J. Math. Soc. Japan
Vol. 25, No. 2, 1973

Finite groups with central Sylow 2-intersections

By Kensaku GOMI

(Received July 6, 1972)

§ 1. Introduction.

The purpose of this paper is to clarify the structure of finite groups which satisfy the following condition:

(CI): The intersection of any two distinct Sylow 2-groups is contained in the center of a Sylow 2-group.

From now on, we call a finite group a (CI)-*group* if it satisfies (CI). The main result is the following:

THEOREM 1. *Let G be a* (CI)-*group. Then one of the following statements holds.*

(1) *G is a solvable group of* 2-*length* 1.

(2) *A Sylow* 2-*group of G is Abelian.*

(3) *G has a normal series $1 \leqq N < M \leqq G$ where N and G/M have odd order and M/N is the central product of an Abelian* 2-*group and a group isomorphic to* $SL(2, 5)$.

(4) *G contains a normal subgroup M of odd index in G which satisfies one of the following conditions:*

(4.1) *M is the direct product of an Abelian* 2-*group and a group isomorphic to* $Sz(q)$, $PSU(3, q)$ *or* $SU(3, q)$, *q a* 2-*power* > 2.

(4.2) *M is the central product of an Abelian* 2-*group and a non-trivial perfect central extension of* $Sz(8)$.

If we combine Theorem 1 with the theorems of Walter [**13**] and Bender [**2**], we obtain the following result.

THEOREM 2. *A non-Abelian simple* (CI)-*group is isomorphic to one of the following groups:*

$PSL(2, q)$, $q \equiv 0, 3, 5 \pmod 8$,

JR,

$Sz(q)$ *or*

$PSU(3, q)$, *q a power of* 2.

Here JR denotes the simple groups with Abelian Sylow 2-groups in which the centralizer of an involution t is a maximal subgroup and isomorphic to $\langle t\rangle \times E$ where $PSL(2, q) \leqq E \leqq P\Gamma L(2, q)$ with odd $q > 3$. This definition is due to [2]. A large amount of work has been done by many authors to classify the groups JR, and the structure of JR is now very well known. In this paper, however, we need no knowledge of the properties of JR except those which are proved easily.

A group satisfying (2) or (4) in Theorem 1, clearly, is a (CI)-group, while a group satisfying (1) or (3) is not necessarily a (CI)-group. Let G be a group which satisfies (1). Then G is 2-closed (resp. a (CI)-group) if and only if $O_{2',2}(G)$ is 2-closed (resp. a (CI)-group). The following result is concerned with this situation.

THEOREM 3. *Let G be a group which is $2'$-closed but not 2-closed and S a Sylow 2-group of G. We set $H=O(G)$ and $C=C_S(H)$. If G is a* (CI)-*group, then one of the following statements is true.*

(a) *S is Abelian.*

(b) *C is contained in $Z(S)$. $Z(S)/C$ is elementary Abelian. S has a subgroup Q with the following properties: $Q \geqq C$, Q/C is a generalized quaternion group and $S=QZ(S)$. Furthermore, every composition S-factor of H is either centralized by S or inverted by the elements of $Z(Q)-C$.*[1]

Conversely, if G satisfies (a) *or* (b), *it is a* (CI)-*group.*

Here, a *composition S-factor* of H means a factor of a composition series of H as a group with the set S of operators. Let G be a group satisfying (3) in Theorem 1, and let S/N be a Sylow 2-group of G/N. It is easy to prove that G is a (CI)-group if and only if S satisfies (CI). Therefore Theorems 1 and 3 give us a necessary and sufficient condition for a group to be a (CI)-group. Theorem 3 also plays an important role in the proof of Theorem 1.

NOTATION AND REMARKS. Unexplained notation is either standard or will be found in [5], especially pp. 4-5, pp. 519-520. All groups are assumed to be finite. The group G is said to be *perfect* if $G'=G$. A perfect group G is called *semisimple* provided $G/Z(G)$ is the direct product of non-Abelian simple groups, and is called *quasisimple* provided $G/Z(G)$ is simple. It is not difficult to show that a semisimple group G is the central product of uniquely determined quasisimple subgroups, which we call the *components* of G. Every group G has the unique maximal normal semisimple subgroup, which we denote by $L(G)$. We define $O_2^*(G)=O_2(G)L(G)$. It is becoming well known that $C_G(O_2^*(G)) \leqq O_2(G)$ if $O(G)=1$ ([6], Theorem 2). A quasisimple group G is said to be *of type* $\mathfrak{F}$ where $\mathfrak{F}$ is a family of simple groups provided $G/Z(G)$ is isomorphic to a member of $\mathfrak{F}$. The 2-element of the group G is called *central in G* if it is contained in the center of a Sylow 2-group of G. A *quaternion group* is a non-Abelian 2-group which contains only one involu-

1) It is not difficult to show that C has index 2 in $Z(Q)$. See Section 4.

tion. The group G is called *π-closed* where π is a set of primes, if the π-elements of G generate a π-group. The symbol $2'$ denotes the set of odd primes. Throughout the paper, we use the fundamental theorem of Feit-Thompson [3] implicitly.

§ 2. Preliminary results.

LEMMA 2.1. *Let S be a 2-group. Assume that $Z(S)$ contains distinct maximal subgroups $C_1, C_2, \cdots, C_n$, $n \geqq 2$, such that $C_1 \cap C_2 \cap \cdots \cap C_n = 1$ and S/C_i is a quaternion group, $i = 1, 2, \cdots, n$. Then S is the direct product of an elementary Abelian group and a quaternion group.*

PROOF. Induction on n. Assume first $n = 2$. In this case we proceed by induction on the order of S. Clearly, $Z(S)$ is a four-group and $\Omega_1(S) = Z(S)$ as $Z(S)/C_1$ is the unique subgroup of order 2 in S/C_1. Assume $|S: Z(S)| = 4$. Inspecting the known list of the groups of order 16, we know that either S is the direct product of a group of order 2 and a quaternion group of order 8, or else S is generated by the elements a and b subject to the relations:

$$a^4 = 1, \qquad b^4 = 1, \qquad a^{-1}ba = b^{-1}.$$

However, the latter does not satisfy the assumption of our lemma. Therefore the assertion is true if $|S: Z(S)| = 4$; so assume $|S: Z(S)| > 4$. Then S contains a maximal subgroup V containing $Z(S)$ such that $V/Z(S)$ is a dihedral group. Since V/C_i is a quaternion group, $i = 1, 2$, we have $Z(V) = Z(S)$. The induction hypothesis now applies to V. In particular, we see that V is not metacyclic. Since $S/Z(S)$ is a dihedral group, there is a maximal subgroup T of S containing $Z(S)$ such that $T/Z(S)$ is cyclic. T is Abelian and, as $\Omega_1(T) = Z(S)$, contains a cyclic maximal subgroup U. Since S/C_i is a quaternion group, $i = 1, 2$, elements of $S - T$ invert U whence U is a normal subgroup of S. As is noticed above, S is not metacyclic, and so there is a subgroup Q of S such that $S = TQ$ and $T \cap Q = U$. Since Q is non-Abelian and contains only one involution, Q is a quaternion group. Thus the assertion is true if $n = 2$.

Assume next $n > 2$. We apply the above argument to the group $S/C_1 \cap C_2$ and find a maximal subgroup T of S containing $C_1 \cap C_2$ such that $T/C_1 \cap C_2$ is a quaternion group. By the inductive hypothesis, there is a quaternion subgroup Q such that $T = Z(T)Q$ (and $|Z(T) \cap Q| = 2$). Since $Z(S)$ is elementary, the assertion is true in this case, too.

LEMMA 2.2. *JR has no non-trivial perfect central 2-extensions. If a 2-automorphism of JR centralizes a Sylow 2-group of JR, then it is inner.*

PROOF. (i) Suppose there is a perfect central extension of JR by a group Z of order 2, and let S be its Sylow 2-group. Since S is not extra-special, as $|S|=16$, and the normalizer of S acts irreducibly on S/Z, S is Abelian and so S is elementary Abelian. A theorem of Gaschütz [8], Hauptsatz I, 17.4, yields a contradiction.

(ii) Let a be a 2-automorphism of $G=JR$ centralizing a Sylow 2-group S of G. We first embed a and G in the semidirect product $G^*=G\langle a\rangle$. Let t be an involution of S and let $H=C_G(t)=\langle t\rangle\times E$. By definition E contains a normal subgroup K of odd index in E isomorphic to $PSL(2, q)$, $q\equiv 3, 5 \pmod 8$. It is immediate that a acts on K as an inner automorphism induced by an element u of $T=S\cap E$. Therefore in the group $E^*=E\langle a\rangle$, we have $au^{-1}\in C_{E^*}(O_2^*(E^*))\leqq O_2(E^*)$ as $O(E^*)=1$. Since $E\cap O_2(E^*)=1$, we conclude that $[E, au^{-1}]=1$. Thus we have $[H, b]=1=[H, bt]$ for $b=au^{-1}$. Let A be a complement of S in $N_G(S)$ and let $B=N_H(S)\cap A$. Counting the conjugates of A in $N_G(S)$ containing B, we see that $A^b=A$ or $A^b=A^t$. Therefore $A^c=A$ for $c=b$ or bt. A Sylow 7-group of $A/C_A(S)$ is centralized by c, because it is regular on S and c centralizes S. So c centralizes A. Since H is a maximal subgroup of G, this implies that a acts on G as an inner automorphism induced by u or tu.

LEMMA 2.3. *Let S be a 2-group of rank 2. If $Z(S)$ contains a maximal subgroup D such that S/D is a quaternion group, then S has a cyclic characteristic subgroup $\neq 1$.*

PROOF. Suppose false. Then every Abelian characteristic subgroup $\neq 1$ of S is a homocyclic group of rank 2. Since $S/Z(S)$ is a dihedral group, $S'/S'\cap Z(S)$ is cyclic and so S' is Abelian. Since both S' and $S'\cap Z(S)$ are homocyclic of rank 2 and $S'/S'\cap Z(S)$ is cyclic, we have $S'\leqq Z(S)$. If $S'=Z(S)$, then $|S: S'|=4$, as $S/Z(S)$ is dihedral, and so S is a 2-group of maximal class and contains a cyclic characteristic subgroup $\neq 1$ ([**5**], Theorem 5.4.5). So we assume $S'\neq Z(S)$. Since both S' and $Z(S)$ are homocyclic groups, S' is contained in the Frattini subgroup of $Z(S)$. Thus, $S'\leqq D$, a contradiction.

LEMMA 2.4. *Let Y be a $2'$-group of automorphisms of an Abelian 2-group T and X a proper subgroup of Y. Assume the following conditions:*

(i) *$T=W\times Q$ where W and Q are X-invariant subgroups of T and Q is elementary Abelian.*

(ii) *$W\cap W^y=1$ for every $y\in Y-X$.*

(iii) *There exists a cyclic normal subgroup R of X which is regular on $Q^\#$ and centralizes W.*

Then W is cyclic (and $|Q|=4$ if $W\neq 1$).

In order to prove this, we can assume that T is elementary Abelian, as

Y acts faithfully on $\Omega_1(T)$ ([5], Theorem 5.2.4). Therefore the last four paragraphs of the proof of [2], (3.8), are applicable without changes.

§ 3. Properties of (CI)-groups.

We will discuss here elementary properties of (CI)-groups. We assume G to be a (CI)-group and S its Sylow 2-group throughout the section.

LEMMA 3.1. *$C_S(x)$ is a Sylow 2-group of $C_G(x)$ for any x in S.*

PROOF. This is obvious, if x is contained in $Z(S)$; so assume that x is not contained in $Z(S)$. Let T be a Sylow 2-group of G such that $C_T(x)$ is a Sylow 2-group of $C_G(x)$ containing $C_S(x)$. If $S \neq T$, then $S \cap T$ is contained in the center $Z(U)$ of a Sylow 2-group U of G. Thus $\langle Z(S), x\rangle \leqq Z(U)$ whence $x \in Z(S)$, contrary to our assumption. Therefore, $S=T$ and so $C_S(x)$ is a Sylow 2-group of $C_G(x)$.

Lemma 3.1 in particular implies that a *central* 2-element of G contained in S is necessarily contained in $Z(S)$. Hence we have the following result.

LEMMA 3.2. *If T is a Sylow 2-group of G different from S, then $S \cap T \leqq Z(S) \cap Z(T)$.*

The proof of the following lemma is easy, if we use the preceding lemma, and is left to the reader.

LEMMA 3.3. *Subgroups and quotient groups of a* (CI)-*group are also* (CI)-*groups.*

LEMMA 3.4. *If x is a 2-element of G, then $C_G(x)$ acts transitively on the Sylow 2-groups of G containing x.*

PROOF. Let S and T be Sylow 2-groups of G containing x. If $S \neq T$, then by Lemma 3.2, S and T are contained in $C_G(x)$, and the assertion follows from Sylow's theorem.

In exactly the same way, we can prove the following:

LEMMA 3.5. *If x is an element of S for which $C_G(x)$ is 2-closed, then S is the only Sylow 2-group that contains x.*

LEMMA 3.6. *Two elements of S which are conjugate in G are already conjugate in $N_G(S)$.*

PROOF. Assume that x and x^g are contained in S where $g \in G$. By Lemma 3.4, we find an element c of $C_G(x)$ such that $gSg^{-1}=c^{-1}Sc$. Then cg normalizes S and $x^{cg}=x^g$.

LEMMA 3.7. *If $C \neq 1$ is a cyclic characteristic subgroup of S, then the involution of C is contained in $Z^*(G)$.*

PROOF. Let c be the involution of C. Lemma 3.6 implies that if $c^g \in S$, $g \in G$, then $c^g=c$. Glauberman's Z^*-theorem [4] yields $c \in Z^*(G)$.

LEMMA 3.8. *If G is 2-constrained, then G is a solvable group of 2-length* 1.

PROOF. We can assume $O(G)=1$. Therefore $C_G(O_2(G))\leqq O_2(G)$ by the definition of 2-constraint. Thus, by (CI), G is 2-closed.

§4. Proof of Theorem 3.

Let G be a group which is $2'$-closed but not 2-closed and S its Sylow 2-group. We set $H=O(G)$ and $C=C_S(H)$. We first assume that G is a (CI)-group with non-Abelian Sylow 2-groups, and prove that G satisfies the condition (b) in Theorem 3. Since C is a normal 2-subgroup of G and G is not 2-closed, C is contained in $Z(S)$ by (CI). Let $\{V_1, \cdots, V_n\}$ be the set of composition S-factors of H not centralized by S. Put $C_i=C_S(V_i)$, $1\leqq i\leqq n$. It is known that $C_1\cap C_2\cap\cdots\cap C_n=C$ (cf. Proof of [**5**], Theorem 5.3.2). Our aim will be to prove the following: *for each* i, $1\leqq i\leqq n$,

C_i is a maximal subgroup of $Z(S)$,
S/C_i is a quaternion group, and
elements of $Z(S)-C_i$ invert V_i.

If this is true, then $Z(S)/C=Z(S/C)$ and we conclude from Lemma 2.1 that S has a subgroup Q containing C with the following properties:

Q/C is a quaternion group,
$S=QZ(S)$, and
$|Q\cap Z(S):C|=2$.

Furthermore, $Z(S)/C$ is elementary Abelian and $Z(Q)=Q\cap Z(S)$. Since C_i does not contain $Z(Q)$ as S/C_i is quaternion, we have $C=C_i\cap Z(Q)$. Therefore elements of $Z(Q)-C$ invert each V_i, and the condition (b) in Theorem 3 holds.

That C_i satisfies the above italicized properties is proved in the following way. Let $V_i=K_i/L_i$ where K_i and L_i are S-invariant subgroups of H and L_i is normal in K_i. We set $G_i=SK_i$ and $\bar{G}_i=G_i/L_i$. Then $\bar{G}_i$ is $2'$-closed but not 2-closed, as $\bar{S}$ does not centralize $\bar{K}_i=V_i$. Clearly, $C_{\bar{S}}(V_i)=\bar{C}_i$ whence $\bar{C}_i\leqq Z(\bar{S})$, because $\bar{C}_i$ is a normal 2-subgroup of a (CI)-group $\bar{G}_i$. Thus, $C_i\leqq Z(S)$. Let $\bar{A}/\bar{C}_i$ be a non-identity subgroup of $\bar{S}/\bar{C}_i$. We argue that $C_{V_i}(\bar{A})=1$. This holds if $\bar{A}\leqq Z(\bar{S})$, because $\bar{S}$ acts irreducibly on V_i. We note that V_i is solvable by the Feit-Thompson theorem [3], and so it is elementary Abelian. If $\bar{A}\nleqq Z(\bar{S})$, then $\bar{S}$ is the only Sylow 2-group of $\bar{G}_i$ that contains $\bar{A}$ (see Lemma 3.2). Since $\bar{S}$ is self-normalizing in $\bar{G}_i$, we again have $C_{V_i}(\bar{A})=1$. Hence $\bar{S}/\bar{C}_i$ is a regular group of automorphisms of V_i whence it is a quaternion group and its unique involution inverts V_i ([**5**], Theorem 10.1.4, Theorem 10.3.1). Thus, S/C_i is a quaternion group and elements of $Z(S)-C_i$, if $Z(S)\neq C_i$, invert V_i. We finally verify that C_i is a maximal subgroup of $Z(S)$. Let Z/C_i be the center of S/C_i. Since S/C_i is a quaternion group, C_i has index 2 in Z. Furthermore Z/C_i is contained in cyclic subgroups A/C_i and B/C_i of S/C_i such that $S=AB$. Since both A and B are Abelian, we conclude that $Z=Z(S)$.

We next prove the converse. Assume that G satisfies (a) or (b) in Theorem 3. Note that if G satisfies (b), then C has index 2 in $Z(Q)=Q\cap Z(S)$. Suppose that G is a counterexample of minimal order to the assertion that G is a (CI)-group. Then S is not Abelian and, as $Z(S/C)=Z(S)/C$, $C=1$ (see Lemma 3.2). Since G is a counterexample, there is an element h of $H-N_H(S)$ such that $S\cap S^h$ is not contained in $Z(S)$. Since $\Omega_1(S)=Z(S)$, $S\cap S^h$ contains an involution which has a square root in S. By the same reason, the involution of $Z(Q)$ is the only one that has a square root in S. Hence $Z(Q)\leqq S\cap S^h$. Let H/K be a composition S-factor of H. Suppose that S^h is contained in SK. In this case we may assume that h is an element of K. If $C_S(K)$ is not contained in $Z(S)$, then we have $Z(Q)\leqq C_S(K)$ in exactly the same way as above. The condition (b) now implies that S centralizes each composition S-factor of K, so even K itself, contradicting $S\neq S^h$. Hence we assume $Z(Q)\nleqq C_S(K)\leqq Z(S)$. In this case, we can apply the inductive hypothesis to SK and conclude that $S\cap S^h\leqq Z(S)$ (see Lemma 3.2), contrary to the choice of h. Therefore S^h is not contained in SK. If S centralizes H/K, equivalently $[S, H]\leqq K$, then we conclude readily that $S^h\leqq SK$. Therefore H/K is not centralized by S, and so is inverted by the involution of $Z(Q)$. Put $\bar{G}=G/K$. Since the involution of $Z(\bar{Q})$ inverts $\bar{H}$, $C_{\bar{S}}(\bar{H})\leqq Z(\bar{S})$. So we can apply the inductive hypothesis to $\bar{G}$, if $K\neq 1$. We conclude that $\bar{S}=\bar{S}^{\bar{h}}$, or equivalently $S^h\leqq SK$, but this is not the case. Hence $K=1$, whence S acts faithfully and irreducibly on H. In particular, every central involution of S acts fixed-point-freely on H. Hence S is a quaternion group and $S=C_G(Z(Q))=S^h$. This contradiction completes the proof.

§ 5. Proof of Theorem 1.

We will begin the proof of Theorem 1. As a matter of fact, we first prove Theorem 2 and obtain Theorem 1 as a corollary of Theorem 2 and a few additional results. Let $\mathfrak{F}$ denote the family of simple groups on the following list:

$PSL(2, q)$, $q\equiv 0, 3, 5 \pmod 8$,
JR,
$Sz(q)$ or
$PSU(3, q)$, q a power of 2.

In some places in this section, we shall use the properties of the automorphism groups and representation groups of these groups. Necessary materials will be found in Lemma 2.2, [1], [8], [9], [10], etc.

LEMMA 5.1. *Let G be a* (CI)*-group with $O(G)=1$. Assume that every component of $L(G)$, if $L(G)\neq 1$, is of type $\mathfrak{F}$. Then G satisfies one of the following conditions:*

(1′) *G is 2-closed.*

(2′) *A Sylow 2-group of G is Abelian.*

(3′) *G contains a normal subgroup M which has odd index in G, and is the central or direct product of an Abelian 2-group and a quasisimple group isomorphic to one of the following groups:*

$SL(2, 5)$,

$\hat{S}z(8)$: *a non-trivial perfect central 2-extension of $Sz(8)$,*

$Sz(q)$ *or*

$PSU(3, q)$, *q a power of 2 and $q>2$.*

PROOF. Let S be a Sylow 2-group of G. We set $L=L(G)$, $T=S\cap O_2^*(G)$ and $U=T\cap L$. If $L=1$, then G is 2-constrained and so 2-closed by Lemma 3.8. So we can assume $L\neq 1$. Since G is a (CI)-group but not 2-closed, we have $O_2(G)\leqq Z(S)$.

Case 1. Assume $U\leqq Z(S)$. Then each component of L is of type $PSL(2, q)$, $q\equiv 0, 3, 5 \pmod 8$, or JR, and is normalized by S. Let K be a component of L, then S induces a 2-group of automorphisms of $\bar{K}=K/Z(K)$ which centralizes a Sylow 2-group of $\bar{K}$. We conclude from Lemma 2.2 and the structure of $P\Gamma L(2, q)$ that S induces a group of inner automorphisms of $\bar{K}$, and even of K. Since K is arbitrary, S induces a group of inner automorphisms of L. Since S as well as L centralizes $O_2(G)$, we have $S\leqq C_G(O_2^*(G))L\leqq O_2^*(G)$. Thus, $S=T$ and (2′) holds.

Case 2. Assume $U\nleqq Z(S)$, then, as G is a (CI)-group, L is quasisimple and $N_G(U)\leqq N_G(S)$ and so G/L is 2-closed. Suppose that U is Abelian. Then we conclude from Lemma 2.2 and the known structure of the representation group of $PSL(2, q)$ that L is isomorphic to $PSL(2, q)$, $q\equiv 0, 3, 5 \pmod 8$, or JR. Since U is normal in S, $U\cap Z(S)\neq 1$. Transitivity of $N_L(U)$ on $U^\#$ and $N_L(U)\leqq N_L(S)$ yield $U\leqq Z(S)$, contrary to assumption. Hence U is non-Abelian. Therefore L is isomorphic to one of the groups mentioned in (3′) above. Note that $O(L)=1$ and that $SL(2, q)$, q odd >5, is not a (CI)-group. Set $C=C_G(L/Z(L))$. Since $C\cap L=Z(L)$ is a 2-group and G/L is 2-closed, C is also 2-closed and $O_2(G)$ is the unique Sylow 2-group of C. It will thus suffice to prove that $|G:CL|$ is odd. We first note that G/C is isomorphic to a subgroup of the automorphism group of $\bar{L}=L/Z(L)$ containing the group of inner automorphisms of $\bar{L}$. If $L\cong SL(2, 5)$, then $G/C\cong PSL(2, 5)$ or $PGL(2, 5)$. However $PGL(2, 5)$ is not a (CI)-group, as is easily verified by Lemma 3.7. Hence $G/C\cong PSL(2, 5)$, or equivalently $G=CL$. If $L\cong \hat{S}z(8)$ or

$Sz(q)$, then $|G:CL|$ is odd, because the outer automorphism group of $Sz(q)$ has odd order. In order to treat the case where $L \cong PSU(3, q)$, q a power of 2, it will suffice to prove the following result.

LEMMA 5.2. *Let X be a subgroup of $P\Gamma U(3, q)$ containing $PSU(3, q)$, q a power of 2. If X is a (CI)-group, then $|X: PSU(3, q)|$ is odd.*

PROOF. Suppose false. We can assume that $X = PSU(3, q)\langle a\rangle$ where a is an involution represented by the involutive automorphism $\neq 1$ of $GF(q^2)$. We find a Sylow 2-group R of $PSU(3, q)$ normalized but not centralized by a such that a has a fixed point b on $PSU(3, q) - N_{PSU(3,q)}(R)$. Since b does not normalize $R\langle a\rangle$, (CI) forces $[R, a] = 1$. This contradiction completes the proof.

LEMMA 5.3. *Let G be a (CI)-group. Assume that $\bar{G} = G/O(G)$ satisfies one of the conditions (1′)—(3′) in Lemma 5.1 where for G we read $\bar{G}$. Then G satisfies one of the conditions (1)—(4) in Theorem 1.*

PROOF. We need only consider the case where $\bar{G}$ contains a normal subgroup which has odd index in $\bar{G}$, and is the central or direct product of an Abelian 2-group and a quasisimple group $\bar{L}$ isomorphic to one of the following groups:

$\hat{S}z(8)$, $Sz(q)$ or $PSU(3, q)$, q a power of 2.

Let S be a Sylow 2-group of G, then S centralizes $O(G)$, otherwise Theorem 3 applied to $SO(G)$ yields that either S is Abelian or $S/Z(S)$ is dihedral, but this is not the case. Hence, if we denote by L the unique minimal normal subgroup of G which covers $\bar{L}$, L also centralizes $O(G)$, because L is perfect and so is generated by its Sylow 2-groups. Therefore L is a quasisimple group of type $Sz(q)$ or $PSU(3, q)$, q a power of 2. Furthermore, $[S, O(G)] = 1$ implies $O_{2',2}(G) = O(G) \times O_2(G)$. Hence $M = O_2(G)L$ is a normal subgroup of G which has odd index in G and satisfies one of the conditions (4.1) or (4.2) in Theorem 1. The proof is complete.

THEOREM 4. *Let G be a (CI)-group with $Z^*(G) = 1$. Assume that the centralizer of every central involution of G is 2-constrained. Then one of the following statements is true.*

(i) *A Sylow 2-group of G is Abelian.*

(ii) *G is a (TI)-group.*

PROOF. We recall from [**11**] that a group is called a (TI)-*group* if two distinct Sylow 2-groups have only the identity element in common. If the centralizer of every central involution of G is 2-closed, then Lemma 3.5 and (CI) imply that G is a (TI)-group. So we assume that the centralizer H of a central involution, say x, is not 2-closed. Let S be a Sylow 2-group of G. Theorem 3 applied to $O_{2',2}(H)$ yields that either S is Abelian or $Z(S)$ contains a maximal subgroup D such that S/D is a quaternion group. In particular, all involutions of G are central. Lemma 3.7 and $Z^*(G)=1$ imply that $Z(S)$ is non-cyclic; so G is connected in the sense of [7]. If $m(G)\geqq 3$, then the "balanced theorem" of Gorenstein-Walter [7], Theorem B, yields $O(H)=1$ and so H is 2-closed, contrary to the choice of H. Hence $m(G)=2$. Suppose that S is non-Abelian, then Lemma 2.3 implies that S contains a cyclic characteristic subgroup $\neq 1$, contradicting $Z^*(G)=1$. Thus, S is Abelian. The proof is complete.

THEOREM 5. *Let G be a non-Abelian simple* (CI)*-group. Assume that not all centralizers of central involutions of G are 2-constrained, and that each non-Abelian composition factor of every proper subgroup of G is isomorphic to a member of $\mathfrak{F}$. Then a Sylow 2-group of G is Abelian.*

PROOF. Let S be a Sylow 2-group of G. We begin with a few remarks. Since $Z^*(G)=1$, Lemma 3.7 implies that S has no cyclic characteristic subgroups $\neq 1$. Lemmas 5.1, 5.3 and the assumption imply that every proper subgroup X of G satisfies one of the conditions (1)—(4) in Theorem 1. However, X does not satisfy (3) if X contains a Sylow 2-group of G, otherwise S' will be a characteristic subgroup of S of order 2. We divide the proof into seven parts. Furthermore, we assume S to be non-Abelian.

(I) *Let x be a central involution of G for which $H=C_G(x)$ is not 2-constrained. Then H contains a normal subgroup M which has odd index in H and is the direct product of a non-cyclic Abelian 2-group and a quasisimple group isomorphic to $Sz(q)$, $PSU(3, q)$ or $SU(3, q)$, q a power of 2 and $q>2$.*

PROOF. Since H is not 2-constrained and contains a Sylow 2-group of G, we conclude from preceding remarks that H satisfies the condition (4) of Theorem 1. We will eliminate the possibility of the condition (4.2). By way of contradiction, we suppose that H contains a normal subgroup which has odd index in H and is the central product of an Abelian 2-group and a group isomorphic to $\hat{S}z(8)$.

Let T be a Sylow 2-group of G different from S such that $S\cap T\neq 1$. We will prove that $Z(S)=Z(T)$. Let y be an involution of $S\cap T$ and set $K=C_G(y)$. Since $S\neq T$, Lemma 3.2 implies $S, T\leqq K$ and so K is not 2-closed. Thus K is not 2-constrained, otherwise Theorem 3 applied to $O_{2',2}(K)$ implies that either S is Abelian or $S/Z(S)$ is dihedral, but this is not the case. So K contains a normal subgroup which has odd index in K and satisfies the condition (4.2) in Theorem 1. It follows immediately that $Z(S)=O_2(K)=Z(T)$, as desired.

We argue that $L=N_G(Z(S))$ is a strongly embedded subgroup of G. If $|L\cap L^g|$ is even where $g\in G$, then there exist Sylow 2-groups P and Q of L such that $P\cap Q^g\neq 1$; so $Z(P)=Z(Q^g)=Z(Q)^g$ as is proved above. Moreover we have $Z(P)=Z(S)=Z(Q)$, because $P\cap Q\geqq Z(S)$. Thus, $Z(S)^g=Z(S)$

and so $g \in L$. This implies that L is a strongly embedded subgroup of G.

So L has only one conjugate class of involutions ([**5**], Theorem 9.2.1), but this is not the case since $S-Z(S)$ contains an involution. Therefore (4.2) does not occur.

We have proved that H contains a normal subgroup M which has odd index in H and is the direct product of an Abelian 2-group P and a group isomorphic to $Sz(q)$, $PSU(3, q)$ or $SU(3, q)$, q a power of 2 and $q>2$. We have to show that P is not cyclic. Since $H=C_G(x)$, $P\neq 1$. Let T be a Sylow 2-group of H, then $T=P\times R$ where R is isomorphic to a Sylow 2-group of $Sz(q)$ or $PSU(3, q)$. If P is cyclic, then $|P|=2$, otherwise the Frattini group of $Z(T)$ is a cyclic characteristic subgroup $\neq 1$ of T. However, if $|P|=2$, then Thompson's fusion lemma [**12**], Lemma 5.38, implies that the involution of P, or x, is conjugate to an element y of R. Since y is a square in T, and x is conjugate to y in $N_G(T)$ by Lemma 3.6, x is also a square in T, contradicting $T^2 \leqq R$. Therefore P is not cyclic.

(II) *S has the form $P\times R$ where P is a non-cyclic Abelian 2-group and R is isomorphic to a Sylow 2-group of $Sz(q)$ or $PSU(3, q)$, q a power of 2 and $q>2$. All involutions of S are contained in $Z(S)$.*

PROOF. This is an immediate consequence of (I).

(III) *Let H be a proper subgroup of G containing a Sylow 2-group of G. Then one of the following statements is true:*

(i) *H is 2-closed.*

(ii) *H contains a normal subgroup which has odd index in H, and is the direct product of a non-cyclic Abelian 2-group and a quasisimple group isomorphic to $Sz(q)$, $PSU(3, q)$ or $SU(3, q)$, q a power of 2.*

PROOF. Since H satisfies (1) or (4) in Theorem 1, (II) implies that H satisfies (ii), or else H is a solvable group of 2-length 1. In the latter case, Theorem 3 implies that H is 2-closed, because $S/Z(S)$ is an elementary Abelian group of order >4.

(IV) *G contains no strongly embedded subgroups.*

PROOF. Suppose that G has a strongly embedded subgroup H. We can assume $S\leqq H$. If H satisfies the condition (ii) of (III), then H has an Abelian normal 2-subgroup $P\neq 1$ such that $H-P$ contains an involution. This is a contradiction, since a strongly embedded subgroup of the group has only one conjugate class of involutions. Consequently, H is 2-closed and so $H=N_G(S)$. However, $N_G(S)\leqq N_G(S')$, and $S-S'$ contains an involution, again a contradiction. The proof is complete.

For each involution x of G, we define $M(x)$ to be the set of maximal subgroups of G containing $C_G(x)$. In the following three steps, let x be an involution of S and H a member of $M(x)$ which is not 2-constrained. Such

x and H exist by assumption. Note that H satisfies the condition (ii) of (III). The argument to be used in (V), (VI) and (VII) below appears in [2], (3.8), (4.4) and (5.1).

(V) *$M(y)=\{H\}$ for every involution y of $O_2(H)$.*

PROOF. Let y be an involution of $O_2(H)$, and let M be an element of $M(y)$. Since $L(H)\leqq C_G(y)\leqq M$, M also satisfies the condition (ii) of (III). Since $M/L(M)$ is 2-closed, we have $L(H)\leqq L(M)$. Also $S\leqq C_G(y)\leqq M$; so x induces an inner automorphism on $L(M)$. Thus, $L(H)$ is a $C_{L(M)}(z)$-invariant non-solvable subgroup of $L(M)$ where z is an involution of $L(M)$. Since $L(M)$ is isomorphic to $Sz(q)$, $PSU(3, q)$ or $SU(3, q)$, q a power of 2 and $q>2$, this forces $L(H)=L(M)$. Therefore, $H=N_G(L(H))=M$.

(VI) $N_G(Z(S))\leqq H$.

PROOF. Suppose false. Let Y and X be the groups of automorphisms of $T=Z(S)$ induced by $N_G(Z(S))$ and $N_H(Z(S))$, respectively. Since $S\leqq C_G(Z(S))\leqq C_G(x)\leqq H$, Y has odd order and X is a proper subgroup of Y. Set $W=O_2(H)$ and $Q=Z(S)\cap L(H)$, then $T=W\times Q$ and Q is elementary Abelian, because Q is the center of the Sylow 2-group $S\cap L(H)$ of $L(H)$. Clearly, both W and Q are X-invariant. Let R be the group of automorphisms of T induced by $N_{L(H)}(Z(S)\cap L(H))$, then R is a cyclic normal subgroup of X acting regularly on $Q^{\#}$. Clearly, R centralizes W. Suppose that $W\cap W^n\neq 1$ where $n\in N_G(Z(S))$. Let w be an involution of $W\cap W^n$. It follows from (V) that $M(w)=\{H\}=M(nwn^{-1})$, whence $H=H^n$ and so $n\in N_G(H)=H$. This implies that $W\cap W^y=1$ if $y\in Y-X$. Therefore all the conditions of Lemma 2.4 are satisfied. We conclude that $O_2(H)=W$ is cyclic, contradicting (III). Therefore, $N_G(Z(S))\leqq H$.

(VII) *S is Abelian.*

PROOF. Suppose false, then we can apply (I)—(VI). Let x and H be as before. There is an involution y of S such that $C_G(y)\nleqq H$, otherwise (VI) implies that H is a strongly embedded subgroup of G, contradicting (IV). Let M be an element of $M(y)$, then $M\nleqq H$ and so M is not 2-closed by (VI). Therefore M satisfies (ii) in (III). Set $K=L(H)$, $U=O_2(H)$, $L=L(M)$ and $V=O_2(M)$. Applying (V) to M, we have $M(v)=\{M\}$ for every involution v of V. As $H\neq M$, $U\cap V=1$. There is a subgroup R of $N_K(S\cap K)$ which has odd order and acts transitively on $Z(S\cap K)^{\#}$. Since $R\leqq N_K(S\cap K)\leqq N_G(S)\leqq M$ by (VI) applied to M, R normalizes V. Since $C_V(R)=C_S(R)\cap V=U\cap V=1$, we have $V=[V, R]\leqq[Z(S), R]\leqq Z(S\cap K)$. Therefore V is elementary Abelian and $V\leqq S^2$. However, on the other hand, we have $S=V\times(S\cap L)$ whence $V\cap S^2=1$. This is a contradiction. Hence S is Abelian, and the proof of Theorem 5 is complete.

It is now not difficult to prove Theorems 1 and 2. We first prove Theorem 2 by induction on the order of G. Here, G is a non-Abelian simple (CI)-group. If the centralizer of every central involution of G is 2-constrained, then, by Theorem 4, either G has Abelian Sylow 2-groups or G is a (TI)-group. By the results of Walter [**13**] and Suzuki [**11**], G is isomorphic to $PSL(2, q)$, $q \equiv 0, 3, 5 \pmod 8$, $Sz(q)$ or $PSU(3, q)$, q a power of 2. If not all centralizers of central involutions of G are 2-constrained, then the inductive hypothesis and Theorem 5 implies that G has Abelian Sylow 2-groups. Thus, $G \cong JR$. Theorem 1 is an immediate consequence of Theorem 2, Lemmas 5.1 and 5.3.

Department of Mathematics
University of Tokyo
Hongo, Bunkyo-ku, Tokyo
Japan

References

[1] J. L. Alperin and D. Gorenstein, The multiplicators of certain simple groups, Proc. Amer. Math. Soc., **17** (1966), 515–519.
[2] H. Bender, On groups with Abelian Sylow 2-subgroups, Math. Z., **117** (1970), 164–176.
[3] W. Feit and J. G. Thompson, Solvability of groups of odd order, Pacific J. Math., **13** (1963), 755–1029.
[4] G. Glauberman, Central elements of core-free groups, J. Algebra, **4** (1966), 403–420.
[5] D. Gorenstein, Finite groups, Harper and Row, New York, 1968.
[6] D. Gorenstein and J. H. Walter, The π-layer of a finite group, Illinois J. Math., **15** (1971), 555–565.
[7] D. Gorenstein and J. H. Walter, Centralizers of involutions in balanced groups, J. Algebra, **20** (1972), 284–319.
[8] B. Huppert, Endliche Gruppen, Springer-Verlag, Berlin/New York, 1968.
[9] R. Steinberg, Automorphisms of finite linear groups, Canad. J. Math., **12** (1960), 606–615.
[10] M. Suzuki, On a class of doubly transitive groups I, Ann. of Math., **75** (1962), 105–145.
[11] M. Suzuki, Finite groups of even order in which Sylow 2-subgroups are independent, Ann. of Math., **80** (1964), 58–77.
[12] J. G. Thompson, Non-solvable finite groups all of whose local subgroups are solvable, Bull. Amer. Math. Soc., **74** (1968), 383–437.
[13] J. H. Walter, The characterization of finite groups with Abelian Sylow 2-subgroups, Ann. of Math., **89** (1969), 405–514.

Added in proof. Recently Goldschmidt [2-Fusion in finite groups (to appear)] has proved the following remarkable result: *Let G be a finite group, T a Sylow 2-group of G and A an Abelian strongly closed subgroup of*

T with respect to G, then non-cyclic composition factors of the normal closure of A in G are isomorphic to one of the groups on the list given in Theorem 2. If G is a (CI)-group with a Sylow 2-group T, then $Z(T)$ is strongly closed in T with respect to G by Lemma 3.1, so we can use this result to shorten the proof of Theorem 1.