J. Math. Soc. Japan
Vol. 26, No. 2, 1974

Monodromy representations of homology of certain elliptic surfaces

By Takao SASAI

(Received Nov. 27, 1972)
(Revised June 12, 1973)

Introduction.

In this paper we shall determine global monodromy representations of certain basic elliptic surfaces over a complex projective line $\boldsymbol{P}^1(\boldsymbol{C})$. Such a surface has a following normal form (Kas [2]); Let $\boldsymbol{P}^2(\boldsymbol{C})$ be a complex projective plane with homogeneous coordinate (x, y, z). We take two copies $W_0=\boldsymbol{P}^2(\boldsymbol{C})\times \boldsymbol{C}_0$ and $W_1=\boldsymbol{P}^2(\boldsymbol{C})\times \boldsymbol{C}_1$ of the product $\boldsymbol{P}^2(\boldsymbol{C})\times \boldsymbol{C}$ and form their union

$$W^k=W_0\cup W_1 \qquad (k=1, 2, \cdots)$$

by identifying $(x, y, z, u)\in W_0$ with $(x_1, y_1, z_1, u_1)\in W_1$ if and only if

$$u^{2k}x_1=x, \quad u^{3k}y_1=y, \quad z_1=z, \quad uu_1=1.$$

Similarly we define

$$\Delta=\boldsymbol{C}_0\cup \boldsymbol{C}_1,$$

where we identify $u\in \boldsymbol{C}_0$ with $u_1\in \boldsymbol{C}_1$ if and only if $uu_1=1$. For a point $(\tau, \sigma)=(\tau_0, \tau_1, \cdots, \tau_{4k}, \sigma_1, \cdots, \sigma_{6k})$ in the space $\boldsymbol{C}^{10k+1}$, we set

$$g_{4k}(u)=\tau_0u^{4k}+\tau_1u^{4k-1}+\cdots+\tau_{4k},$$

$$h_{6k}(u)=u^{6k}+\sigma_1u^{6k-1}+\cdots+\sigma_{6k}.$$

Then the basic elliptic surface $B_k(\tau, \sigma)$ over $\Delta=\boldsymbol{P}^1(\boldsymbol{C})$ is defined by

$$y^2z-4x^3+g_{4k}(u)xz^2+h_{6k}(u)z^3=0 \qquad \text{in } W_0,$$

$$y_1^2z_1-4x_1^3+u_1^{4k}g_{4k}(1/u_1)x_1z_1^2+u_1^{6k}h_{6k}(1/u_1)z_1^3=0 \qquad \text{in } W_1.$$

The projection Ψ of $B_k(\tau, \sigma)$ onto Δ is defined by

$$\Psi: (x, y, z, u)\longmapsto u$$

$$(x_1, y_1, z_1, u_1)\longmapsto u_1.$$

We simply denote by $u=\infty$ the point $u_1=0$ on Δ.

We define two polynomials $D_k(u)$ and $\tilde{D}_k(u_1)$, respectively, by

$$D_k(u)=g_{4k}^3(u)-27h_{6k}^2(u)$$

and

$$\tilde{D}_k(u_1)=u_1^{12k}D_k(1/u_1)\,.$$

We can easily verify that $C_u=\Psi^{-1}(u)$ ($C_\infty=\Psi^{-1}(\infty)$) is a non-singular elliptic curve if $D_k(u)\neq 0$ ($\tilde{D}_k(0)\neq 0$). Such a fibre C_u is called *regular.* If $D_k(u)=0$ ($\tilde{D}_k(0)=0$), we call C_u (C_∞) a *singular* fibre.

Let $\{a_j\}$ be a finite set of all points a_j ($j=1, 2, \cdots, r$) such that C_{a_j} are singular. Let $\varDelta'=\varDelta-\{a_j\}$. Then $B_k(\tau, \sigma)|\varDelta'$ is a differentiable fibre bundle over $\varDelta'$ with tori as fibres. We fix a base point $\mathcal{O}\in\varDelta'$ and choose a basis for the first homology group $H_1(C_o, \boldsymbol{Z})$. This determines a representation of the fundamental group $\pi_1(\varDelta')$ into the group $SL(2, \boldsymbol{Z})$. We call this representation a *monodromy representation of homology* of $B_k(\tau, \sigma)$ or, simply, a monodromy of $B_k(\tau, \sigma)$. This representation determines a sheaf over $\varDelta$ (the homological invariant of $B_k(\tau, \sigma)$) which is locally constant over $\varDelta'$ with the general stalk $\boldsymbol{Z}\oplus\boldsymbol{Z}$.

Now, for each point $u\in\varDelta'$, we represent the elliptic curve C_u as a complex torus with the periods ($\omega(u)$, 1), $\operatorname{Im}\omega(u)>0$, and denote by $J(\omega)$ the elliptic modular function defined on the upper half plane $H=\{\omega|\operatorname{Im}\omega>0\}$. Then defining $\mathcal{J}(u)=J(\omega(u))$, it follows that

$$\mathcal{J}(u)=\frac{g_{4k}^3(u)}{g_{4k}^3(u)-27h_{6k}^2(u)}\;.$$

This is called the functional invariant of $B_k(\tau, \sigma)$. Thus we obtain two invariants, functional and homological, of $B_k(\tau, \sigma)$. Conversely Kodaira proved the following important theorem.

THEOREM. *When a meromorphic function $\mathcal{J}(u)$ on $\varDelta$ and a sheaf G over $\varDelta$ belonging to $\mathcal{J}(u)$ are given, it is possible to construct a basic elliptic surface over $\varDelta$ having $\mathcal{J}(u)$ and G as its functional and homological invariants.*

REMARK. This theorem is valid for arbitrary compact Riemann surface $\varDelta$. For our purpose we have only to consider the case in which the base space is $\varDelta=\boldsymbol{P}^1(\boldsymbol{C})$; for details, see Kodaira [3] p. 578-603.

Though his method of constructing the basic elliptic surface gives us many detailed results, it is not so easy to obtain global expressions such as a Picard-Fuchs differential equation of a basic elliptic surface. So we take the global form $B_k(\tau, \sigma)$ of a basic elliptic surface mentioned above and determine the monodromy of $B_k(\tau, \sigma)$ (consequently the homological invariant of $B_k(\tau, \sigma)$).

In §1 we construct an analytic fibre space F over $\boldsymbol{C}^2$ which induces $B_k(\tau, \sigma)$ by a certain holomorphic mapping. In §2 we calculate the monodromies of F and some $B_k(\tau, \sigma)$. In §3 we determine monodromies for certain classes of basic elliptic surfaces (Theorem 3.1). As a corollary to this theorem we obtain the monodromy representation groups of Picard-Fuchs differential equations.

The author wishes to thank Professor H. Omori for his valuable suggestions. He also expresses his hearty thanks to Professor M. Obata for his encouragement and careful reading.

§ 1. Analytic fibre space F over C^2.

We define an analytic fibre space F over $\boldsymbol{C}^2$ as follows; Let (X, Y, Z) be a homogeneous coordinate of $\boldsymbol{P}^2(C)$ and (G, H) a complex euclidean coordinate of $\boldsymbol{C}^2$. Then an analytic fibre space F is defined in $\boldsymbol{P}^2(C)\times\boldsymbol{C}^2$ by

$$Y^2Z-4X^3+GXZ^2+HZ^3=0\,.$$

The projection Φ of F onto $\boldsymbol{C}^2$ is defined by

$$\Phi:\,(X, Y, Z, G, H)\longmapsto(G, H)\,.$$

Let $E=\boldsymbol{C}^2-\{G^3-27H^2=0\}$. Then $F|E$ is a differentiable fibre bundle over E with tori as fibres.

Let $\boldsymbol{C}_0=\Delta-\{u=\infty\}$ and $\Delta''=\boldsymbol{C}_0-\{a_j\}$. We define two holomorphic mappings φ of $\boldsymbol{C}_0$ into $\boldsymbol{C}^2$ and $\bar{\varphi}$ of $B_k(\tau, \sigma)|\boldsymbol{C}_0$ into F by

$$\varphi:\,u\longmapsto(g_{4k}(u),\,h_{6k}(u))$$

and

$$\bar{\varphi}:\,(x, y, z, u)\longmapsto(x, y, z, g_{4k}(u), h_{6k}(u))\,,$$

respectively. Then we can easily verify

LEMMA 1.1. (1) *$B_k(\tau, \sigma)|\boldsymbol{C}_0$ is a fibre space induced by φ from F and $\bar{\varphi}$ is a fibre mapping induced by φ. In particular,*

(2) *$B_k(\tau, \sigma)|\Delta''$ is a differentiable fibre bundle induced by $\varphi|\Delta''$ from $F|E$ and $\bar{\varphi}$ is a fibre mapping induced by φ.*

Since $F|E$ is a differentiable fibre bundle over E with tori as fibres, we can define a monodromy of F in the same manner as in the case of $B_k(\tau, \sigma)$. It is a representation of $\pi_1(E)$ into $SL(2, \boldsymbol{Z})$. We fix a point $\mathcal{O}\in\Delta''$ and choose a basis $\{\gamma_1, \gamma_2\}$ for $H_1(C_o, \boldsymbol{Z})$. Let ρ be the corresponding monodromy of $B_k(\tau, \sigma)$. We denote $\bar{\varphi}_*$ the natural homomorphism of $H_1(C_o, \boldsymbol{Z})$ into $H_1(C_{\varphi(o)}, \boldsymbol{Z})$ induced by $\bar{\varphi}$, where $C_{\varphi(o)}$ is a fibre of F over $\varphi(\mathcal{O})$. $\bar{\varphi}_*$ is an isomorphism by Lemma 1.1. Therefore $\{\bar{\varphi}_*(\gamma_1), \bar{\varphi}_*(\gamma_2)\}$ is a basis for $H_1(C_{\varphi(o)}, \boldsymbol{Z})$ and determines the monodromy $\tilde{\rho}$ of F.

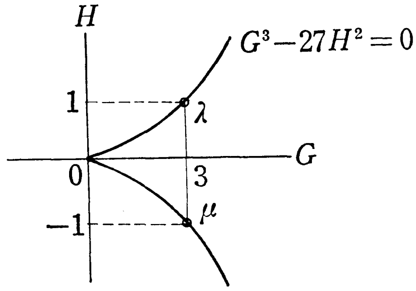
LEMMA 1.2. *The following diagram is commutative:*

$$\begin{tikzcd} \pi_1(\Delta'', \mathcal{O}) \arrow[rr, "\varphi_*"] \arrow[dr, "\rho"'] & & \pi_1(E, \varphi(\mathcal{O})) \arrow[dl, "\tilde{\rho}"] \\ & SL(2, \boldsymbol{Z}) & \end{tikzcd}$$

where φ_* *is the natural homomorphism induced by* φ.

Next we state Lemma 1.3, which can be seen in [**5**] p. 102. Choose $(G, H) = (3, 0)$ as a base point and let λ, μ be simple loops associated to two real branches of a curve $G^3-27H^2=0$.

LEMMA 1.3. *$\pi_1(E)$ is generated by λ, μ and subject only to the fundamental relation $\lambda\mu\lambda=\mu\lambda\mu$.*



REMARK 1.4. In the sequel we use the same letter to denote a loop and its homotopy class.

§ 2. Monodromy of F.

Let a be a point in $\boldsymbol{C}_0$ such that the fibre C_a is singular. We take a sufficiently small oriented disk D_a around a and put $\alpha=\partial D_a$.

LEMMA 2.1. *C_a is a singular fibre of type I_1, namely it is a rational curve with one ordinary double point, if $\varphi(\alpha)$ is homotopic to either λ or μ.*

PROOF. An easy computation shows that C_a has no singular point of the surface if $u=a$ is a simple root of the equation $D_k(u)=0$. On the other hand we obtain from our assumption that $g_{4k}(a)\ h_{6k}(a)\neq 0$ and rank $(\varphi(u))_{u=a} = \text{rank}\,(g'_{4k}(a), h'_{6k}(a))=1$, where g'_{4k}, h'_{6k} are the derivatives of g_{4k}, h_{6k} with respect to u. Thus $u=a$ is a simple root of $D_k(u)=0$. Therefore C_a is a singular fibre of type I_1 ([**3**], Theorem 6.2). q. e. d.

By Lemma 1.3, Lemma 2.1 and the results of Kodaira [**3**, p. 604] we can normalize the monodromy $\tilde{\rho}$ of F in such a way that $\tilde{\rho}(\lambda)=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$. Then, by Lemma 1.3, $\tilde{\rho}(\mu)$ is equal to either $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ or $\begin{pmatrix}2-d & (d-1)^2\\ -1 & d\end{pmatrix}$, where d is an integer. Let $S=\begin{pmatrix}1 & 1-d\\ 0 & 1\end{pmatrix}$. Then

$$S\cdot\tilde{\rho}(\lambda)\cdot S^{-1}=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$$

and

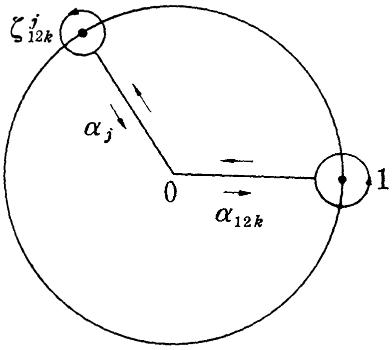
$$S\cdot\begin{pmatrix}2-d & (d-1)^2\\ -1 & d\end{pmatrix}\cdot S^{-1}=\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}.$$

Thus we can take either $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ or $\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}$ for $\tilde{\rho}(\mu)$. On the other hand since it must be determined uniquely by F, we shall show the form of $\tilde{\rho}(\mu)$ by the calculation of the following example.

EXAMPLE A. $g_{4k}(u)=3$, $h_{6k}(u)=u^{6k}$.

REMARK 2.2. If C_∞ is regular, it suffices to consider a monodromy over C_0. In fact, the matrix which corresponds to $u=\infty$ is $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

Now $D_k(u)=27-27u^{12k}$. Then the singular fibres of this surface exist over $u=\zeta_{12k}^j$ ($j=1, 2, \cdots, 12k$), where $\zeta_{12k}=\exp(2\pi i/12k)$. Choose loops α_j ($j=1, 2, \cdots, 12k$) which start at the origin 0, round ζ_{12k}^j in the positive direction and return to the origin.



We note that $\varphi(u)=(3, u^{6k})$. Therefore $\varphi(\alpha_{2m})=\lambda$, $\varphi(\alpha_{2m-1})=\mu$ ($m=1, 2, \cdots, 6k$). If we take $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ for $\tilde{\rho}(\mu)$, then

$$\alpha_1\cdot\alpha_2\cdot\cdots\cdot\alpha_{12k}=\begin{pmatrix}1 & 12k\\ 0 & 1\end{pmatrix},$$

while $\alpha_1\cdot\alpha_2\cdot\cdots\cdot\alpha_{12k}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ on Δ' (Remark 2.2). This is a contradiction. Therefore $\tilde{\rho}(\mu)=\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}$. Thus we obtain the following two results.

THEOREM 2.3. *If*

$$\tilde{\rho}(\lambda)=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},$$

then

$$\tilde{\rho}(\mu)=\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}.$$

LEMMA 2.4. *When $g_{4k}(u)=3$, $h_{6k}(u)=u^{6k}$, the monodromy of $B_k(\tau, \sigma)$ is determined by*

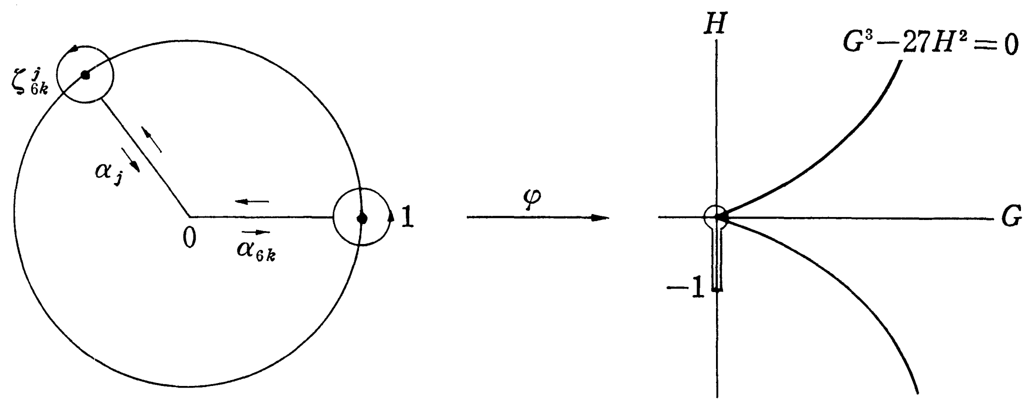
$$\rho(\alpha_{2m})=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},\qquad (m=1, 2, \cdots, 6k)$$
$$\rho(\alpha_{2m-1})=\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}.$$

Moreover $\rho(\pi_1(\Delta', 0))=\rho(\pi_1(\Delta'', 0))=SL(2, \boldsymbol{Z})$.

REMARK 2.5. $SL(2, \boldsymbol{Z})$ is generated by $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}$ and subject only to the fundamental relation $\left\{\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\right\}^6=1$.

Now we examine some other examples.

EXAMPLE B (Elliptic surface of Fermat type, Sasakura [4]). $g_{4k}(u)=0$, $h_{6k}(u)=u^{6k}-1$. Then $D_k(u)=-27(u^{6k}-1)^2$. The singular fibres of this surface exist over $u=\zeta_{6k}^j$ $(j=1, 2, \cdots, 6k)$, where $\zeta_{6k}=\exp(2\pi i/6k)$. An easy computation shows that all singular fibres are of type II ([3] Theorem 6.2). Choose loops α_j $(j=1, 2, \cdots, 6k)$ which start at the origin, round ζ_{6k}^j in the positive direction and return to the origin. We note that $\varphi(u)=(0, u^{6k}-1)$.



Then $\varphi(\alpha_j)$ is homotopic to $\lambda\mu$. Therefore by an appropriate choice of a basis $\{\gamma_1, \gamma_2\}$ for $H_1(C_0, \boldsymbol{Z})$,

$$\rho(\alpha_j)=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}=\begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix}\qquad \text{for all } j=1, 2, \cdots, 6k.$$

REMARK 2.6. Before examining the next example, we note that we can make the same discussion over $\boldsymbol{C}_1=\Delta-\{u=0\}$ as in § 1. More generally, let

$$u_2=\frac{au+b}{cu+d},$$

where $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in GL(2, \boldsymbol{C})$ and $c\neq 0$. We consider u_2 as a local coordinate over $C_2=\Delta-\{u=-d/c\}$ with center $u=-b/a$ or ∞ according as $a\neq 0$ or $a=0$, respectively. Let $W_2=P^2(C)\times C_2$. We identify $(x, y, z, u)\in W_0$ and $(x_2, y_2, z_2, u_2)\in W_2$ if and only if

$$u_2=\frac{au+b}{cu+d},\ \Big(\frac{cu+d}{bc-ad}\Big)^{2k}x_2=x,\ \Big(\frac{cu+d}{bc-ad}\Big)^{3k}y_2=y,\ z_2=z.$$

Then this system of coordinate transformations determines exactly the same analytic fibre space as W^k defined in the introduction and the defining equation of $B_k(\tau, \sigma)$ in W_2 is

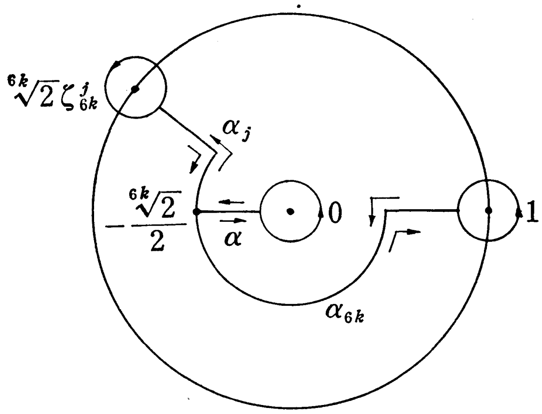
$$y_2^2z_2-4x_2^3+(cu_2-a)^{4k}g_{4k}\Big(\frac{du_2-b}{a-cu_2}\Big)x_2z_2^2+(cu_2-a)^{6k}h_{6k}\Big(\frac{du_2-b}{a-cu_2}\Big)z_2^3=0.$$

We set

$$\tilde\varphi(u_2)=\Big((cu_2-a)^{4k}\cdot g_{4k}\Big(\frac{du_2-b}{a-cu_2}\Big),\ (cu-a)^{6k}\cdot h_{6k}\Big(\frac{du_2-b}{a-cu_2}\Big)\Big)$$

for φ. Then obviously the same discussion as in §1 is valid over C_2.

EXAMPLE C. $g_{4k}(u)=3u^{4k}$, $h_{6k}(u)=u^{6k}-1$. By the above remark we calculate a monodromy over C_1. We note that the fibre over $u=0$ is regular (cf. Remark 2.2). The singular fibres of this surface exist over $u_1=0$, $\sqrt[6k]{2}\,\zeta_{6k}^j$ $(j=1, 2, \cdots, 6k)$. Choose a base point $u_1=-\sqrt[6k]{2}/2$ and loops α, α_j $(j=1, 2, \cdots, 6k)$ as shown below.



We note that $\tilde\varphi(u_1)=(3, 1-u_1^{6k})$. Therefore $\tilde\varphi(\alpha)$ is homotopic to λ^{6k} and $\tilde\varphi(\alpha_j)$ to $\lambda^{-3k+j}\mu\lambda^{3k-j}$. Thus

$$\rho(\alpha)=\begin{pmatrix}1 & 6k\\ 0 & 1\end{pmatrix}$$

and

$$\rho(\alpha_j)=\begin{pmatrix}1+(3k-j) & 3k-j\\ -1 & 1-(3k-j)\end{pmatrix}.$$

We obtain that the singular fibre over $u_1=0$ is of type I_{6k} and the singular fibres over $u_1=\sqrt[6k]{2}\,\zeta_{6k}^j$ $(j=1, 2, \cdots, 6k)$ are of type I_1.

3. Monodromy and Picard-Fuchs equation of $B_k(\tau, \sigma)$.

In this section we prove the following theorem.

THEOREM 3.1. *Let $B_k(\tau, \sigma)$ be an arbitrary basic elliptic surface over $\boldsymbol{P}^1(\boldsymbol{C})$ which satisfies the following condition* (*).

(*) *The roots of the equation $D_k(u)=0$, $\tilde{D}_k(u_1)=0$ are all simple.*

Then, by an appropriate choice of a base point $\mathcal{O}$ of $\pi_1(\Delta')$, a basis for $H_1(C_{\mathcal{O}}, \boldsymbol{Z})$ and loops β_j $(j=1, 2, \cdots, 12k)$ generating $\pi_1(\Delta', \mathcal{O})$, the global monodromy of $B_k(\tau, \sigma)$ is determined by

$$\left.\begin{aligned} \rho(\beta_{2j}) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ \rho(\beta_{2j-1}) &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \end{aligned}\right. \qquad (j=1, 2, \cdots, 6k).$$

PROOF. By an appropriate choice of a coordinate over Δ (cf. Remark 2.6), it suffices to consider the case in which the fibre C_∞ over $u=\infty$ is regular. First we fix our notation. In the $(10k+1)$-dimentional complex space $\boldsymbol{C}^{10k+1} \ni (\tau, \sigma)=(\tau_0, \tau_1, \cdots, \tau_{4k}, \sigma_1, \cdots, \sigma_{6k})$, we set

$$X_1=\{(\tau, \sigma) \mid \tau_0^3-27=0\}.$$

For a point $(\tau, \sigma) \in \boldsymbol{C}^{10k+1}-X_1$, we denote the discriminant of the algebraic equation $D_k(u)=0$ by $\delta(\tau, \sigma)$. We may consider it as a polynomial in τ, σ. We set

$$X_2=\{(\tau, \sigma) \in \boldsymbol{C}^{10k+1}-X_1 \mid \delta(\tau, \sigma)=0\}$$

and

$$\tilde{X}=\boldsymbol{C}^{10k+1}-X_1 \cup X_2.$$

If a point (τ, σ) belongs to $\tilde{X}$, $B_k(\tau, \sigma)$ satisfies the condition (*), and the fibre C_∞ over $u=\infty$ is regular. We note that the point $(\tau_0, \cdots, \tau_{4k-1}, \tau_{4k}, \sigma_1, \cdots, \sigma_{6k}) =(0, \cdots, 0, 3, 0, \cdots, 0)$ belongs to $\tilde{X}$ (Example A). From now on we denote this point by Σ_0. Let Σ_1 be an arbitrary point in $\tilde{X}$. Since $\tilde{X}$ is arc-wise connected, we can choose a path $\alpha(t)=(\tau(t), \sigma(t))$ $(0 \leqq t \leqq 1)$ in $\tilde{X}$ such that

(1) $\alpha(t)$ depends continuously on t,

(2) $\alpha(0)=\Sigma_0$, $\alpha(1)=\Sigma_1$.

Henceforth we denote by $B_k(t)$ and $\Delta(t)$, respectively, the basic elliptic surface and its base space which correspond to $(\tau(t), \sigma(t))$. Similarly we denote $D_k(u)$, $\boldsymbol{C}_0$, $\varphi(u)$, ρ, $\cdots$ by $D_{k,t}(u)$, $\boldsymbol{C}_0(t)$, $\varphi_t(u)$, ρ_t, $\cdots$, respectively.

Now we define a fibre space $\mathcal{F}$ over the unit interval $I=\{0 \leqq t \leqq 1\}$ as follows: we denote a point in $\boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{C}$ by (u, t), where u is a non-homogeneous coordinate of $\boldsymbol{P}^1(\boldsymbol{C})$. Let $a_j(t)$ be the root of $D_{k,t}(u)=0$ such that $a_j(0)=\zeta_{12k}^j$. Then $\mathcal{F}$ is a subset of $\boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{C}$ such that the fibre $\mathcal{F}_t$ over $t \in I$ is defined by

$$\mathcal{F}_t=\{\boldsymbol{P}^1(\boldsymbol{C})\times t\}-\{(a_1(t),t),\cdots,(a_{12k}(t),t),(\infty,t)\}$$
$$=\Delta''(t).$$

The projection ϕ of $\mathcal{F}$ onto I is defined by

$$\phi:(u,t)\longmapsto t.$$

Obviously $\mathcal{F}$ is a locally trivial fibre space. Since I is contractible, $\mathcal{F}$ is trivial. Thus there exists a fibre mapping ψ of $\Delta''(0)\times I$ onto $\mathcal{F}$ such that

(1) ψ is a homeomorphism,

(2) the following diagram is commutative:

$$\begin{tikzcd} \Delta''(0)\times I \arrow[rr, "\psi"] \arrow[dr, "p"'] & & \mathcal{F} \arrow[dl, "\phi"] \\ & I & \end{tikzcd}$$

where p is the natural projection. Let ψ_t be the homeomorphism of $\Delta''(0)$ onto $\mathcal{F}_t=\Delta''(t)$ induced by ψ i.e.

$$\psi_t(u)=\psi(u,t).$$

We note that ψ_0 is the identity mapping.

We set $O=\psi_1(0)$ and $\beta_j=\psi_1(\alpha_j)$. Then $\varphi_0(\alpha_j)=\varphi_0\circ\psi_0(\alpha_j)$ is homotopic to $\varphi_1(\beta_j)$, for we can take $(\varphi_t\circ\psi_t)(\alpha_j)$ as a homotopy. This completes the proof of Theorem 3.1.

Now we refer to Griffiths ([1], p. 1305) for the general definition of the Picard-Fuchs differential equation. He stated it as a higher order differential equation, and it is easy to modify it to a system of equations of the first order. Then an easy computation shows that the Picard-Fuchs differential equation of $B_k(\tau,\sigma)$ is

$$(\sharp)\qquad \frac{d}{du}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}=\begin{pmatrix} -\dfrac{1}{12}\dfrac{d}{du}\log D_k(u) & \dfrac{3\delta_k(u)}{2D_k(u)} \\ -\dfrac{g_{4k}(u)\delta_k(u)}{8D_k(u)} & \dfrac{1}{12}\dfrac{d}{du}\log D_k(u) \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

where

$$\delta_k(u)=3h_{6k}(u)\frac{d}{du}g_{4k}(u)-2g_{4k}(u)\frac{d}{du}h_{6k}(u).$$

As a corollary to Theorem 3.1 we obtain the following.

THEOREM 3.2. *If the roots of the algebraic equations* $D_k(u)=0$, $\tilde{D}_k(u_1)=0$ *are all simple, then, by an appropriate choice of a base point, a system of fundamental solutions of* ($\sharp$) *and loops* β_j *on* $\boldsymbol{P}^1(\boldsymbol{C})$, *the global monodromy representation* ρ *of* ($\sharp$) *is determined by*

$$\left.\begin{aligned}\rho(\beta_{2j}) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ \rho(\beta_{2j-1}) &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\end{aligned}\right. \qquad (j=1, 2, \cdots, 6k).$$

References

[1] P. A. Griffiths, The residue calculus and some transcendental results in algebraic geometry, I, Proc. Nat. Acad. Sci. U. S. A., **55** (1966), 1303–1309.

[2] A. Kas, Deformations of elliptic surfaces, Dissertation, Stanford University, 1966.

[3] K. Kodaira, On compact analytic surface II, Ann. of Math., **77** (1963), 563–626.

[4] N. Sasakura, Periods of rational forms on certain elliptic surfaces, J. Math. Soc. Japan, **21** (1969), 638–671.

[5] F. Pham, Introduction à l'étude topologique des singularités de Landau, Mémorial des Sc. Math., N°164, Gauthier-Villars, 1967.

Takao SASAI
Department of Mathematics
Faculty of Science
Tokyo Metropolitan University
Fukazawa, Setagaya-ku
Tokyo, Japan