak{O}_K=\mathfrak{o}$, (Lemma 4.6, [**6**]). q. e. d.

COROLLARY. *When* $\mathfrak{o}=N\mathfrak{O}_K+\mathrm{Tr}\,\mathfrak{O}_K$, $W^{\perp} \supsetneqq W$, *and when both L and $L \cap W^{\perp}/L \cap W$ are modular and normal, we have* $\Sigma_H(L)=\Sigma_H(L, W)$.

When $\dim_K V$ is odd, or when K is either $k \times k$ or a unramified quadratic extension of k, every modular lattice in V is known to be normal (Ch. I, 4.3 [4]). Hence, in such a case, if $\mathfrak{o}=N\mathfrak{O}_K+\mathrm{Tr}\,\mathfrak{O}_K$ and $W^{\perp}\supsetneqq W$, then $\Sigma_H(L)=\Sigma_H(L, W)$.

From this point, whenever K/k is a quadratic extension of a number field we shall assume that $\mathfrak{o}_k$ is the ring of integers in k.

1.5. Given an $\mathfrak{O}_K$-lattice L in a non-degenerate Hermitian vector space (V, H) over K/k we shall set:

$$C_H(L)=\{\det(g)\,;\ g\in\tilde{G}_L^{\mathrm{I}}\}\ ,$$

where $\tilde{G}^{\mathrm{I}}=U(V, H)$. When, furthermore, we are given a totally isotropic subspace W of V we set:

$$C_H(L, W)=\{\det(g)\,;\ g\in\tilde{G}_L^{\mathrm{I}}\cap\tilde{G}_W^{\mathrm{I}}\}\ .$$

When g belongs to $\tilde{G}_L^{\mathrm{I}}$ we have $d(gL, L)=d(L, L)=(\det g)=\mathfrak{O}_K$ and hence $C_H(L)$ is a subgroup of the following group:

$$U(K/k)=\{\alpha\in U_K\,;\ N\alpha=1\}\ ,$$

where U_K is the group of invertible elements of $\mathfrak{O}_K$. In particular, if L has an orthogonal summand of rank 1 we have $C_H(L)=U(K/k)$.

Let W be a totally isotropic subspace of V and let L be an $\mathfrak{J}$-modular lattice in V admitting the following W-decomposition:

$$L=\sum_{i=1}^{s}(\mathfrak{A}_i^{-\sigma}\mathfrak{J}w_i+\mathfrak{A}_i w_1')\oplus L'\ .$$

Then, for $u\in U(K/k)$, the element $g\in GL(V)$ given by:

$$g(w_1)=uw_1,\ g(w_1')=u^{-\sigma}w_1'=uw_1',\ g(w_i)=w_i,\ g(w_i')=w_i',\quad(2\leqq i\leqq s)\,,$$
$$g(x)=x\,,\qquad \text{for}\quad x\in L'\,,$$

belongs to $\tilde{G}_L^{\mathrm{I}}\cap\tilde{G}_W^{\mathrm{I}}$. Hence, $U(K/k)\supset C_H(L)\supset C_H(L, W)\supset U(K/k)^2$. Particularly when $H(w_1')=0$, (e.g. when $\Sigma_H(L)=\Sigma_H(L, W)$) one has:

$$C_H(L, W)\supset B(K/k)=\{u^{1-\sigma}\,;\ u\in U_K\}\ .$$

Suppose that K/k is a quadratic extension of a number field and let r_1, r_2 be the numbers of real and complex prime spots of K. Let $\{\zeta_m\}$ be the group of all the roots of unity in K, (ζ_m denotes a primitive m-th root of unity; m is even). Then we have $U_K\cong\{\zeta_m\}\times\boldsymbol{Z}^{r_1+r_2-1}$ and hence $[U_K: U_K^2]=2^{r_1+r_2}$. There exists, on the other hand, a natural homomorphism f: $U(K/k)/U(K/k)^2\to U_K/U_K^2$ induced by the inclusion mapping $U(K/k)\to U_K$. It is, then, easily seen that the homomorphism f is injective if there exists no element ε in U_K such that $N\varepsilon=-1$; whereas if there exists such an element ε the kernel of f is represented by 1 and ε^2. Therefore, according as whether we are in the first or second of the cases above we have

$$[U(K/k): U(K/k)^2]=2^g, \quad \text{with} \quad g \leqq r_1+r_2 \quad \text{or} \quad g \leqq r_1+r_2+1.$$

In particular when K/k is a totally imaginary quadratic extension of a totally real number field we have

$$[U(K/k): U(K/k)^2]=2^g, \qquad g \leqq d=[k:\boldsymbol{Q}].$$

Hence, when K/k is a quadratic extension of a number field and when L is modular, $C_H(L)/C_H(L, W)$ may be regarded as a quotient group of a subgroup of the finite Abelian group of type $(2, 2, \cdots, 2)$, $U(K/k)/U(K/k)^2$. In particular, when $\Sigma_H(L)=\Sigma_H(L, W)$, $C_H(L)/C_H(L, W)$ is regarded as a quotient group of a subgroup of $H^1(\mathfrak{g}(K/k), U_K)$, where $\mathfrak{g}(K/k)$ denotes the Galois group of K/k.

When K/k is a quadratic extension of a number field the order of the cohomology group $H^1(\mathfrak{g}(K/k), U_K)$, denoted by $h(K/k)$, is known to be finite and is equal to $[U_k: NU_K]/2^{d-1}$, where d is the number of real conjugates of k which are contained in complex conjugates of K, (cf. [**14**] § 13.2; [**4**] Ch. II, 1.7). When, furthermore, K/k is a totally imaginary quadratic extension of a totally real number field, we have $U_k \cong \{\pm 1\} \times \boldsymbol{Z}^{d-1}$, $(d=[k:\boldsymbol{Q}])$, and hence, (in view of the fact that $NU_K \supset U_k^2$), $h(K/k)$ is equal to $2/[NU_K: U_k^2]$ which is either 1 or 2. In particular when $k=\boldsymbol{Q}$ and K is an imaginary quadratic number field we have $U(K/k)=U_K$, $U(K/k)^2=U_K^2=B(K/k)$, and $h(K/k)=[U(K/k): U(K/k)^2]=2$.

We shall note here that when K/k is a quadratic extension of a number field it is known that $h(K/k)=h(k)2^q/|C(K/k)|$, where $h(k)=$ class number of k, $q=$ the number of prime ideals in k which ramify in K, and $C(K/k)$ is the subgroup of the $\mathfrak{O}_K$-ideal class group $C(K)$ consisting of the classes represented by σ-invariant ideals, ([**14**]).

REMARKS. 1) When K/k is a totally imaginary quadratic extension of a totally real number field, and when $h(K)=1$, it is known that $h(K/k)=2$, or, equivalently, $NU_K=U_k^2$, ([**4**]).

2) When $k=\boldsymbol{Q}(\sqrt{m})$, where m is a square-free integer such that $m \equiv 5 \pmod 8$ and $K=k(\sqrt{-1})$, one has $NU_K=U_k^2$. Particularly when $m>0$ one has $h(K/k)=2$.

3) There are examples of K/k such that $h(K/k)=1$; e.g. when $k=\boldsymbol{Q}(\sqrt{3})$ and $K=k(\sqrt{-1})$ one has $[NU_K: U_k^2]=2$ and $h(K/k)=1$.

1.6. Let K/k be a quadratic extension of a number field, (V, H) a non-degenerate Hermitian vector space over K/k, W a non-zero totally isotropic subspace of V and let L be an $\mathfrak{J}$-modular lattice in V. Let U be a totally isotropic subspace of V with $U \underset{G^{\mathrm{I}}}{\sim} W$. Then $U^{\perp}/U$ may be identified with

$W^{\perp}/W$. The association:

$$\varphi_L^{\mathrm{I}}(U)=(c(L\cap U),\ \gamma(L\cap U^{\perp}/L\cap U)),$$

where $\gamma(L\cap U^{\perp}/L\cap U)=$unitary-equivalence class of $L\cap U^{\perp}/L\cap U$, induces the mapping:

$$\Phi_{L,W}^{\mathrm{I}}: G_L^{\mathrm{I}}\backslash G^{\mathrm{I}}/G_W^{\mathrm{I}}\ni(U)_{G_L^{\mathrm{I}}}\longrightarrow\varphi_L^{\mathrm{I}}(U)\in C(K)\times C(W^{\perp}/W,\mathfrak{J}),$$

where $C(W^{\perp}/W,\mathfrak{J})$ denotes the set of unitary-equivalence classes among $\mathfrak{J}$-modular lattices in $W^{\perp}/W$. For the purpose of investigating the properties of $\Phi_{L,W}^{\mathrm{I}}$, which is sometimes written as Φ^{I}, it is convenient to utilize the following mapping:

$$\psi_W^{\mathrm{I}}(M)=(c(M\cap W),\ \ \gamma(M\cap W^{\perp}/M\cap W)),\quad M\in(L)_{G^{\mathrm{I}}}.$$

This mapping induces the following mapping:

$$\Psi_{W,L}^{\mathrm{I}}: G_W^{\mathrm{I}}\backslash G^{\mathrm{I}}/G_L^{\mathrm{I}}\ni(M)_{G_W^{\mathrm{I}}}\longrightarrow\psi_W^{\mathrm{I}}(M)\in C(K)\times C(W^{\perp}/W,\mathfrak{J}),$$

which is simply written as Ψ^{I}. We have:

$$\Psi^{\mathrm{I}}(G_W^{\mathrm{I}}gG_L^{\mathrm{I}})=\Phi^{\mathrm{I}}(G_L^{\mathrm{I}}g^{-1}G_W^{\mathrm{I}}),\qquad(g\in G^{\mathrm{I}}).$$

We denote the canonical projection: $C(K)\times C(W^{\perp}/W,\mathfrak{J})\to C(K)$ by π and set $\tilde{\Phi}^{\mathrm{I}}=\pi\circ\Phi^{\mathrm{I}}$, $\tilde{\Psi}^{\mathrm{I}}=\pi\circ\Psi^{\mathrm{I}}$.

1.7. Let us set $n=\dim_K V$, $r=\mathrm{ind}\,(H)$, $s=\dim_K W>0$ and consider the following cases:

I) $s<r$; in this case $W^{\perp}\supsetneqq W$ and $W^{\perp}/W$ is isotropic,

II) $s=r<n/2$; in this case $W^{\perp}/W$ is anisotropic and of dimension $n-2r$,

II′) $s=r$, $n=2r+1$; this is a special case of (II) and $\dim_K W^{\perp}/W=1$,

III) $s=r=n/2$; in this case $W^{\perp}=W$ and when K/k is a totally imaginary quadratic extension of a totally real number field, the rational boundary component B_W^{I} is a point.

Let us first investigate the properties of the images $\mathrm{Im}\,\Psi^{\mathrm{I}}$, $\mathrm{Im}\,\tilde{\Psi}^{\mathrm{I}}$.

Cases I), II): 1) $\tilde{\Psi}_{W,L}^{\mathrm{I}}$ is surjective.

PROOF. We shall use the "local argument". Given a prime ideal $\mathfrak{p}$ in k we set $k_{\mathfrak{p}}=$the completion of k at $\mathfrak{p}$, $\mathfrak{o}_{\mathfrak{p}}=$the ring of $\mathfrak{p}$-adic integers, $K_{\mathfrak{p}}=K\otimes k_{\mathfrak{p}}$, $V_{\mathfrak{p}}=V\otimes k_{\mathfrak{p}}$, $\mathfrak{O}_{K_{\mathfrak{p}}}=\mathfrak{O}_K\otimes\mathfrak{o}_{\mathfrak{p}}$, $L_{\mathfrak{p}}=L\otimes\mathfrak{o}_{\mathfrak{p}}$, etc. The Hermitian form H naturally induces a non-degenerate Hermitian form $H_{\mathfrak{p}}$ on $V_{\mathfrak{p}}$ and $L_{\mathfrak{p}}$ is an $\mathfrak{O}_{K_{\mathfrak{p}}}$-lattice in $V_{\mathfrak{p}}$. We have $\mu^{\mathrm{I}}(L_{\mathfrak{p}})=\mu^{\mathrm{I}}(L)_{\mathfrak{p}}$, $\mu_0^{\mathrm{I}}(L_{\mathfrak{p}})=\mu_0^{\mathrm{I}}(L)_{\mathfrak{p}}$ and for lattices L and M in V, $d(L_{\mathfrak{p}},M_{\mathfrak{p}})=d(L,M)_{\mathfrak{p}}$ holds for every prime ideal $\mathfrak{p}$ in k. It is known that L is $\mathfrak{J}$-modular if and only if $L_{\mathfrak{p}}$ is $\mathfrak{J}_{\mathfrak{p}}$-modular for every $\mathfrak{p}$, [4]. Lattices L and M in V are said to be of the same unitary (resp. special unitary)-genus when $L_{\mathfrak{p}}$ and $M_{\mathfrak{p}}$ belong to the same unitary (resp. special unitary)-equivalence class for every $\mathfrak{p}$. It is known that when V is indefinite, (and of dimension greater than 1), a special unitary genus of a lattice in V consists of only one special unitary class, [13].

Now let c be an arbitrary element of $C(K)$. Then there exists a prime ideal $\mathfrak{P}$ in K not dividing the different $\mathfrak{D}(K/k)$ such that $c(\mathfrak{P})c(L\cap W)=c$. Since the prime ideal $\mathfrak{p}=\mathfrak{P}\cap k$ does not ramify in K/k we have $\mathfrak{o}_{\mathfrak{p}}=\mathrm{Tr}\,\mathfrak{O}_{K_{\mathfrak{p}}}=N\mathfrak{O}_{K_{\mathfrak{p}}}+\mathrm{Tr}\,\mathfrak{O}_{K_{\mathfrak{p}}}$. Since, furthermore, every lattice in $V_{\mathfrak{p}}$ or in $W_{\mathfrak{p}}^{\perp}/W_{\mathfrak{p}}$ is normal, we have $\Sigma_{H_{\mathfrak{p}}}(L_{\mathfrak{p}})=\Sigma_{H_{\mathfrak{p}}}(L_{\mathfrak{p}}, W_{\mathfrak{p}})$. Hence $L_{\mathfrak{p}}$ admits the following $W_{\mathfrak{p}}$-decomposition

$$L_{\mathfrak{p}}=\sum(\mathfrak{A}_i^{-\sigma}\mathfrak{J}_{\mathfrak{p}}w_i+\mathfrak{A}_iw_i')\oplus L_{\mathfrak{p}}', \qquad H_{\mathfrak{p}}(w_i')=0\,, \qquad (1\leqq i\leqq s)\,.$$

Now, since $W^{\perp}/W$ is of a positive dimension and $L_{\mathfrak{p}}'$ is modular and normal, it admits an orthogonal summand of rank 1, [4]; $L_{\mathfrak{p}}'=\mathfrak{O}_{K_{\mathfrak{p}}}v\oplus L_{\mathfrak{p}}''$. Let us set for the above $\mathfrak{p}$,

$$M_{\mathfrak{p}}=\mathfrak{P}\mathfrak{A}_1^{-\sigma}\mathfrak{J}_{\mathfrak{p}}w_1+\mathfrak{P}^{-\sigma}\mathfrak{A}_1w_1'+\sum_{i=2}^{s}(\mathfrak{A}_i^{-\sigma}\mathfrak{J}_{\mathfrak{p}}w_i+\mathfrak{A}_iw_i')+(\Pi^{\sigma-1})v+L_{\mathfrak{p}}''\,,$$

where Π is a prime element of $\mathfrak{P}$. Then there exists a lattice N in V with $N_{\mathfrak{p}}=M_{\mathfrak{p}}$, $N_{\mathfrak{q}}=L_{\mathfrak{q}}$, (for every prime ideal $\mathfrak{q}$ different from the above $\mathfrak{p}$). N is $\mathfrak{J}$-modular, $\mu^{\mathrm{I}}(N)=\mu^{\mathrm{I}}(L)$ and since $d(L, N)=\mathfrak{O}_K$ they belong to the same special unitary-genus, (Proposition 5.2, Ch. I, [4]). Therefore, $N\underset{G^{\mathrm{I}}}{\sim}L$. We have, on the other hand, $c(N\cap W)=c(\mathfrak{P})c(L\cap W)=c$. q. e. d.

Next, let M and N be elements of $(L)_{G^{\mathrm{I}}}$ such that $c(M\cap W)=c(N\cap W)$. Let us investigate the relation between $M'=M\cap W^{\perp}/M\cap W$ and $N'=N\cap W^{\perp}/N\cap W$. We have, by Proposition 3.5 of [6], the following W-decompositions:

$$M=\sum_{i=1}^{s}(\mathfrak{A}_i^{-\sigma}\mathfrak{J}w_i+\mathfrak{A}_iw_i')\oplus M'\,, \qquad N=\sum_{i=1}^{s}(\mathfrak{A}_i^{-\sigma}\mathfrak{J}u_i+\mathfrak{A}_iu_i')\oplus N'\,.$$

Hence, $c(d(M, N))=c(d(M', N'))=1$ and therefore $c(M')=c(N')$ and there exists $g'\in GL(W^{\perp}/W)$ sending $M\cap W^{\perp}/M\cap W$ onto $N\cap W^{\perp}/N\cap W$. Particularly, when $\Sigma_H(M)=\Sigma_H(M, W)$ and $\Sigma_H(N)=\Sigma_H(N, W)$ both hold we may assume, in the above W-decompositions, that $H(w_i')=H(u_i')=0$ for $1\leqq i\leqq s$. Hence, in this case, it is obvious that there exists $\alpha\in K$ such that $N\alpha=1$ and $d(M, N)=\alpha\cdot d(M', N')=\mathfrak{O}_K$. Now, since $W^{\perp}\supsetneqq W$ there exists an element $h\in U(W^{\perp}/W, H_W)$ with $\det(h)=\alpha^{-1}$. We then have $d(hM', N')=\mathfrak{O}_K$.

When M' and N' are both normal the above assumption is automatically satisfied and furthermore hM' and N' belong to the same special unitary genus, [4].

Generally, when we are given $\mathfrak{O}_K$-lattices L and M in a non-degenerate Hermitian vector space (V, H), we shall write $L\equiv M$ if and only if there exists an element $h\in U(V, H)$ such that hL and M belong to the same special unitary genus. We shall set, furthermore,

$$\tilde{c}(L)=\text{number of unitary-equivalence classes among } \{M;\, M\equiv L\}\,.$$

$\tilde{c}(L)$ is known to be finite, and, in particular, when L is modular and normal and when $C(K)=C(K/k)$, $\tilde{c}(L)$ is equal to the unitary class number $u(L)$ of L, (cf. [5], 1.7).

The above consideration shows that when $M, N\in(L)_{G^{\mathrm{I}}}$, $c(M\cap W)=c(N\cap W)$, and when $M'=M\cap W^{\perp}/M\cap W$ and $N'=N\cap W^{\perp}/N\cap W$ are both normal, one has $M'\equiv N'$.

We shall now show the following:

2) Let M be an element of $(L)_{G^{\mathrm{I}}}$ and assume that $M'=M\cap W^{\perp}/M\cap W$ is normal. If $N\in(L)_{G^{\mathrm{I}}}$ and if $N'=N\cap W^{\perp}/N\cap W$ is normal and $c(M\cap W)=c(N\cap W)$ one has $M'\equiv N'$. Conversely, if N' is a lattice in $W^{\perp}/W$ such that $N'\equiv M'$ there exists a lattice N in V such that $N\in(L)_{G^{\mathrm{I}}}$ and $c(M\cap W)=c(N\cap W)$.

PROOF. It is sufficient to show the latter half of the assertion. Let N' be a lattice in $W^{\perp}/W$ with $N'\equiv M'$. We shall again utilize the W-decomposition:

$$M=\sum_{i=1}^{s}(\mathfrak{A}_i^{-\sigma}\mathfrak{J}w_i+\mathfrak{A}_iw_i')\oplus M', \qquad H(w_i')=0, \quad (1\leqq i\leqq s).$$

N' is, by the assumption, modular and normal, and furthermore $d(M', N')=(b^{1-\sigma})$, for an element $b\in K^*$. Let us set:

$$N=(b^{-1}\mathfrak{A}_1^{-\sigma}\mathfrak{J}w_1+b^{\sigma}\mathfrak{A}_1w_1'+\sum_{i=2}^{s}(\mathfrak{A}_i^{-\sigma}\mathfrak{J}w_i+\mathfrak{A}_iw_i')\oplus N'.$$

Then N is $\mathfrak{J}$-modular normal and $d(M, N)=\mathfrak{O}_K$. Hence M and N belong to the same G^{I}-genus. Since V is indefinite they are mutually G^{I}-equivalent.

q. e. d.

As corollaries to 2) above we obtain:

2′) Suppose that we are in case I) and that every $\mathfrak{J}$-modular lattice in $W^{\perp}/W$ is normal; e.g. $\dim_K V$ is odd or K/k is unramified. Then for each $c\in C(K)$ there exists one and only one $\gamma_c\in C(W^{\perp}/W, \mathfrak{J})$ with $(c, \gamma_c)\in \operatorname{Im}\Psi^{\mathrm{I}}$.

2″) Suppose that we are in case II′). Then for each $c\in C(K)$ there exists one and only one $\gamma_c\in C(W^{\perp}/W, \mathfrak{J})$ with $(c, \gamma_c)\in \operatorname{Im}\Psi^{\mathrm{I}}$.

Next we shall consider the case where $L'=L\cap W^{\perp}/L\cap W$ admits an orthogonal summand of rank 1. Case II′) is contained in such a case. L and L' are, in this case, both modular and normal and hence $\Sigma_H(L)=\Sigma_H(L, W)$. Furthermore, in this case we have $C_H(L, W)=U(K/k)$. We now have W-decomposition:

$$L=\sum_{i=1}^{s}(\mathfrak{A}_i^{-\sigma}\mathfrak{J}w_i+\mathfrak{A}_iw_i')\oplus L', \quad L'=\mathfrak{A}v\oplus L'', \quad H(w_i')=0, \quad (1\leqq i\leqq s).$$

Suppose that we are given $M\in(L)_{G^{\mathrm{I}}}$ and that $c(M\cap W)=c(\mathfrak{B})c(L\cap W)$. Then by setting:

$$N=\mathfrak{B}\mathfrak{A}_1^{-\sigma}\mathfrak{J}w_1+\mathfrak{B}^{-\sigma}\mathfrak{A}_1w_1'+\sum_{i=2}^{s}(\mathfrak{A}_i^{-\sigma}\mathfrak{J}w_i+\mathfrak{A}_iw_i')+\mathfrak{B}^{\sigma-1}\mathfrak{A}v+L'',$$

we obtain an element $N\in(L)_{G^{\mathrm{I}}}$ such that $c(N\cap W)=c(M\cap W)$. Since $N\cap W^{\perp}/N\cap W$ is normal, if $M\cap W^{\perp}/M\cap W$ is also normal we have, by 2), $M\cap W^{\perp}/M\cap W\equiv N\cap W^{\perp}/N\cap W$. If in particular $\tilde{c}(M\cap W^{\perp}/M\cap W)=1$; e.g. cases I) or II′), we have $C_H(M, W)=U(K/k)$.

Case III): Given $M\in(L)_{G^{\mathrm{I}}}$ we have W-decompositions:

$$L=\sum_{i=1}^{s}(\mathfrak{A}_i^{-\sigma}\mathfrak{J}w_i+\mathfrak{A}_iw_i'),\qquad M=\sum_{i=1}^{s}(\mathfrak{B}_i^{-\sigma}\mathfrak{J}u_i+\mathfrak{B}_iu_i').$$

Hence, $c(d(L, M))=c(\prod_{i=1}^{s}(\mathfrak{A}_i^{-1}\mathfrak{B}_i)^{1-\sigma})=1$, and therefore $c(L\cap W)^{-1}c(M\cap W)=c(\prod_{i=1}^{s}(\mathfrak{A}_i\mathfrak{B}_i^{-1})^{\sigma})$ is σ-invariant. We shall denote the subgroup of $C(K)$ consisting of σ-invariant elements by $C_0(K/k)$.

When in particular $\mu^{\mathrm{I}}(L)=\mathfrak{O}_K\operatorname{Tr}\mu_0^{\mathrm{I}}(L)$, the same condition holds for every $M\in(L)_{G^{\mathrm{I}}}$, and hence, in the W-decomposition above, we may assume that $H(w_i')=H(u_i')=0$, for $1\leqq i\leqq s$. Hence, in this case, the linear transformation g of V determined by $g(w_i)=u_i$, $g(w_i')=u_i'$, $(1\leqq i\leqq s)$ is unitary. Therefore there exists $a\in K^*$ with $Na=1$ and $d(L, M)=a\prod_{i=1}^{s}(\mathfrak{A}_i^{-1}\mathfrak{B}_i)^{1-\sigma}$. There exists $b\in K^*$ such that $a=b^{1-\sigma}$. Since $d(L, M)=\mathfrak{O}_K$ the ideal $b\prod_{i=1}^{s}(\mathfrak{A}_i^{-1}\mathfrak{B}_i)$ is σ-invariant. Hence, in this case, $c(L\cap W)^{-1}c(M\cap W)\in C(K/k)$. Conversely, suppose that $c_0\in C(K/k)$ and let $c=c(L\cap W)c_0$, $c_0=c(\mathfrak{A})$ with $\mathfrak{A}^{\sigma}=\mathfrak{A}$. Then, since $H(w_i')=0$, $(1\leqq i\leqq s)$, we obtain by setting:

$$N=\mathfrak{A}\mathfrak{A}_1^{-\sigma}\mathfrak{J}w_1+\mathfrak{A}^{-\sigma}\mathfrak{A}_1w_1'+\sum_{i=2}^{s}(\mathfrak{A}_i^{-\sigma}\mathfrak{J}w_i+\mathfrak{A}_iw_i'),$$

an $\mathfrak{J}$-modular lattice N such that $\mu_0^{\mathrm{I}}(N)=\mu_0^{\mathrm{I}}(L)$, $\mu^{\mathrm{I}}(N)=\mu^{\mathrm{I}}(L)$ and $d(L, N)=\mathfrak{O}_K$. If in the above the ideal $\mathfrak{A}$ may be expressed as the quotient of two integral ideals both of which being relatively prime to the different $\mathfrak{D}$, then $N\in(L)_{G^{\mathrm{I}}}$. Indeed, for a prime ideal $\mathfrak{p}$ in k such that $\mathfrak{A}_{\mathfrak{p}}=(1)$ we have $N_{\mathfrak{p}}=L_{\mathfrak{p}}$, whereas when $A_{\mathfrak{p}}\neq(1)$, $\mathfrak{p}$ is unramified in K/k and therefore both $L_{\mathfrak{p}}$ and $N_{\mathfrak{p}}$ are normal. Therefore N and L belong to the same G^{I}-genus which now consists of only one G^{I}-class.

LEMMA. *Suppose that* $c=c(\mathfrak{A})\in C(K/k)$ *with* $\mathfrak{A}^{\sigma}=\mathfrak{A}$. *Then* $\mathfrak{A}$ *may be expressed as the quotient of two integral ideals both of which being relatively prime to the different* $\mathfrak{D}$ *if and only if there exists an ideal* $\mathfrak{a}$ *in* k *such that* $c=c(\mathfrak{a})$, (*where* $\mathfrak{a}$ *and* $\mathfrak{a}\mathfrak{O}_K$ *are identified*).

PROOF. A σ-invariant integral ideal $\mathfrak{B}$ may be written as $\mathfrak{B}=\mathfrak{B}_0\mathfrak{B}_1$ where $\mathfrak{B}_0$ is an ideal in k and $\mathfrak{B}_1=\prod\mathfrak{Q}_i$ with $\mathfrak{Q}_i$ ranging over the ramifying prime ideals in K dividing $\mathfrak{B}$. On the other hand, for a given prime ideal $\mathfrak{p}$ in k

(ramifying in K/k), there exists an ideal $\mathfrak{a}$ in k relatively prime to any ramifying ideal such that $\mathfrak{p}$ and $\mathfrak{a}$ are mutually equivalent. Lemma is, thus, evident. q. e. d.

We set:

$$C_1(K/k)=\{c(\mathfrak{a})\in C(K)\,;\ \mathfrak{a} \text{ is an ideal in } k\}\,.$$

When, in particular, every ramifying prime ideal in K is principal we have $C_1(K/k)=C(K/k)$. We have thus obtained the following:

3) When we are in case III) one has:

$$C(L\cap W)^{-1}\operatorname{Im}\tilde{\Psi}^{\mathrm{I}}\subset C_0(K/k)\,.$$

When furthermore $\mu^{\mathrm{I}}(L)=\mathfrak{O}_K \operatorname{Tr}\mu_0^{\mathrm{I}}(L)$ one has:

$$C_1(K/k)\subset c(L\cap W)^{-1}\operatorname{Im}\tilde{\Psi}^{\mathrm{I}}\subset C(K/k)\,.$$

As we have seen in 1.5 of this chapter one has:

$$|C(K/k)|=2^q h(k)/h(K/k)\,,$$

where q is the number of prime ideals in k ramifying in K. In particular, when K/k is a totally imaginary quadratic extension of a totally real number fied one has:

$$|C(K/k)|=2^{q-1}h(k)[NU_K:U_k^2]\,.$$

When furthermore $h(k)=1$, $C(K/k)$ is an Abelian group of type $(2,2,\cdots,2)$ generated by $c(\mathfrak{O}_1),\cdots,c(\mathfrak{O}_q)$, where $\mathfrak{O}_i$ are the prime ideals in K ramifying in K/k. Furthermore in this case $|C(K/k)|=2^q$ or 2^{q-1} according as whether $h(K/k)=1$ or 2. Specifically when $q=1$, $|C(K/k)|=2$ or 1 according as whether $h(K/k)=1$ or 2.

Particularly when $k=\boldsymbol{Q}$ and K is an imaginary quadratic number field, one has $h(K/k)=2$ and $|C(K/k)|=2^{q-1}$. Let $K=\boldsymbol{Q}(\sqrt{-m})$, where m is a squarefree positive integer. Then $|C(K/k)|=|C_1(K/k)|=1$ if and only if m is either 1 or prime number $p\equiv 3 \pmod 4$,

We shall now investigate the properties of inverse images $(\Psi^{\mathrm{I}})^{-1}(c,\gamma)$, $((c,\gamma)\in C(K)\times C(W^{\perp}/W,\mathfrak{J}))$. Suppose that M and N are elements of $(L)_{G^{\mathrm{I}}}$ such that $c(M\cap W)=c(N\cap W)$, $\gamma(M\cap W^{\perp}/M\cap W)=\gamma(N\cap W^{\perp}/N\cap W)$. Then, by Proposition 3.5 of [6] we have the following W-decompositions:

$$M=\sum_{i=1}^{s}(\mathfrak{A}_i^{-\sigma}\mathfrak{J}w_i+\mathfrak{A}_i w_i')\oplus M'\,,\qquad N=\sum_{i=1}^{s}(\mathfrak{A}_i^{-\sigma}\mathfrak{J}u_i+\mathfrak{A}_i u_i')\oplus N'\,,$$

and $\gamma(M')=\gamma(N')$, i. e. there exists a unitary transformation g' sending M' onto N'. If furthermore we have $H(w_i')=H(u_i')$, $(1\leqq i\leqq s)$ then the element $g\in GL(V)$ given by: $g(w_i)=u_i$, $g(w_i')=u_i'$, $(1\leqq i\leqq s)$, $g|M'=g'$, belongs to $\tilde{G}_W^{\mathrm{I}}$. Since $M,N\in(L)_{G^{\mathrm{I}}}$ we have, furthermore, $d(M,N)=(\det g)=\mathfrak{O}_K$ and hence $\det g\in U(K/k)$. It is, on the other hand, clear that when we are given an $\mathfrak{O}_K$-lattice M in V and an element $g \in \tilde{G}^{\mathrm{I}}_W$ one has $gM \underset{G^{\mathrm{I}}_W}{\sim} M$ if and only if $\det g \in C_H(M, W)$. It is known that $[\Sigma_H(M) : \Sigma_H(M, W)] \leqq (N_{K/Q}\mathfrak{D})^{\frac{1}{2}}$, (Proposition 4.9, [6]). Hence we have:

$$|(\Psi^{\mathrm{I}})^{-1}(c, \gamma)| \leqq (N_{K/Q}\mathfrak{D})^{s/2}[U(K/k) : U(K/k)^2],$$

$(c = c(M \cap W), \gamma = \gamma(M'))$. When, particularly, $\Sigma_H(M) = \Sigma_H(M, W)$ one has $|(\Psi^{\mathrm{I}})^{-1}(c, \gamma)| \leqq h(K/k)$. Furthermore, when $U(K/k) = C_H(M, W)$ there exists only one point in $(\Psi^{\mathrm{I}})^{-1}(c, \gamma)$.

When we are in case III) and $M \in (L)_{G^{\mathrm{I}}}$, we have $C_H(M, W) \subset B(K/k)$. Indeed, we have W-decomposition:

$$M = \sum_{i=1}^{s} (\mathfrak{A}_i^{-\sigma}\mathfrak{J}w_i + \mathfrak{A}_i w_i'),$$

and with respect to the base $\{w_1, \cdots, w_s, w_1', \cdots, w_s'\}$ of V an element g of $\tilde{G}^{\mathrm{I}}_M \cap \tilde{G}^{\mathrm{I}}_W$ is expressed as:

$$g = \begin{pmatrix} A & * \\ 0 & {}^tA^{-\sigma} \end{pmatrix}.$$

Furthermore, since g sends $M \cap W$ onto itself, $\det A$ belongs to U_K, and hence $\det g = \det A \cdot \det {}^tA^{-\sigma} = (\det A)^{1-\sigma} \in B(K/k)$.

If, in particular, $B(K/k) = U(K/k)^2$, (e.g. $k = \boldsymbol{Q}$, K = imaginary quadratic number field), or if $\Sigma_H(M) = \Sigma_H(M, W)$, one has $C_H(M, W) = B(K/k)$.

Combining the above we obtain the following:

1.8. THEOREM. *Let K/k be a totally imaginary quadratic extension of a totally real number field, $\mathfrak{o}_k$ the ring of integers in k, and let (V, H) be a non-degenerate Hermitian vector space over K/k. We set $n = \dim_K V$, $r = \mathrm{ind}(H)$ and assume that $r > 0$. Let W be a totally isotropic subspace of V of dimension $s > 0$ and suppose that we are given an $\mathfrak{J}$-modular lattice L in V. Then the mappings:*

$$\Phi^{\mathrm{I}} : C^{\mathrm{I}}_s(L) = G^{\mathrm{I}}_L \backslash G^{\mathrm{I}} / G^{\mathrm{I}}_W \longrightarrow C(K) \times C(W^{\perp}/W, \mathfrak{J}),$$

$$\tilde{\Phi}^{\mathrm{I}} : C^{\mathrm{I}}_s(L) \longrightarrow C(K)$$

have the following properties:

1) *For $(c, \gamma) \in C(K) \times C(W^{\perp}/W, \mathfrak{J})$ one has*
$|(\Phi^{\mathrm{I}})^{-1}(c, \gamma)| \leqq N_{K/Q}(\mathfrak{D})^{s/2} \cdot [U(K/k) : U(K/k)^2]$;

2) *Case: $s < r$:* i) *$\tilde{\Phi}^{\mathrm{I}}$ is surjective;*

ii) *When n is odd or K/k is unramified, for each $c \in C(K)$ there exists one and only one $\gamma_c \in C(W^{\perp}/W, \mathfrak{J})$ such that $(c, \gamma_c) \in \mathrm{Im}\,\Phi^{\mathrm{I}}$. Furthermore, in this case, one has $|(\Phi^{\mathrm{I}})^{-1}(c, \gamma_c)| = |(\tilde{\Phi}^{\mathrm{I}})^{-1}(c)| \leqq h(K/k)$;*

iii) *If $L \cap W^{\perp}/L \cap W$ admits an orthogonal summand of rank 1, and if n is odd or K/k is unramified, then the mapping $\tilde{\Phi}^{\mathrm{I}}$ is bijective;*

Case: $s=r<n/2$: i) $\tilde{\Phi}^{\mathrm{I}}$ *is surjective;*

ii) *When n is odd or K/k is unramified, for each (c, γ) one has:*

$$|(\Phi^{\mathrm{I}})^{-1}(c, \gamma)| \leqq h(K/k).$$

Furthermore, in this case, for $U \in (W)_{G^{\mathrm{I}}}$ and an $\mathfrak{J}$-modular lattice M' in $U^{\perp}/U(=W^{\perp}/W)$, one has:

$$(c(L \cap U), \gamma(M')) \in \operatorname{Im} \Phi^{\mathrm{I}} \iff M' \equiv L', \qquad (L' = L \cap U^{\perp}/L \cap U).$$

Case: $s=r$, $n=2r+1$: $\tilde{\Phi}^{\mathrm{I}}$ is bijective.

Case: $s=r=n/2$: One has $c(L \cap W)^{-1} \operatorname{Im} \tilde{\Phi}^{\mathrm{I}} \subset C_0(K/k)$. When, in particular, $\mu^{\mathrm{I}}(L) = \mathfrak{O}_K \operatorname{Tr} \mu_0^{\mathrm{I}}(L)$, one has:

$$C_1(K/k) \subset c(L \cap W)^{-1} \operatorname{Im} \tilde{\Phi}^{\mathrm{I}} \subset C(K/k),$$

and furthermore, $|(\tilde{\Phi}^{\mathrm{I}})^{-1}(c)| \leqq h(K/k)$.

COROLLARY 1. *Let $k=\boldsymbol{Q}$ and $K=$imaginary quadratic number field. If $(u, |U_K|)=1$ then Φ^{I} is injective.*

PROOF. U_k is, in our case, a finite cyclic group generated by a primitive m-th root of unity ζ_m with $m=2$, 4 or 6. Hence n is odd and therefore $\Sigma_H(M)=\Sigma_H(M, W)$ holds for every $M \in (L)_{G^{\mathrm{I}}}$. Furthermore, the condition "$(n, |U_K|)=1$" implies the existence of d such that $\zeta_m^{nd}=\zeta_m$. Since the scalar mapping $\zeta_m^d 1_V$ belongs to $\tilde{G}_M^{\mathrm{I}} \cap \tilde{G}_W^{\mathrm{I}}$ for every $M \in (L)_{G^{\mathrm{I}}}$, one has $C_H(M, W) = U_K = U(K/k)$. In view of 1.7 this proves the Corollary. q. e. d.

COROLLARY 2. *Suppose that $k=\boldsymbol{Q}$, $K=\boldsymbol{Q}(\sqrt{-m})$, where m is a square-free positive integer, and that $\dim_K V=2$, $\dim_K W=1$, L is $\mathfrak{O}_K$-modular and $c(L \cap W)=1$. Then the following assertions hold:*

1) *Case $m=1$:* $\operatorname{Im} \tilde{\Phi}^{\mathrm{I}}=\{1\}$, *and* $|(\tilde{\Phi}^{\mathrm{I}})^{-1}(1)|=1$ *or* 2 *according as whether* $\mu^{\mathrm{I}}(L)=(2)$ *or* $\mathfrak{O}_K$;

2) *Case $m=2$:* $\operatorname{Im} \tilde{\Phi}^{\mathrm{I}}=\{1\}$, $|(\tilde{\Phi}^{\mathrm{I}})^{-1}(1)|=2$;

3) *Case $m=3$:* $\operatorname{Im} \tilde{\Phi}^{\mathrm{I}}=\{1\}$, $\tilde{\Phi}^{\mathrm{I}}$ *is injective;*

4) *Case $m \equiv 1$ or 2 (mod 4), $C_0(K/k)=\{1\}$, L is normal:* $\operatorname{Im} \tilde{\Phi}^{\mathrm{I}}=\{1\}$, $|(\tilde{\Phi}^{\mathrm{I}})^{-1}(1)|=2$;

5) *Case $m \equiv 3$ (mod 4), $m>3$:* $|(\tilde{\Phi}^{\mathrm{I}})^{-1}(c)|=2$ *for every* $c \in \operatorname{Im} \tilde{\Phi}^{\mathrm{I}}$.

Particularly when m is a prime number $p \equiv 3$ (mod 4), $(p>3)$, $\operatorname{Im} \tilde{\Phi}^{\mathrm{I}}=\{1\}$.

For the proof of the above we need some preparations.

1.9. Suppose, generally, that we are given a non-degenerate Hermitian vector space (V, H) over K/k and an $\mathfrak{O}_K$-lattice L in V. Then, for $g \in \tilde{G}^{\mathrm{I}}$ we obviously have:

$$gL \underset{G^{\mathrm{I}}}{\sim} L \iff \det g \in C_H(L).$$

Suppose, in particular, that $n=2s$; i.e. V admits a totally isotropic subspace W such that $\dim_K V=2\dim_K W$. Suppose, furthermore, that L is an $\mathfrak{J}$-modular lattice and let $M\in(L)_{G^{\mathrm{I}}}$. We have a W-decomposition:

$$M=\sum_{i=1}^{s}(\mathfrak{A}_i^{-\sigma}\mathfrak{J}w_i+\mathfrak{A}_iw_i').$$

Suppose, now, that for every $N\in(L)_{G^{\mathrm{I}}}$ such that $c(N\cap W)=c(M\cap W)$ we have a W-decomposition:

$$N=\sum_{i=1}^{s}(\mathfrak{A}_i^{-\sigma}\mathfrak{J}u_i+\mathfrak{A}_iu_i'),$$

with $H(u_i')=H(w_i')$, $(1\leqq i\leqq s)$. Let $\{\alpha_1,\cdots,\alpha_t\}$ be a complete system of representatives of $C_H(M)/C_H(M,W)$. If, in this case, there exist $g_1,\cdots,g_t\in\tilde{G}^{\mathrm{I}}$ with $\det(g_i)=\alpha_i$, $(1\leqq i\leqq t)$ such that $g_1M,\cdots,g_tM$ are mutually distinct, then we have:

$$|(\tilde{\Phi}^{\mathrm{I}})^{-1}(c(M\cap W))|=[C_H(M):C_H(M,W)].$$

For the purpose of explicitly calculating $C_H(M)$ we utilize the following:

(*) For $\mathfrak{J}$-modular lattices M, N in V and $g\in\tilde{G}^{\mathrm{I}}$, if $gM\subset N$ one has $gM=N$. This assertion immediately follows from the following:

LEMMA. *Let $\mathfrak{o}$ be a Dedekind domain with the characteristic not 2, k its quotient field, K its quadratic extension, (V,H) a non-degenerate Hermitian vector space over K/k and let L, M be $\mathfrak{J}$-modular lattices in V. If $M\subset L$ then $L=M$.*

PROOF. $\mathfrak{J}L^{\#}\subset\mathfrak{J}M^{\#}=M\subset L=\mathfrak{J}L^{\#}$. q. e. d.

1.10. PROOF OF COROLLARY 2: We have, in our case, $U_K=U(K/k)$, $B(K/k)=U(K/k)^2$, $[U(K/k):U(K/k)^2]=2$. On the other hand, since $\mathfrak{O}_K\mathrm{Tr}\,\mu_0^{\mathrm{I}}(L)\subset\mu^{\mathrm{I}}(L)\subset\mu_0^{\mathrm{I}}(L)$ and $\mu_0^{\mathrm{I}}(L)=\mathfrak{O}_K$, $\mu^{\mathrm{I}}(L)$ is either $\mathfrak{O}_K$ or (2); particularly when $m\equiv3\pmod 4$, $\mu^{\mathrm{I}}(L)$ must be $\mathfrak{O}_K$.

Case $m=1$: Since $h(K)=1$, $\mathrm{Im}\,\tilde{\Phi}^{\mathrm{I}}=\{1\}$.

i) $\mu^{\mathrm{I}}(L)=(2)$: In this case, $\mu^{\mathrm{I}}(L)=\mathfrak{O}_K\mathrm{Tr}\,\mu_0^{\mathrm{I}}(L)$. For an element $M\in(L)_{G^{\mathrm{I}}}$ we have a W-decomposition:

$$M=\mathfrak{O}_Kw+\mathfrak{O}_Kw',\qquad H(w')=0. \tag{1}$$

In view of 1.9 an element $g\in GL(V)$, represented with respect to the base $\{w,w'\}$ as $g=\begin{pmatrix}a&c\\b&d\end{pmatrix}$ belongs to $\tilde{G}_M^{\mathrm{I}}$ if and only if the following conditions are satisfied:

$$a^\sigma b+ab^\sigma=0,\quad c^\sigma d+cd^\sigma=0,\quad a^\sigma d+b^\sigma c=1,\quad a,b,c,d\in\mathfrak{O}_K. \tag{2}$$

For the purpose of showing that $\tilde{\Phi}^{\mathrm{I}}$ is injective it suffices to show that $C_H(M)=U_k^2=\{\pm1\}$, (cf. 1.9). Suppose, in the above, that $a=0$. Then $b^\sigma c=1$ and hence $b,c\in U_K$, $\det(g)=-bc=-b^{1-\sigma}=\pm1$. Similarly if $b=0$ we have $\det(g)=\pm1$. Suppose therefore that $ab\neq0$. We then have $a^\sigma\det(g)=a^\sigma(ad-bc)=a(a^\sigma d+b^\sigma c)=a$, whence $\det(g)=a^{1-\sigma}$. Similarly $\det(g)=-b^{1-\sigma}$.

Suppose now that $\det(g)=\zeta=\sqrt{-1}$. Then we must have $a=a_0(1+\zeta)$, $b=b_0(1+\zeta)^3$, $(a_0, b_0\in \boldsymbol{Z})$ and hence $\det(g)\in(1+\zeta)\mathfrak{O}_K$. This implies that $\det(g) \notin U_K$ which is a contradiction.

ii) $\mu^{\mathrm{I}}(L)=\mathfrak{O}_K$: We now have $\boldsymbol{Q}\cap\mu^{\mathrm{I}}(M)=\boldsymbol{Z}$ and $\mathrm{Tr}\,\mu_0^{\mathrm{I}}(M)=(2)$. Replacing, if necessary, the element w' by $\alpha w+w'$, $(\alpha\in\mathfrak{O}_K)$ we may assume that, in W-decomposition (1), $H(w')=1$. A linear transformation $g=\begin{pmatrix} a & c \\ b & d\end{pmatrix}$ then belongs to $\tilde{G}_M^{\mathrm{I}}$ if and only if the following conditions are satisfied:

$$(3)\qquad a^\sigma b+ab^\sigma+Nb=0,\quad c^\sigma d+cd^\sigma+Nd=1,\quad a^\sigma d+b^\sigma c+b^\sigma d=1,\qquad a, b, c, d\in\mathfrak{O}_K .$$

Since, on the other hand,

$$(4)\qquad H(w-w')=-1\,,\qquad H(w-w', w')=0\,,$$

the linear transformation h given by:

$$(5)\qquad h(w-w')=w-w'\,,\qquad h(w')=\zeta w'\,,$$

belongs to $\tilde{G}_M^{\mathrm{I}}$ and $\det(h)=\zeta$. Hence, $C_H(M)=U_K$ and $C_H(M)/C_H(M, W)$ is represented by ζ. For the purpose of showing that $|(\tilde{\Phi}^{\mathrm{I}})^{-1}(1)|=2$ it suffices to show the existence of $g\in\tilde{G}^{\mathrm{I}}$ such that $\det(g)=\zeta$ and $gM\neq M$. Such an element is given by setting:

$$g=\begin{pmatrix} 1/(1-\sqrt{-1}) & -1/2 \\ 0 & 1+\sqrt{-1}\end{pmatrix}.$$

Case $m=2$: Since $h(K)=1$ we have $\mathrm{Im}\,\tilde{\Phi}^{\mathrm{I}}=\{1\}$. $U_K=\{\pm 1\}$, $U_K^2=\{1\}$.

i) $\mu^{\mathrm{I}}(L)=(2)$: An element $M\in(L)_{G^{\mathrm{I}}}$ admits a W-decomposition (1) with $H(w')=0$. The linear transformation $\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$ then belongs to $\tilde{G}_M^{\mathrm{I}}$ and hence $C_H(M)=U_K$. Furthermore, $g=\begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 0\end{pmatrix}\in\tilde{G}^{\mathrm{I}}$, $\det(g)=-1$ and $gM\neq M$. Hence $|(\tilde{\Phi}^{\mathrm{I}})^{-1}(1)|=2$.

ii) $\mu^{\mathrm{I}}(L)=\mathfrak{O}_K$: M admits a W-decomposition (1) with $H(w')=1$. Hence, similarly as in the case ii) of the previous case, $C_H(M)=U_K$. By setting:

$$(6)\qquad g=\begin{pmatrix} -1/\sqrt{-m} & (m-1)/2\sqrt{-m} \\ 0 & \sqrt{-m}\end{pmatrix},$$

we obtain an element g of $\tilde{G}^{\mathrm{I}}$ such that $\det(g)=-1$ and $gM\neq M$. Hence, $|(\tilde{\Phi}^{\mathrm{I}})^{-1}(1)|=2$.

Case $m=3$: Since $h(K)=1$, $\mathrm{Im}\,\tilde{\Phi}^{\mathrm{I}}=1$. $U_K=\{\zeta\}$, $\zeta=e^{\pi\sqrt{-1}/3}$, $U_K^2=\{\zeta^2\}$. Since $\mathrm{Tr}\,\mathfrak{O}_K=\boldsymbol{Z}$, L must be normal. We also have $\mu^{\mathrm{I}}(L)=\mathfrak{O}_K\mathrm{Tr}\,\mu_0^{\mathrm{I}}(L)$. Hence an element $M\in(L)_{G^{\mathrm{I}}}$ admits a W-decomposition (1) with $H(w')=0$. We assert that $C_H(M)=U_K^2$ which implies the injectivity of $\tilde{\Phi}^{\mathrm{I}}$. Suppose $g=\begin{pmatrix} a & c \\ b & d\end{pmatrix}\in\tilde{G}_M^{\mathrm{I}}$

and that $\det(g)=\zeta$. Then, similarly as in the case of $m=1$ we must have:

$$ab\neq 0\,,\quad a=a_0(1+\zeta)\,,\quad b=b_0(1+\zeta)^4\,,\qquad (a_0,\, b_0\in \boldsymbol{Z})$$

whence $\det(g)\in(1+\zeta)\mathfrak{O}_K$, which is a contradiction.

Case $m\equiv 1$ or $2 \pmod 4$, $C_0(K/k)=1$ and L is normal: By Theorem 1.8, $\mathrm{Im}\,\tilde{\Phi}^{\mathrm{I}}=\{1\}$. Since, in this case, $\mathrm{Tr}\,\mathfrak{O}_K=(2)$ an element $M\in(L)_{G^{\mathrm{I}}}$ admits a W-decomposition (1) with $H(w')=1$. We may assume that $m>1$. Hence, similarly as in the case of $m=1$, $C_H(M)=U_K=\{\pm 1\}$. An element g described in (6) belongs to $\tilde{G}^{\mathrm{I}}$ and $\det(g)=-1$, $gM\neq M$. Hence, $|(\tilde{\Phi}^{\mathrm{I}})^{-1}(1)|=2$.

Case $m\equiv 3 \pmod 4$, $m>3$: $U_K=\{\pm 1\}$, $U_K^2=\{1\}$, $\mathrm{Tr}\,\mathfrak{O}_K=\boldsymbol{Z}$. Hence, $\mu^{\mathrm{I}}(L)=\mathfrak{O}_K\mathrm{Tr}\,\mu_0^{\mathrm{I}}(L)$, and $M\in(L)_{G^{\mathrm{I}}}$ admits a W-decomposition $M=\mathfrak{A}^{-\sigma}w+\mathfrak{A}w'$, $H(w')=0$. An element $g=\begin{pmatrix} a & c\\ b & d\end{pmatrix}\in GL(V)$ belongs to $\tilde{G}_M^{\mathrm{I}}$ if and only if it satisfies the following:

$$(7)\qquad a^\sigma b+ab^\sigma=0\,,\ c^\sigma d+cd^\sigma=0\,,\ a^\sigma d+b^\sigma c=1\,,\ a,d\in\mathfrak{O}_K\,,\ b\in N\mathfrak{A}\,,\ c\in N\mathfrak{A}^{-1}\,.$$

Let $N\mathfrak{A}=(\alpha)$, $(\alpha\in\boldsymbol{Z})$. Then $g=\begin{pmatrix} 0 & \alpha^{-1}\\ \alpha & 0\end{pmatrix}$ belongs to $\tilde{G}_M^{\mathrm{I}}$ and $\det(g)=1$. Hence, $C_H(M)=U_K$. Since, furthermore, $h=\begin{pmatrix} 0 & 2\alpha^{-1}\\ \alpha/2 & 0\end{pmatrix}$ belongs to $\tilde{G}^{\mathrm{I}}$ and $\det(h)=-1$, $hM\neq M$, one has, for every $c\in\mathrm{Im}\,\tilde{\Phi}^{\mathrm{I}}$, $|(\tilde{\Phi}^{\mathrm{I}})^{-1}(c)|=2$.

When, in particular, $m=p\equiv 3 \pmod 4$, p is the only prime which ramifies in K. Therefore, as we saw in 1.7, $C(K/k)=\{1\}$, whence $\mathrm{Im}\,\tilde{\Phi}^{\mathrm{I}}=\{1\}$. q.e.d.

§ 2. Cusps of D^{III}.

2.1. Let k be a totally real number field and let (V, A) be a non-degenerate alternating vector space over k. Let $G^{\mathrm{III}}=Sp(V, A)$ and let $D^{\mathrm{III}}=D^{\mathrm{III}}(\tilde{V}, \tilde{A})$. Denoting, as in the previous section, the ring of integers in k by $\mathfrak{o}$ or $\mathfrak{o}_k$, we let L be an $\mathfrak{o}$-lattice in V. Given a totally isotropic subspace W of V with $\dim W=s$, the G^{III}-orbit $C_s^{\mathrm{III}}=\bigcup_{\dim W=s} B_W^{\mathrm{III}}$ is decomposed into disjoint union of G_L^{III}-orbits each of which is called a "cusp of level s (with respect to G_L^{III})". We denote the space of cusps of level s with respect to G_L^{III} by $C_s^{\mathrm{III}}(L)$. $C_s^{\mathrm{III}}(L)$ is in a bijective correspondence with the double coset space $G_L^{\mathrm{III}}\backslash G^{\mathrm{III}}/G_W^{\mathrm{III}}$.

2.2. Generally, let $\mathfrak{o}$ be a Dedekind domain and let k be its quotient field. Let (V, A) be a non-degenerate alternating vector space over k and let L be an $\mathfrak{o}$-lattice in V. We set:

$$\mu_0^{\mathrm{III}}(L)=\mathfrak{o}\text{-ideal generated by } A(x, y) \text{ for } x, y\in L\,,$$

$$L^{\natural}=\{x\in V\,;\ A(x, L)\subset\mathfrak{o}\}\,.$$

We call L $\mathfrak{J}$-*maximal* if $\mu_0^{\mathrm{III}}(L)=\mathfrak{J}$ and $L=\mu_0^{\mathrm{III}}(L)L$. Two maximal lattices L

and M in V are G^{III}-equivalent if and only if $\mu_0^{\mathrm{III}}(L)=\mu_0^{\mathrm{III}}(M)$, [**12**].

2.3. The situation being the same as in 2.2 we assume that W is a totally isotropic subspace of V and that L is an $\mathfrak{J}$-maximal lattice in V. Then L admits a "W-decomposition":

$$L=\sum_{i=1}^{s}(\mathfrak{a}_i^{-1}\mathfrak{J}w_i+\mathfrak{a}_iw_i')\oplus L', \quad w_i\in L \quad (1\leqq i\leqq s), \qquad \mathfrak{a}_i\supset\mathfrak{a}_{i+1}, \quad (1\leqq i\leqq s-1),$$

$$A(w_i, w_j')=\delta_{ij} \text{ and } \{w_1, \cdots, w_s\} \text{ is a base of } W,$$

(Proposition 3.3, [**6**]).

When in the above $W\neq\{0\}$, we obtain for any $\mathfrak{o}$-ideal $\mathfrak{a}$, an $\mathfrak{J}$-maximal lattice M in V by setting:

$$M=(\mathfrak{a}\mathfrak{a}_1^{-1}\mathfrak{J}w_1+\mathfrak{a}^{-1}\mathfrak{a}_1w_1'+\sum_{i=2}^{s}(\mathfrak{a}_i^{-1}\mathfrak{J}w_i+\mathfrak{a}_iw_i'))\oplus L'.$$

$\mathfrak{J}$-maximal lattices L_1 and L_2 in V are G_W^{III}-equivalent to each other if and only if $c(L_1\cap W)=c(L_2\cap W)$, (Proposition 3.5, [**6**]). We thus obtain:

THEOREM. *Situation being the same as in* 2.1, *we let W be a totally isotropic subspace of dimension $s>0$ of V and let L be an $\mathfrak{J}$-maximal lattice in V. Then the mapping:*

$$\Phi^{\mathrm{III}}: C_s^{\mathrm{III}}(L) \longrightarrow C(k),$$

given by $\Phi^{\mathrm{III}}:(U)_{G_L^{\mathrm{III}}}\to c(L\cap U)$, $(U\in(W)_{G^{\mathrm{III}}})$ *is bijective.*

§ 3. The mapping $\rho: C_s^{\mathrm{I}}\to C_{2s}^{\mathrm{III}}$.

3.1. Let K/k be a totally imaginary quadratic extension of a totally real number field, $\mathfrak{o}_k$ the ring of integers in k and let (V, H) be a non-degenerate Hermitian vector space over K/k. Then, as explained in Ch. I, the functor $\mathfrak{R}_{K/k}:(V, H)\to(V', A')$ induces a holomorphic imbedding $\rho: D^{\mathrm{I}}(\tilde{V}, \tilde{H})\to D^{\mathrm{III}}(\tilde{V}', \tilde{A}')$ which sends a rational boundary component B_W^{I} into $B_{\mathfrak{R}(W)}^{\mathrm{III}}$, where W is a totally isotropic subspace of V. Let L be an $\mathfrak{O}_K$-lattice in V. L is then an $\mathfrak{o}_k$-lattice in V' which we shall denote by $\mathfrak{R}_{K/k}L$ or simply by L. Let $G^{\mathrm{I}}=SU(V, H)$ and $G^{\mathrm{III}}=Sp(V', A')$. The homomorphism $\rho: G^{\mathrm{I}}\to G^{\mathrm{III}}$ induced by $\mathfrak{R}_{K/k}$ then sends G_L^{I} into $G_{\mathfrak{R}L}^{\mathrm{III}}$ and hence the mapping $\rho:\bar{D}^{\mathrm{I}}\to\bar{D}^{\mathrm{III}}$ naturally induces the mapping $\rho: C_s^{\mathrm{I}}(L)\to C_{2s}^{\mathrm{III}}(\mathfrak{R}_{K/k}L)$, which may be identified with the mapping: $G_L^{\mathrm{I}}\backslash G^{\mathrm{I}}/G_W^{\mathrm{I}}\to G_{\mathfrak{R}L}^{\mathrm{III}}\backslash G^{\mathrm{III}}/G_{\mathfrak{R}W}^{\mathrm{III}}$ naturally induced by $\mathfrak{R}_{K/k}$.

Let $K=k+kw$, with $w^\sigma=-w$, and $A'=\mathrm{Im}_w H$. We set $\mathrm{Im}_w(\mathfrak{O}_K)=\mathrm{Im}_w(x)=\{(x-x^\sigma)/2w;\ x\in\mathfrak{O}_K\}$. Suppose that L is an $\mathfrak{J}$-modular lattice in V. Then $\mathfrak{R}_{K/k}L=L$ is maximal in (V', A') if and only if $\mathfrak{J}$ is an ideal in k, and in such a case $\mu_0^{\mathrm{III}}(L)=\mathfrak{J}\,\mathrm{Im}_w(\mathfrak{O}_K)$, (Theorem 3.8, [**7**]). When furthermore $U\in(W)_{G^{\mathrm{I}}}$, one has $\mathfrak{R}U\in(\mathfrak{R}W)_{G^{\mathrm{III}}}$ and the mappings $\tilde{\Phi}^{\mathrm{I}}: C_s^{\mathrm{I}}(L)\to C(K)$ and $\Phi^{\mathrm{III}}: C_{2s}^{\mathrm{III}}(\mathfrak{R}_{K/k}L)\to C(k)$ are respectively given by:

$$\tilde{\Phi}^{\mathrm{I}}:(U)_{G_L^{\mathrm{I}}}\longrightarrow c_K(L\cap U)\,,\qquad \Phi^{\mathrm{III}}:(\mathfrak{R}U)_{G_{\mathfrak{R}L}^{\mathrm{III}}}\longrightarrow c_k(\mathfrak{R}L\cap\mathfrak{R}U)=c_k(L\cap U)\,.$$

For the purpose of investigating the relation between $c_K(L\cap U)$ and $c_k(L\cap U)$ we need subsequent lemmas.

3.2. LEMMA A. *Let $\mathfrak{o}$ be a Dedekind domain, k its quotient field, K either a quadratic extension of k or $k\times k$ and let $\mathfrak{O}_K$ be the ring of $\mathfrak{o}$-integers in K. Let V be a free K-module of a finite rank and let L, M be $\mathfrak{O}_K$-lattices in V. V is then a finite dimensional vector space over k and L, M are $\mathfrak{o}$-lattices in V. We have, furthermore, the following:*

(1) $d_k(L, M)=N(d_K(L, M))$,

(2) $c_k(L)N(c_K(L))^{-1}=c_k(M)N(c_K(M))^{-1}$.

PROOF. By virtue of formula (3) of Ch. II, 1.2 the (2) above follows immediately from (1). We shall use the local argument to show (1). Since for every prime ideal $\mathfrak{p}$ in k the completion $\mathfrak{o}_\mathfrak{p}$ is a principal ideal domain there exists $g_\mathfrak{p}\in GL(V_\mathfrak{p})$ such that $g_\mathfrak{p}L_\mathfrak{p}=M_\mathfrak{p}$. We have $d_{k_\mathfrak{p}}(L_\mathfrak{p}, M_\mathfrak{p})=(N_{K_\mathfrak{p}/k_\mathfrak{p}}\det g_\mathfrak{p})$ $=N_{K_\mathfrak{p}/k_\mathfrak{p}}(d_{K_\mathfrak{p}}(L_\mathfrak{p}, M_\mathfrak{p}))$. Hence, $d_k(L, M)=N_{K/k}(d_K(L, M))$. q. e. d.

LEMMA B. *Suppose that $\mathfrak{o}$ has the characteristic different from 2 and that we are given a finite dimensional vector space U over k supplied with a non-degenerate symmetric bilinear form S. Given an $\mathfrak{o}$-lattice L in U we set $L^S=\{x\in U\,;\ S(x, L)\subset\mathfrak{o}\}$. When L is decomposed as $L=\mathfrak{a}_1v_1+\cdots+\mathfrak{a}_nv_n$, ($\mathfrak{a}_i$ are $\mathfrak{o}$-ideals), we have $d_k(L^S, L)=\det(S(v_i, v_j))\prod_{i=1}^{n}\mathfrak{a}_i^2$.*

PROOF. It is known that $L^S=\mathfrak{a}_1^{-1}u_1+\cdots+\mathfrak{a}_n^{-1}u_n$, where $S(v_i, u_j)=\delta_{ij}$, [8]. The lemma follows easily from this. q. e. d.

Setting, specifically, $U=K$, (and assuming that ch. $k\neq 2$), one has a non-degenerate symmetric bilinear form S given by $S(x, y)=\mathrm{Tr}\,(xy)$. The $\det(S(v_i, v_j))$ appearing in Lemma B is, then the discriminant of K/k, mod k^{*2}, denoted by $\Delta(K/k)$. We have, furthermore, $\mathfrak{O}_K^S=\mathfrak{D}^{-1}$, where $\mathfrak{D}=\mathfrak{D}(K/k)$ denotes the relative different of K/k. Hence,

$$d_k(\mathfrak{O}_K^S, \mathfrak{O}_K)=N(d_K(\mathfrak{D}^{-1}, \mathfrak{O}_K))=N(\mathfrak{D}(K/k))=d(K/k)$$

$$\equiv\Delta(K/k)\prod_{i=1}^{2}\mathfrak{a}_i^2\ (\mathrm{mod}\ k^{*2})\,,$$

where $\mathfrak{O}_K=\mathfrak{a}_1v_1+\mathfrak{a}_2v_2$. Therefore, $\prod_{i=1}^{2}\mathfrak{a}_i^2\equiv d(K/k)\cdot\Delta(K/k)^{-1}\ (\mathrm{mod}\ k^{*2})$. Hence we may write:

$$c_k(\mathfrak{O}_K)=c_k((d(K/k)\cdot\Delta(K/k)^{-1})^{\frac{1}{2}})\,.$$

Returning to the situation described in Lemma A one may choose, for the lattice M, the following:

$$M=\mathfrak{O}_Kv_1+\cdots+\mathfrak{O}_Kv_n\,,\qquad (n=\mathrm{rank}\ V)\,,$$

and obtain the following:

LEMMA C. *The situation being the same as in Lemma* A, *we assume that* ch. $k\neq 2$. *Then, for a given* $\mathfrak{O}_K$*-lattice L in V we have:*

$$c_k(L)=N_{K/k}(c_K(L))c_K(\mathfrak{O}_K)^n\,, \qquad (n=\operatorname{rank} V)\,.$$

REMARK. The above Lemmas A, B and C and their proofs remain valid if we take as K a finite separable extension of k, (ch. $k\neq 2$). The formula for $c_k(\mathfrak{O}_K)$ for a finite separable extension K/k was also obtained by E. Artin [**1**].

The following Lemma D and its proof are due to S-N. Kuroda.

LEMMA D. *Let k be an algebriac number field of finite degree and let K/k be a finite extension. Suppose that there exists a prime spot $\mathfrak{p}$ of k which completely ramifies in K, (i.e. the ramification exponent of $\mathfrak{p}$ in K/k is equal to $[K:k]$). Then the homomorphism $N_{K/k}: C(K)\to C(k)$ is surjective.*

PROOF. Let $\tilde{k}$ denote the absolute class field of k, (in Hilbert's sense). Our assumption implies that $\tilde{k}\cap K=k$. We have the diagram:

$$\begin{tikzcd}
 & K\cdot\tilde{k} \arrow[d, dash] \\
K \arrow[ur, dash] \arrow[d, dash] & \tilde{k} \\
k \arrow[ur, dash] &
\end{tikzcd}$$

where, by Translation Theorem, $K\tilde{k}/K$ is the class field associated to the group of ideals $\mathfrak{A}$ in K such that $N_{K/k}\mathfrak{A}$ are principal in k. Hence, $[C(K):\operatorname{Ker} N_{K/k}]=[K\tilde{k}:K]=[\tilde{k}:k]=h(k)$. This implies that $N_{K/k}: C(K)\to C(k)$ is surjective. q. e. d.

COROLLARY. *When K/k is a totally imaginary quadratic extension of a totally real number field the homomorphism $N: C(K)\to C(k)$ is surjective.*

3.3. Combining 1.8, 2.3, 3.1 of this chapter and the above Lemmas C and D we obtain the following:

THEOREM. *Let K/k be a totally imaginary quadratic extension of a totally real number field, $\mathfrak{o}_k$ the ring of integers in k, (V, H) a non-degenerate Hermitian vector space over K/k with* $\operatorname{ind}(H)>0$, $(V', A')=\mathfrak{R}_{K/k}(V, H)$, $A'=\operatorname{Im}_w H$, $(w\in K^*,\ w^\sigma=-w)$ *and let W be an s-dimensional totally isotropic subspace of V, $(s>0)$. Let $\mathfrak{J}$ be an ideal in k and suppose that we are given an $\mathfrak{J}$-modular lattice L in V. Then we have:*

(1) *$L=\mathfrak{R}_{K/k}L$ is $\mathfrak{J}\operatorname{Im}_w(\mathfrak{O}_K)$-maximal in V';*

(2) *The following diagram is commutative:*

$$\begin{tikzcd}
C_s^{\mathrm{I}}(L) \arrow[r, "\rho"] \arrow[d, "\tilde{\Phi}^{\mathrm{I}}"'] & C_{2s}^{\mathrm{III}}(\mathfrak{R}_{K/k}L) \arrow[d, "\wr", "\Phi^{\mathrm{III}}"'] \\
C(K) \arrow[r, "\nu_s"] & C(k)
\end{tikzcd}\,,$$

where $\nu_s: C(K)\to C(k)$ *is given by* $\nu_s(c)=N_{K/k}(c)c_k((d(K/k)\cdot\Delta(K/k)^{-1})^{\frac{1}{2}})^s$, *and is surjective;*

(3) *When* $W^\perp \supsetneqq W$, *the above* $\rho: C_s^{\mathrm{I}}(L)\to C_{2s}^{\mathrm{III}}(\mathfrak{R}L)$ *is surjective;*

(4) *When* $W^\perp=W$, *or equivalently*, $2s=\dim_K V$, *boundary components* B_W^{I} *and* $B_{\mathfrak{R}W}^{\mathrm{III}}$ *are point components. When, furthermore* $\mu^{\mathrm{I}}(L)=\mathfrak{O}_K \mathrm{Tr}\,\mu_0^{\mathrm{I}}(L)$, *for each* $\kappa\in C_s^{\mathrm{I}}(L)$ *one has:*

$$|\rho^{-1}(\rho(\kappa))|\geqq|\mathrm{Ker}\,\lambda|\ ,$$

where λ *is given by* $\lambda: C_1(K/k)\ni c_K(\mathfrak{a}\mathfrak{O}_K)\to c_k(\mathfrak{a}^2)\in C(k)$, ($\mathfrak{a}$ *is an ideal in* k).

REMARK. By Corollary 2 of Theorem 1.8 we obtain examples of $\rho: C_s^{\mathrm{I}}(L)\to C_{2s}^{\mathrm{III}}(\mathfrak{R}L)$ with $2s=\dim_K V$ which are not injective; e. g. $k=\boldsymbol{Q}$, $K=\boldsymbol{Q}(\sqrt{-m})$, m is square-free positive integer such that $m\equiv 3 \pmod 4$, $m>3$, $s=1$, $\mathfrak{J}=\mathfrak{O}_K$, and $c(L\cap W)=1$.

Appendix, Examples.

1. Let $k=\boldsymbol{Q}$, $K=\boldsymbol{Q}(\sqrt{-1})$, $V=\{v_1, v_2\}_K$, and let H be a non-degenerate Hermitian form on V given by $H(v_i, v_j)=\varepsilon_i\delta_{ij}$, $\varepsilon_1=1$, $\varepsilon_2=-1$. Let $L=\mathfrak{O}_K v_1+\mathfrak{O}_K v_2$, ($\mathfrak{O}_K=\boldsymbol{Z}[\sqrt{-1}]$). L is, then, a normal $\mathfrak{O}_K$-modular lattice in V. Let $W=\{w\}_K$, $w=v_1+v_2$. W is a totally isotropic subspace of V.

The symmetric bounded domain $D^{\mathrm{I}}=D^{\mathrm{I}}(\tilde{V}, \tilde{H})$ is realized as:

$$D^{\mathrm{I}}=\{z\in\boldsymbol{C};\ |z|<1\}\ ,$$

and we have:

$\bar{D}^{\mathrm{I}}=\{z\in\boldsymbol{C};\ |z|\leqq 1\}$, $\bar{D}^{\mathrm{I}}=C_0^{\mathrm{I}}\cup C_1^{\mathrm{I}}$, $C_0^{\mathrm{I}}=D^{\mathrm{I}}$, $C_1^{\mathrm{I}}=\{z\in\boldsymbol{C};\ |z|=1\}$.
Boundary components of $\bar{D}^{\mathrm{I}}$ are point components. Particularly, $B_W^{\mathrm{I}}=\{1\}$.

In view of Corollary 2 of Theorem 1.8 of Ch. II, an easy calculation shows that $C_1^{\mathrm{I}}(L)$ is represented by 1 and $\sqrt{-1}$.

Let $(V', A')=\mathfrak{R}_{K/k}(V, H)$, $D^{\mathrm{III}}=D^{\mathrm{III}}(\tilde{V}', \tilde{A}')$. D^{III} is a subdomain of $\boldsymbol{C}^3$ consisting of symmetric 2×2 complex matrices Z with $1-{}^t\bar{Z}Z>0$. We have:

$$\bar{D}^{\mathrm{III}}=C_0^{\mathrm{III}}\cup C_1^{\mathrm{III}}\cup C_2^{\mathrm{III}}\ ,$$

and the mapping $\rho: \bar{D}^{\mathrm{I}}\to\bar{D}^{\mathrm{III}}$ sends C_1^{I} into C_2^{III}; the latter space consists of point components.

The lattice $L=\mathfrak{R}_{K/k}L$ is maximal in V' and each cusp $C_s^{\mathrm{III}}(\mathfrak{R}L)$, $(s=1, 2)$ consists of one element. Therefore the images $\rho(1)$ and $\rho(\sqrt{-1})$ are $G_{\mathfrak{R}L}^{\mathrm{III}}$-equivalent to each other.

2. Let $k=\boldsymbol{Q}$, $K=\boldsymbol{Q}(\sqrt{-m})$, $m\equiv 1$ or 2, $(m\neq 1)$ and suppose that $C_0(K/k)=\{1\}$. Let $V=\{v_1, v_2\}_K$, $H(v_i, v_j)=\varepsilon_i\delta_{ij}$, $\varepsilon_1=1$, $\varepsilon_2=-1$ and let $L=\mathfrak{O}_K v_1+\mathfrak{O}_K v_2$, $W=\{v_1+v_2\}_K$. Then the situation is exactly alike the previous case 1, and

$C_1^{\mathrm{I}}(L)$ is represented by $B_W^{\mathrm{I}}=\{1\}$ and $\{-1\}$. The images $\rho(1)$ and $\rho(-1)$ are point components of $\bar{D}^{\mathrm{III}}$ which are mutually $G_{\mathfrak{R}L}^{\mathrm{III}}$-equivalent.

3. Let $k=\boldsymbol{Q}$, $K=\boldsymbol{Q}(\sqrt{-5})$, $V=\{v_1, v_2, v_3\}_K$, $H(v_i, v_j)=\varepsilon_i\delta_{ij}$, $\varepsilon_1=\varepsilon_2=1$, $\varepsilon_3=-1$ and let $L=\mathfrak{O}_K v_1+\mathfrak{O}_K v_2+\mathfrak{O}_K v_3$, $W=\{w\}_K$, $w=v_1+v_3$, $(\mathfrak{O}_K=\boldsymbol{Z}[\sqrt{-5}])$. L is a normal $\mathfrak{O}_K$-modular lattice in V and W is a totally isotropic subspace of V. The symmetric bounded domain $D^{\mathrm{I}}=D^{\mathrm{I}}(\tilde{V}, \tilde{H})$ is realized as:

$$D^{\mathrm{I}}=\left\{\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\in \boldsymbol{C}^2\,;\ |z_1|^2+|z_2|^2<1\right\},$$

and we have:

$$\bar{D}^{\mathrm{I}}=\left\{\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\in \boldsymbol{C}^2\,;\ |z_1|^2+|z_2|^2\leqq 1\right\}=C_0^{\mathrm{I}}\cup C_1^{\mathrm{I}}\,,$$

$$C_0^{\mathrm{I}}=D^{\mathrm{I}},\ C_1^{\mathrm{I}}=\left\{\begin{pmatrix} z_1 \\ z_2 \end{pmatrix};\ |z_1|^2+|z_2|^2=1\right\}.$$

Boundary components of $\bar{D}^{\mathrm{I}}$ are point components. Particularly, $B_W^{\mathrm{I}}=\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$.

The mapping $\tilde{\Phi}^{\mathrm{I}}: C_1^{\mathrm{I}}(L)\to C(K)$, given by $\tilde{\Phi}^{\mathrm{I}}((U)_{G_L^{\mathrm{I}}})=c(L\cap U)$, (where U is a totally isotropic subspace of dimension 1 of V) is bijective.

$C(K)$ is of order 2 and consists of 1 and the ideal class represented by $\mathfrak{p}=(2, 1+\sqrt{-5})$. We have $N\mathfrak{p}=(2)$, $\mathfrak{p}^{\sigma}=\mathfrak{p}$; whence $\mathfrak{p}^{-\sigma}=2^{-1}\mathfrak{p}$.

L admits the following W-decomposition:

$$L=(\mathfrak{O}_K w+\mathfrak{O}_K w')\oplus\mathfrak{O}_K v\,,\qquad w'=-(v_2+v_3)\,,\qquad v=v_1+v_2+v_3\,.$$

Let us set:

$$M=(\mathfrak{p}^{-\sigma}w+\mathfrak{p}w')\oplus\mathfrak{p}^{\sigma-1}v=(2^{-1}\mathfrak{p}w+\mathfrak{p}w')\oplus\mathfrak{O}_K v\,.$$

As explained in Ch. II, 1.7, L and M are mutually G^{I}-equivalent. Indeed, utilizing Lemma in Ch. II, 1.9 we can calculate and obtain $x, y, z\in V$ with:

$$M=\mathfrak{O}_K x+\mathfrak{O}_K y+\mathfrak{O}_K z\,,\qquad H(x)=H(y)=0\,,\qquad H(x, y)=1\,,\qquad H(z)=1\,;$$

e.g. we may set:

$$\begin{aligned} x&=(1+\sqrt{-5})/2w-4w'+2v\,,\\ y&=-(3+\sqrt{-5})/2w+(3-\sqrt{-5})w'-2v\,,\\ z&=2w-2(1-\sqrt{-5})w'+(2-\sqrt{-5})v\,. \end{aligned}$$

An element $g\in G^{\mathrm{I}}$ sending L onto M is given, with respect to $\{v_1, v_2, v_3\}$ as follows:

$$g=\begin{bmatrix} (1-3\sqrt{-5})/2 & (-3+3\sqrt{-5})/2 & 2+2\sqrt{-5} \\ -1-2\sqrt{-5} & 2+3\sqrt{-5} & 7+2\sqrt{-5} \\ (-1-5\sqrt{-5})/2 & (1+7\sqrt{-5})/2 & 7+3\sqrt{-5} \end{bmatrix}.$$

$C_1^{\mathrm{I}}(L)$ consists of γ_1, γ_2 which are G_L^{I}-orbits of B_W^{I} and $g^{-1}(B_W^{\mathrm{I}}) = \frac{1-\sqrt{-5}}{5-\sqrt{-5}}\begin{pmatrix}1\\-2\end{pmatrix}$.

We set $(V', A') = \mathfrak{R}_{K/k}(V, H)$, $D^{\mathrm{III}} = D^{\mathrm{III}}(\tilde{V}', \tilde{A}')$. D^{III} is a subdomain of $\boldsymbol{C}^6$ consisting of symmetric 3×3 complex matrices Z such that $1-{}^t\bar{Z}Z>0$. We have:

$$\bar{D}^{\mathrm{III}} = C_0^{\mathrm{III}} \cup C_1^{\mathrm{III}} \cup C_2^{\mathrm{III}} \cup C_3^{\mathrm{III}},$$

and the mapping $\rho: \bar{D}^{\mathrm{I}} \to \bar{D}^{\mathrm{III}}$ sends C_1^{I} into C_2^{III}. Each space of cusps $C_s^{\mathrm{III}}(\mathfrak{R}L)$ consists of one element. Hence the points $\rho(B_W^{\mathrm{I}})$ and $\rho(g^{-1}(B_W^{\mathrm{I}}))$ must find their $G_{\mathfrak{R}L}^{\mathrm{III}}$-equivalents in $B_{\mathfrak{R}W}^{\mathrm{III}}$. The boundary component $B_{\mathfrak{R}W}^{\mathrm{III}}$ may be identified with $D^{\mathrm{III}}((\mathfrak{R}W^{\perp}/\mathfrak{R}W)^{\sim}, \tilde{A}'_{\mathfrak{R}W})$ which is the open unit disc $\boldsymbol{D}$ on which operates $Sp(1) = SU(2)$. $Sp(1)$ contains an arithmetic subgroup Γ which is equal to $Sp(\mathfrak{R}W^{\perp}/\mathfrak{R}W, A'_{\mathfrak{R}W})_{\mathfrak{R}L\cap\mathfrak{R}W^{\perp}/\mathfrak{R}L\cap\mathfrak{R}W} \cong G_{\mathfrak{R}W}^{\mathrm{III}} \cap G_{\mathfrak{R}L}^{\mathrm{III}}$.

We utilize the following base of $V'_{\boldsymbol{R}}$:

$$\{v_1, v_2, v_3, v'_1, v'_2, v'_3\}, \qquad v'_i = \sqrt{-1}\,v_i, \qquad (i=1, 2), \qquad v'_3 = -\sqrt{-1}\,v_3;$$

and the following base of the complexification $V'_{\boldsymbol{C}}$:

$$\{e_i, \bar{e}_i\}_{i=1,2,3}, \qquad e_i = (v_i - \sqrt{-1}\,v'_i)/(1+\sqrt{-1}),$$

where, of course, $\sqrt{-1}\,v'_i$ are elements of the complexification $V'_{\boldsymbol{C}}$. $\bar{D}^{\mathrm{III}}$ is then identified with $\bar{\boldsymbol{D}}^{\mathrm{III}}$ which consists of the subspaces U of $V'_{\boldsymbol{C}}$ spanned by $z_i = \sum_j z_{ij}e_j + \bar{e}_i$, $(i=1,2,3)$ such that the matrices $Z = Z(U) = (z_{ij})$ are symmetric and $1-{}^t\bar{Z}Z \geqq 0$.

The totally isotropic subspace $\mathfrak{R}W$ of V' is spanned by v_1+v_3 and $\sqrt{5}(v'_1-v'_3)$. Hence its complexification W' is spanned by $e_1-\sqrt{-1}\,\bar{e}_3$ and $\bar{e}_1+\sqrt{-1}\,e_3$. From this it follows easily that:

$$B_{\mathfrak{R}W}^{\mathrm{III}} = \left\{\begin{pmatrix} 0 & 0 & -1 \\ 0 & z & 0 \\ -1 & 0 & 0 \end{pmatrix};\ 1-|z|^2>0\right\};$$

the mapping: $\begin{pmatrix} 0 & 0 & -1 \\ 0 & z & 0 \\ -1 & 0 & 0 \end{pmatrix} \to z$ induces the isomorphism sending $B_{\mathfrak{R}W}^{\mathrm{III}}$ onto the open unit disc $\boldsymbol{D}$.

$\mathfrak{R}W^{\perp}/\mathfrak{R}W$ is spanned by classes represented by v_2, $\sqrt{5}\,v'_2$, and we have:

$$\mathfrak{R}L \cap \mathfrak{R}W^{\perp}/\mathfrak{R}L \cap \mathfrak{R}W \cong \boldsymbol{Z}v_2 + \sqrt{5}\,\boldsymbol{Z}v'_2.$$

The arithmetic subgroup Γ of $SU(2)$ is, then, represented as follows:

$$\Gamma=\left\{\frac{1}{2}\begin{pmatrix}(a+d)-\sqrt{-1}(\sqrt{5}b-\sqrt{5}^{-1}c) & -(\sqrt{5}b+\sqrt{5}^{-1}c)+\sqrt{-1}(a-d)\\ -(\sqrt{5}b+\sqrt{5}^{-1}c)-\sqrt{-1}(a-d) & (a+d)+\sqrt{-1}(\sqrt{5}b-\sqrt{5}^{-1}c)\end{pmatrix};\right.$$

$$\left.\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL(2, \boldsymbol{Z})\right\}.$$

We have, on the other hand,

$$\rho(B_W^{\mathrm{I}})=\begin{pmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix},\qquad \rho(g^{-1}(B_W^{\mathrm{I}}))=\begin{pmatrix}0 & 0 & \alpha\\ 0 & 0 & -2\alpha\\ \alpha & -2\alpha & 0\end{pmatrix},$$

$$\alpha=\sqrt{-1}(1-\sqrt{-5})/(5-\sqrt{-5}).$$

An element $h\in G_L^{\mathrm{III}}$ sending $\rho(g^{-1}(B_W^{\mathrm{I}}))$ to a point of $B_{\mathfrak{R}W}^{\mathrm{III}}$ is obtained by observing $\mathfrak{R}W$-decompositions of $\mathfrak{R}L$ and $\rho(g^{-1})\mathfrak{R}L$; with respect to $\{v_1, v_2, v_3, \sqrt{5}v_1', \sqrt{5}v_2', \sqrt{5}v_3'\}$ we have:

$$h=\begin{bmatrix}0 & 0 & 0 & -2 & 0 & -1\\ 0 & 2 & 0 & 0 & 5 & -6\\ 2 & -4 & -4 & -3 & 2 & 4\\ 8 & -18 & -15 & 0 & 0 & 30\\ 2 & -1 & -4 & 0 & 10 & -4\\ -5 & 12 & 10 & 0 & 0 & -20\end{bmatrix}.$$

Expressing this h with respect to the base $\{e_i, \bar{e}_i\}_{i=1,2,3}$ in the form

$$h=\begin{pmatrix}A & B\\ \bar{B} & \bar{A}\end{pmatrix},$$

and writing $X=\rho(g^{-1}(B_W^{\mathrm{I}}))$, we have:

$$(AX+B)(\bar{B}X+\bar{A})^{-1}=\begin{pmatrix}0 & 0 & -1\\ 0 & z & 0\\ -1 & 0 & 0\end{pmatrix},\qquad (z\in\boldsymbol{D}),$$

or, equivalently,

$$AX+B=\begin{pmatrix}0 & 0 & -1\\ 0 & z & 0\\ -1 & 0 & 0\end{pmatrix}(\bar{B}X+\bar{A}).$$

From this follows easily that $z=(-10\sqrt{5}+8\sqrt{-1})/(22+5\sqrt{-5})$.

We can now verify that the above z and 0 are not mutually Γ-equivalent Indeed, let $\gamma=\begin{pmatrix}\alpha & \beta \\ \bar{\beta} & \bar{\alpha}\end{pmatrix}$ be an element of Γ and suppose that γ sends 0 onto z. $\Big(\alpha=\frac{1}{2}\{(a+d)-\sqrt{-1}(\sqrt{5}\,b-\sqrt{5}^{-1}c)\},\ \beta=\frac{1}{2}\{-(\sqrt{5}\,b+\sqrt{5}^{-1}c)+\sqrt{-1}(a-d)\},$ $\begin{pmatrix}a & b \\ c & d\end{pmatrix}\in SL(2, \boldsymbol{Z})\Big)$. We must have, then, $\gamma(0)=\beta\cdot\bar{\alpha}^{-1}=z$ or, equivalently,

$$\begin{cases} 5a-14b-6c+15d=0\,, \\ 14a+25b-15c-30d=0\,, \\ ad-bc=1\,. \end{cases}$$

The first two relations above imply that:

$$8a-b-9c=0\,,$$

whence, replacing b by $8a-9c$ we obtain:

$$\begin{cases} -107a+120c+15d=0\,, \\ ad-8ac+9c^2=1\,. \end{cases}$$

The first equation implies that $a\equiv 0 \pmod 3$, whence, by the last equation

$$1=a(d-8c)+9c^2\equiv 0 \pmod 3\,,$$

which is a contradiction.

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