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Approximations of nonlinear evolution equations

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§ 1. Introduction.

In this paper we are concerned with approximation of the solution to the Cauchy initial value problem

$$(1.1) \qquad 0 \in u'(t)+A(t)u(t), \qquad u(0)=x.$$

The basic tool of this investigation is a theorem by Crandall and Liggett [4] which provides conditions sufficient for the existence of the infinite product, $u(t)=\lim_{n\to\infty}\prod_{i=1}^{n}(I+(t/n)A(it/n))^{-1}x$. A product of the foregoing type is often referred to as a product integral. It is not always possible to obtain a solution to (1.1), however, it may be possible to associate a product integral with a Cauchy problem. As we shall see, solutions to given Cauchy problems may often be represented by product integrals. Our main result concerns the convergence of product integrals associated with a class of approximate Cauchy problems. Several authors have studied questions of this nature (e.g., see Oharu [15], Miyadera [13], [14], [15], Brezis and Pazy [1], [2], [3] and Mermin [10], [11], and Crandall and Pazy [5]).

The author is grateful for the opportunity to see preprints of the forementioned manuscripts of Brezis and Pazy, and Crandall and Liggett. Appreciation is due G. F. Webb for suggesting the problems considered and for his invaluable criticisms. The author also wishes to thank the referee whose suggestions strengthened Theorems 2.11 and 3.1.

§ 2. Preliminaries.

Throughout this paper X will be a real Banach space. It is often useful to consider "multivalued" operators. Cauchy problems associated with these operators assume the form $0 \in u'(t)+A(t)u(t)$. We shall refer to "multivalued" operators as subsets of $X\times X$. The term operator will be exclusively reserved for operators in the usual sense.

If S is a set, let $|S|=\inf\{\|x\| \mid x\in S\}$. A subset of $X\times X$ is said to be accretive if for each $\lambda\geqq 0$ and $[x_i, y_i]\in A$, $i=1, 2$, we have $\|(x_1+\lambda y_1)-$

$(x_2+\lambda y_2)\| \geqq \|x_1-x_2\|$. This definition is equivalent to the statement that $(y_1-y_2, f) \geqq 0$ for some $f \in F(x_1-x_2)$ where F is the duality map from X to X^*. If $A(t)$ is an accretive set, it is easily seen that $J_\lambda(t)=(I+\lambda A(t))^{-1}$ is a function having domain $D_\lambda(t)=R(I+\lambda A(t))$. Clearly if $x, y \in D_\lambda(t)$, $\|J_\lambda(t)x-J_\lambda(t)y\| \leqq \|x-y\|$. If $x \in D_\lambda(t)$ define $A_\lambda(t)x=\lambda^{-1}(I-J_\lambda(t))x$.

The following are well-known facts concerning an accretive set $A(t)$.

$$\|J_\lambda(t)x-x\| \leqq \lambda |A(t)x| \qquad \text{for } x \in D_\lambda(t) \cap D(A(t)). \tag{2.1}$$

$$A_\lambda(t)x \in A(t)J_\lambda(t)x, \ |A(t)J_\lambda(t)x| \leqq \|A_\lambda(t)x\| \qquad \text{for } x \in D_\lambda(t).$$

$$\|A_\lambda(t)x\| \leqq |A(t)x| \qquad \text{for } x \in D_\lambda(t) \cap D(A(t)).$$

$$\|J_{\lambda_1}(t)x-J_{\lambda_2}(t)x\| \leqq |\lambda_1-\lambda_2|\,|A(t)x| \qquad \text{for } x \in D_{\lambda_1}(t) \cap D_{\lambda_2}(t) \cap D(A(t)).$$

If $\{T_i\}$ is any collection of functions on X we introduce our product notation as follows:

$$\prod_{i=j}^{j} T_i x = T_j x \tag{2.2}$$

$$\prod_{i=j}^{k+1} T_i x = T_{k+1}\Big(\prod_{i=j}^{k} T_i x\Big)$$

$$\prod_{i=j}^{k} T_i x = x \qquad \text{if } k<j.$$

We now introduce some conditions which will be used throughout the paper.

DEFINITION 2.3. Let $\{A(t): t \in [0, T]\}$ be a family of accretive subsets of $X \times X$, then $\{A(t)\}$ is said to satisfy condition R provided that the following are true:

(i) $D(A(t))$ is independent of t.

(ii) $R(I+\lambda A(t)) \supseteq \overline{D(A(0))}$ for $\lambda \geqq 0$ and $t \in [0, T]$.

(iii) $|A(t)x| \leqq |A(\tau)x| + |t-\tau|\, L(\|x\|)(1+|A(\tau)x|)$ for $t, \tau \in [0, T]$ and $x \in D(A(0))$.

(iv) $\|(I+\lambda A(t))^{-1}x-(I+\lambda A(\tau))^{-1}x\| \leqq \lambda |t-\tau|\, L(\|x\|+|A(\tau)x|)$ for $t, \tau \in [0, T]$ and $x \in D(A(0))$.

Here $L: [0, \infty) \to [0, \infty)$ is an increasing function.

DEFINITION 2.4. Let $\{A(t): t \in [0, T]\}$ be a family of accretive operators, then $\{A(t)\}$ is said to satisfy condition C provided that the following are true:

(i) $D(A(t))$ is independent of t.

(ii) $R(I+\lambda A(t)) \supseteq \overline{D(A(0))}$ for $\lambda \geqq 0$ and $t \in [0, T]$.

(iii) $\|A(\tau)x-A(t)x\| \leqq |t-\tau| L(\|x\|)(1+\|A(\tau)x\|)$, $t, \tau \in [0, T]$ and $x \in D(A(0))$.

Here L is the same as in Condition R.

It is not difficult to see that condition C implies condition R in the case of accretive operators. Condition R was introduced by Crandall and Liggett in [4] and condition C was essentially introduced by Kato in [**10**]. The next theorem is due to Crandall and Liggett [**4**].

THEOREM 2.5. *Let $\{A(t): t \in [0, T]\}$ be a family of accretive subsets of $X \times X$ satisfying Condition R. Then $u(t)=\lim_{n\to\infty} \prod_{i=1}^{[t/\varepsilon_n]} (I+\varepsilon_n A(i\varepsilon_n))^{-1}x$ exists for $x \in \overline{D(A(0))}$, $\varepsilon_n \downarrow 0$ such that $\varepsilon_n \leqq T/n$ and $0 \leqq t \leqq T$.*

In the course of proving Theorem 2.5 one needs to establish the following lemma:

LEMMA 2.6. *Let $\{A(t): t \in [0, T]\}$ satisfy condition R. If $x \in D(A(0))$, then there is a constant $B(x)$ such that:*

(1) $\|\prod_{i=1}^{l} (I+\varepsilon_n A(i\varepsilon_n))^{-1}x-x\| \leqq l\varepsilon_n B(x)$.

(2) $\|\prod_{i=1}^{l} (I+\varepsilon_n A(i\varepsilon_n))^{-1}x\| \leqq B(x)$.

(3) $|A(u)\prod_{i=1}^{l} (I+\varepsilon_n A(i\varepsilon_n))^{-1}x| \leqq B(x)$ *whenever* $0 \leqq l \leqq [t/\varepsilon_n]$, $t, u \in [0, T]$ *and ε_n is sufficiently small.*

Let $\varepsilon_n \downarrow 0$ be a sequence such that $\varepsilon_n \leqq T/n$ and let $x \in D(A(0))$. We define a sequence of step functions

$$u_n(t)=\prod_{i=1}^{[t/\varepsilon_n]} (I+\varepsilon_n A(i\varepsilon_n))^{-1}x \qquad \text{for } t \in [0, T] \tag{2.7}$$

and a sequence of piecewise linear functions,

$$v_n(t)=\begin{cases} u_n(j\varepsilon_n)+(1/\varepsilon_n)(t-j\varepsilon_n)[u_n((j+1)\varepsilon_n)-u_n(j\varepsilon_n)] \\ \qquad\qquad\qquad\qquad \text{if } j\varepsilon_n \leqq t < (j+1)\varepsilon_n . \\ u_n(t) \qquad \text{if } \varepsilon_n[T/\varepsilon_n] \leqq t \leqq T . \end{cases} \tag{2.8}$$

These functions originated with Mermin [**10**] and are used by Brezis and Pazy in [**1**].

LEMMA 2.9. *Let $\{A(t): t \in [0, T]\}$ satisfy condition R. If we choose ε_n small enough and $v_n(t)$ and $u_n(t)$ are as above, then there exists a $B(x)$ such that:*

(1) $\|dv_n(t)/dt\| \leqq B(x)$ *for a.e.* $t \in [0, T]$.

(2) $\|v_n(t)-u_n(t)\| \leqq \varepsilon_n B(x)$ *for* $t \in [0, T]$.

Here $B(x)$ *is the constant of Lemma* 2.5.

PROOF. Clearly $v_n(t)$ is differentiable for all but a finite number of points. If $j\varepsilon_n<t<(j+1)\varepsilon_n$, $dv_n(t)/dt=(1/\varepsilon_n)[u_n((j+1)\varepsilon_n)-u_n(j\varepsilon_n)]=(1/\varepsilon_n)[\prod_{i=1}^{j+1} J_{\varepsilon_n}(i\varepsilon_n)x-\prod_{i=1}^{j} J_{\varepsilon_n}(i\varepsilon_n)x]=-A_{\varepsilon_n}((j+1)\varepsilon_n)\prod_{i=1}^{j} J_{\varepsilon_n}(i\varepsilon_n)x$. Thus by Lemma 2.6, $\|dv_n(t)/dt\| \leqq B(x)$. To establish the second assertion, observe that $\|v_n(t)-u_n(t)\| \leqq \|u_n((j+1)\varepsilon_n)-u_n(j\varepsilon_n)\| = \|\prod_{i=1}^{j+1} J_{\varepsilon_n}(i\varepsilon_n)x-\prod_{i=1}^{j} J_{\varepsilon_n}(i\varepsilon_n)x\| \leqq \varepsilon_n |A((j+1)\varepsilon_n)\prod_{i=1}^{j} J_{\varepsilon_n}(i\varepsilon_n)x| \leqq \varepsilon_n B(x)$. Thus $\lim u_n(t)=\lim v_n(t)$.

DEFINITION 2.10. By a strong solution to the Cauchy problem (1.1) we mean a function $u(t)$ Lipschitz continuous in t such that $u(0)=x$, $u'(t)$ exists a. e. on $[0, T]$ and $0 \in u'(t)+A(t)u(t)$ a. e. $t \in [0, T]$ (for accretive operators $du(t)/dt+A(t)u(t)=0$).

We have the following representation theorem for strong solutions $u(t)$. This result is an extension of Brezis and Pazy [1] to the time dependent case.

THEOREM 2.11. *Let* $\{A(t): t \in [0, T]\}$ *be a family of accretive sets satisfying condition R and let* $x \in D(A(0))$. *If the Cauchy problem has a strong solution* $u(t)$ *on* $[0, T]$ *then the sequences* $u_n(t)$ *and* $v_n(t)$ *converge uniformly to* $u(t)$ *for* $t \in [0, T]$.

PROOF. Suppose that K is the Lipschitz constant for $u(t)$ on $[0, T]$. Then it is easily seen that $\|du(t)/dt\| \leqq K$ for a. e. $t \in [0, T]$. Let $M(x)=\max\{K, B(x)\}$ where $B(x)$ is obtained from Lemma 2.9.

From the definition of $u_n(t)$ we see that $\varepsilon_n^{-1}[u_n(t)-u_n(t-\varepsilon_n)]=-A_{\varepsilon_n}([t/\varepsilon_n]\varepsilon_n)\prod_{i=1}^{[t/\varepsilon_n]-1} J_{\varepsilon_n}(i\varepsilon_n)x$. Thus defining $y_n(t)=A_{\varepsilon_n}([t/\varepsilon_n]\varepsilon_n)\prod_{i=1}^{[t/\varepsilon_n]-1} J_{\varepsilon_n}(i\varepsilon_n)x=A_{\varepsilon_n}([t/\varepsilon_n]\varepsilon_n)u_n(t-\varepsilon_n)$ we have,

$$\varepsilon_n^{-1}[u_n(t)-u_n(t-\varepsilon_n)]+y_n(t)=0\,. \tag{2.12}$$

In order for (2.12) to hold for all $t \in [0, T]$, it is convenient to define $u_n(t)$ for $t<0$ as $u_n(t)=x+\varepsilon_n y$ where $y \in A(0)x$. Let $y(t)=-du(t)/dt \in A(t)u(t)$ for a. e. $t \in [0, T]$. Extend $u(t)$ as x for $t<0$.

Now we observe that

$$\begin{aligned} &\left\|du(t)/dt-\frac{u(t)-u(t-\varepsilon_n)}{\varepsilon_n}\right\| \\ &=\left\|\frac{u_n(t)-u_n(t-\varepsilon_n)}{\varepsilon_n}-\frac{u(t)-u(t-\varepsilon_n)}{\varepsilon_n}+y_n(t)-y(t)\right\| \\ &\geqq\left\|\frac{1}{\varepsilon_n}u_n(t)+y_n(t)-\left(\frac{1}{\varepsilon_n}u(t)+y(t)\right)\right\|-\left\|\frac{u_n(t-\varepsilon_n)-u(t-\varepsilon_n)}{\varepsilon_n}\right\| \end{aligned} \tag{2.13}$$

$$\geqq \left\| \frac{1}{\varepsilon_n} u_n(t) + A_{\varepsilon_n}(t) u_n(t-\varepsilon_n) - \left(\frac{1}{\varepsilon_n} u(t) + y(t) \right) \right\| - \left\| \frac{u_n(t-\varepsilon_n) - u(t-\varepsilon_n)}{\varepsilon_n} \right\|$$
$$- \| A_{\varepsilon_n}(t) u_n(t-\varepsilon_n) - A_{\varepsilon_n}([t/\varepsilon_n]\varepsilon_n) u_n(t-\varepsilon_n) \|$$
$$\geqq \left\| \frac{1}{\varepsilon_n} u_n(t) + A_{\varepsilon_n}(t) u_n(t-\varepsilon_n) - \left(\frac{1}{\varepsilon_n} u(t) + y(t) \right) \right\| - \left\| \frac{u_n(t-\varepsilon_n) - u(t-\varepsilon_n)}{\varepsilon_n} \right\|$$
$$-O(\varepsilon_n).$$

The right-most inequality of (2.13) follows from the definition of $A_\varepsilon(t)$ together with part (iv) of condition R.

Observing that $\|u_n(t) - J_{\varepsilon_n}(t) u_n(t-\varepsilon_n)\| = \| J_{\varepsilon_n}([t/\varepsilon_n]\varepsilon_n) u_n(t-\varepsilon_n) - J_{\varepsilon_n}(t) u_n(t-\varepsilon_n)\| \leqq \varepsilon_n^2 L(\|u_n(t-\varepsilon_n)\| + |A(t) u_n(t-\varepsilon_n)|)$ we obtain,

(2.14) $\quad \|du(t)/dt - (u(t) - u(t-\varepsilon_n))/\varepsilon_n\|$

$$\geqq \left\| \frac{1}{\varepsilon_n} J_{\varepsilon_n}(t) u_n(t-\varepsilon_n) + A_{\varepsilon_n}(t) u_n(t-\varepsilon_n) - \left(\frac{1}{\varepsilon_n} u(t) + y(t) \right) \right\|$$
$$- \left\| \frac{u_n(t-\varepsilon_n)}{\varepsilon_n} - \frac{u(t-\varepsilon_n)}{\varepsilon_n} \right\| - O(\varepsilon_n)$$
$$\geqq \left\| \frac{1}{\varepsilon_n} J_{\varepsilon_n}(t) u_n(t-\varepsilon_n) - \frac{1}{\varepsilon_n} u(t) \right\| - \left\| \frac{u_n(t-\varepsilon_n) - u(t-\varepsilon_n)}{\varepsilon_n} \right\| - O(\varepsilon_n)$$
$$\geqq \left\| \frac{u_n(t) - u(t)}{\varepsilon_n} \right\| - \left\| \frac{u_n(t-\varepsilon_n) - u(t-\varepsilon_n)}{\varepsilon_n} \right\| - O(\varepsilon_n).$$

The second inequality of (2.14) follows from the accretiveness of $A(t)$ and the facts that $A_{\varepsilon_n}(t) u_n(t-\varepsilon_n) \in A(t) J_{\varepsilon_n}(t) u_n(t-\varepsilon_n)$ and that $y(t) \in A(t) u(t)$.

We now refer the reader to Brezis and Pazy [**1**]. In this paper they establish the theorem for time invariant A. Their methods are readily extendable and we shall not reproduce them in detail. The left and right-most side of inequality (2.14) are integrated on $(0, \theta)$ where $\varepsilon_n < \theta < T$ and integration techniques are applied to deduce that $u_n(t)$ and $v_n(t)$ converge to $u(t)$ in $L^1(0, T; X)$. This together with the fact that $\frac{d}{dt}\|u(t) - v_n(t)\| \leqq 2M(x)$ yield the uniform convergence of $v_n(t)$ to $u(t)$ for $t \in [0, T]$. Hence $u_n(t)$ converges to $u(t)$ for $t \in [0, T]$.

Since $u(t) = \lim_{n\to\infty} \prod_{i=1}^{[t/\varepsilon_n]} (I + \varepsilon_n A(i\varepsilon_n))^{-1} x$ holds for any $\varepsilon_n \downarrow 0$ we have $u(t) = \lim_{n\to\infty} \prod_1^n (I + (t/n) A(it/n))^{-1} x$.

§ 3. Main results—approximation methods.

The first result of this section deals with the approximations of the evolution operator $U(t)x = \lim_{n\to\infty} \prod_1^n (I + (t/n) A(it/n))^{-1} x$ where $\{A(t)\}$ is a time dependent family of accretive sets.

THEOREM 3.1. *For each integer k let $\{A_k(t): t\in[0, T]\}$ be a family of accretive subsets of $X\times X$ satisfying condition R with L independent of k. Let $\{A(t): t\in[0, T]\}$ also be an accretive subset satisfying R. Then if we denote*

$$U_k(t)x=\lim_{n\to\infty}\prod_{i=1}^{n}(I+(t/n)A_k(it/n))^{-1}x \qquad \textit{for } x\in\overline{D(A_k(0))}$$

and

$$U(t)x=\lim_{n\to\infty}\prod_{i=1}^{n}(I+(t/n)\,A(it/n))^{-1}x \qquad \textit{for } x\in\overline{D(A(0))}\,,$$

we have $\lim_{k\to\infty}U_k(t)x=U(t)x$ for $x\in\overline{D(A(0))}$ uniformly for $t\in[0, T]$ provided that the following are true:

(1) $\overline{D(A_k(0))}\supseteqq\overline{D(A(0))}$

(2) $\lim_{k\to\infty}(I+\lambda A_k(t))^{-1}x=(I+\lambda A(t))^{-1}x$ *m for* $\lambda\geqq 0$, $x\in D(A(0))$ *and* $t\in[0, T]$.

PROOF. The basic idea of the proof is simple. We prove the following: if $x\in\overline{D(A(0))}$, then for each $\lambda>0$ $U(t)J_\lambda(0)x=\lim_{k\to\infty}U_k(t)J_\lambda(0)x$. Then $U(t)x=\lim_{k\to\infty}U_k(t)x$ uniformly on $[0, T]$ follows from the facts that $U_k(t)$ and $U(t)$ are contractions and that $\|J_\lambda(0)x-x\|\to 0$ as $\lambda\to 0$. For simplicity we denote $J_\lambda(0)x=x_\lambda$. Since $(A_k)_\lambda(0)x\in A_k(0)x_\lambda$ and $(A_k)_\lambda(0)x\to A_\lambda(0)x$ we see that $\sup_{k\in Z^+}|A_k(0)x_\lambda|<\infty$.

To show that $U_k(t)x_\lambda$ converges uniformly to $U(t)x_\lambda$ we need to show that: $\prod_{i=1}^{n}(I+(t/n)\,A_k(it/n))^{-1}x_\lambda$ converges uniformly to $U_k(t)x_\lambda$ independently of k; and that $\prod_{i=1}^{n}(I+(t/n)A_k(it/n))^{-1}x_\lambda$ converges to $\prod_{i=1}^{n}(I+(t/n)A(it/n))^{-1}x_\lambda$ uniformly with respect to $t\in[0, T]$. Then we will be able to choose n so that $\|U(t)x_\lambda-\prod_{i=1}^{n}(I+(t/n)\,A(it/n))^{-1}x_\lambda\|<\varepsilon/3$ and $\|U_k(t)x_\lambda-\prod_{i=1}^{n}(I+(t/n)\,A_k(it/n))^{-1}x_\lambda\|<\varepsilon/3$; and we can choose K so that $k>K$ implies that $\|\prod_{i=1}^{n}(I+(t/n)A_k(it/n))^{-1}x_\lambda-\prod_{i=1}^{n}(I+(t/n)A(it/n))^{-1}x_\lambda\|<\varepsilon/3$. It then becomes immediate that if $k>K$ and $t\in[0, T]$ then $\|U(t)x_\lambda-U_k(t)x_\lambda\|\leqq\|U(t)x_\lambda-\prod_{i=1}^{n}(I+(t/n)A(it/n))^{-1}x_\lambda\|+\|\prod_{i=1}^{n}(I+(t/n)A(it/n))^{-1}x_\lambda-\prod_{i=1}^{n}(I+(t/n)A_k(it/n))^{-1}x_\lambda\|+\|U_k(t)x_\lambda-\prod_{i=1}^{n}(I+(t/n)A_k(it/n))^{-1}x_\lambda\|<\varepsilon$. Thus the theorem will be established by the sequence of lemmas below. We make the further notational simplification of replacing x_λ by x.

LEMMA 3.2. *Under the conditions of the theorem there exists a constant $B'(x)$ such for n, $k\in Z^+$:*

(1) $\|\prod_{1}^{l}(I+(t/n)A_k(it/n))^{-1}x-x\|\leqq(l/n)B'(x)\,,\qquad 0\leqq l\leqq n\,.$

(2) $\quad \| \prod_1^l (I+(t/n)\, A_k(it/n))^{-1}x\| \leqq B'(x)\,, \qquad 0 \leqq l \leqq n\,.$

(3) $\quad | A_k(u) \prod_1^l (I+(t/n)\, A_k(it/n))^{-1}x| \leqq B'(x)\,, \qquad 0 \leqq l \leqq n,\ u \in [0, T]\,.$

PROOF. Assertion (2) follows immediately from (1). To establish (1) we observe that $\|\prod_1^l (I+(t/n)A_k(it/n))^{-1}-x\| \leqq \sum_{i=1}^l \| \prod_i^l(I+(t/n)A_k(jt/n))^{-1}x-\prod_{i+1}^l(I+(t/n) A_k(jt/n))^{-1}x\| \leqq \sum_{i=1}^l (t/n)| A_k(it/n)x| \leqq \sum_{i=1}^l (t/n)(L(\|x\|)(1+|A_k(0)x|))$. The last term of the preceding inequality is bounded because $\sup_k |A_k(0)x| < \infty$. To verify (3) choose $L' \geqq \sup_k \{L(\| \prod_1^l (I+(t/n)\, A_k(it/n))^{-1}x\|)\}$. We now apply an estimate originating with J. Mermin [**10**] to obtain $|A_k(lt/n) \prod_1^l (I+(t/n)\, A_k(it/n))^{-1}x| \leqq 4e^{L'}(2L'+|A_k(0)x|)$. We can now utilize part (iii) of condition R to insure that $|A_k(u) \prod_{i=1}^l (I+(t/n)\, A_k(it/n))^{-1}x|$ is bounded independent of k.

LEMMA 3.3. *Under the hypotheses of the theorem* $\lim_{k\to\infty} \prod_1^n (I+(t/n)A_k(it/n))^{-1}x = \prod_1^n (I+(t/n)\, A(it/n))^{-1}x$ *uniformly with* $t \in [0, T]$.

PROOF. The pointwise convergence will follow by induction if we can verify the case $j=2$. Observe that

$$\begin{aligned}
&\| \prod_1^2 (I+(t/n)\, A_k(it/n))^{-1}x - \prod_1^2 (I+(t/n)\, A(it/n))^{-1}x\| \\
&\quad \leqq \| \prod_1^2 (I+(t/n)A_k(it/n))^{-1}x-(I+(t/n)A_k(2t/n))^{-1}(I+(t/n)A(t/n))^{-1}x\| \\
&\qquad + \|(I+(t/n)A_k(2t/n))^{-1}(I+(t/n)A(t/n))^{-1}x - \prod_1^2 (I+(t/n)A(it/n))^{-1}x\| \\
&\quad \leqq \|(I+(t/n)A_k(t/n))^{-1}x-(I+(t/n)A(t/n))^{-1}x\| \\
&\qquad + \|(I+A_k(2t/n))^{-1}(I+(t/n)A(t/n))^{-1}x - \prod_1^2 (I+(t/n)A(it/n))^{-1}x\|\,.
\end{aligned}$$

Both of the final terms of the above inequality converge to zero by assumption (2) of the theorem.

The uniform convergence will be established if we can demonstrate the equicontinuity of $f_k(t) = \prod_1^n (I+(t/n)A_k(it/n))^{-1}x$.

$$\begin{aligned}
\|f_k(t)-f_k(\tau)\| &= \| \prod_1^n (I+(t/n)A_k(it/n))^{-1}x - \prod_1^n (I+(\tau/n)A_k(i\tau/n))^{-1}x\| \\
&\leqq \| \prod_1^{n-1} (I+(t/n)A_k(it/n))^{-1}x - \prod_1^{n-1} (I+(\tau/n)A_k(i\tau/n))^{-1}x\|
\end{aligned}$$

$$+\|(I+(t/n)A_k(t))^{-1}\prod_1^{n-1}(I+(\tau/n)A_k(i\tau/n))^{-1}x$$

$$-(I+(t/n)A_k(\tau))^{-1}\prod_1^{n-1}(I+(\tau/n)A(i\tau/n))^{-1}x\|$$

$$+\|(I+(t/n)A_k(\tau))^{-1}\prod_1^{n-1}(I+(\tau/n)A(i\tau/n))^{-1}x-\prod_1^{n}(I+(\tau/n)A(i\tau/n))^{-1}x\|$$

$$\leqq\|\prod_1^{n-1}(I+(t/n)A_k(it/n))^{-1}x-\prod_1^{n-1}(I+(\tau/n)A_k(i\tau/n))^{-1}x\|$$

$$+(t/n)|t-\tau|L(\|\prod_1^{n-1}(I+(\tau/n)A_k(i\tau/n))^{-1}x\|$$

$$+|A_k(\tau)\prod_1^{n-1}(I+(\tau/n)A_k(i\tau/n))^{-1}x|)$$

$$+|t/n-\tau/n||A_k(\tau)\prod_1^{n-1}(I+(\tau/n)A_k(i\tau/n))^{-1}x|.$$

The second term of the right-most side of the above inequality follows from Condition R and the third term follows from (2.1). By continuing the above process an additional $(n-1)$ times we obtain the inequality,

$$\|\prod_1^{n}(I+(t/n)A_k(it/n))^{-1}x-\prod_1^{n}(I+(\tau/n)A(i\tau/n))^{-1}x\|$$

$$\leqq\sum_{j=1}^{n}\{(t/n)(|t-\tau|)L(\|\prod_1^{j-1}(I+(\tau/n)A_k(i\tau/n))^{-1}x\|$$

$$+|A_k(i\tau/n)\prod_1^{j-1}(I+(\tau/n)A_k(i\tau/n))^{-1}x|)$$

$$+(1/n)|t-\tau||A_k(j\tau/n)\prod_1^{j-1}(I+(\tau/n)A_k(i\tau/n))^{-1}x|\}$$

$$\leqq|t-\tau|M,\qquad\text{for some positive } M.$$

The existence of the bound M follows by applications of Lemma 3.2.

LEMMA 3.4. *The convergence of* $\prod_1^{n}(I+(t/n)A_k(t))^{-1}x$ *to* $u_k(t)$ *is independent of* k.

PROOF. This lemma is based on the estimates of Crandall and Liggett [4] and we shall only outline its proof. If $n>m$ and

$$a_{m,n}=\|\prod_1^{n}(I+(t/n)A_k(it/n))^{-1}x-\prod_1^{m}(I+(t/m)A_k(it/m))^{-1}x\|$$

we wish to show that $\lim\{a_{m,n}\}\to 0$ independently of k. If $t\in[0,T]$, $0\leqq p\leqq 1+n-m$, $0\leqq l\leqq m$, and $a_{l,p}=\|\prod_1^{l}(I+(t/n)A_k(it/n))^{-1}x-\prod_1^{p}(I+(t/m)A_k(it/m))^{-1}x\|$, then $a_{l,p}\leqq(m/n)a_{l-1,p-1}+(n-m/n)a_{l,p-1}+\|(I+(t/n)A_k(pt/n))^{-1}\prod_1^{p-1}(I+(t/n)A_k(it/n))^{-1}x$

$-(I+(t/n)A_k(lt/m))^{-1}\prod_1^{p-1}(I+(t/m)A_k(it/n))^{-1}x\| \leqq (n/m)a_{l-1,p-1}+(m-n/n)a_{l,p-1}+$ $(t/n)(pt/n-lt/m)M$ for some $M>0$. The induction techniques of Crandall and Liggett yield,

$$a_{m,n} \leqq \sum_{j=0}^{m}\binom{n}{j}\left(\frac{m}{n}\right)^j\left(\frac{m-n}{n}\right)^{n-j}\left\|\prod_1^{m-j}(I+(t/m)A_k(it/m))^{-1}x-x\right\|$$

$$+\sum_{j=m}^{n}\left(\frac{m}{n}\right)^n\left(\frac{n-m}{n}\right)^{j-m}\binom{j-1}{m-1}\left\|\prod_1^{n-j}(I+(t/n)A_k(it/n))^{-1}x-x\right\|$$

$$+\frac{Ht^2}{mn}\sum_{j=0}^{n-1}\sum_{i=0}^{j}\left(\frac{m}{n}\right)^i\left(\frac{n-m}{n}\right)^{j-i}\binom{j}{i}\left|i-j\left(\frac{m}{n}\right)\right|.$$

The first two terms on the right of the above inequality are bounded by a multiple of $t/\sqrt{m}$ and the third term is bounded by a multiple of $t^2/\sqrt{m}$.

By establishing Lemma 4.4 we have completed the proof of the theorem.

REMARK. The author has recently learned that J. Goldstein in [7] has independently proved a theorem which is quite similar to Theorem 3.1. His methods are completely different and he has an additional requirement concerning the $A_k(t)$'s. After submitting this paper the author received a preprint of [5] by Crandall and Pazy. Parts of their paper are similar to this paper.

The next theorem deals with the convergence of strong solutions to approximate Cauchy problems.

THEOREM 3.5. *For each integer k, let $\{A_k(t): t\in[0, T]\}$ be a family of accretive sets satisfying condition R with L independent of k. Let $u_k(t)$ be a solution to the approximate Cauchy problem $0\in u_k'(t)+A_k(t)u_k(t)=0$, $u_k(0)=x$. Let $\{A(t): t\in[0, T]\}$ also be a family of accretive sets satisfying R and suppose that $u(t)$ is the solution to the Cauchy problem $0\in u'(t)+A(t)u(t)$, $u(0)=x$. If $\{A_k(t)\}$ and $\{A(t)\}$ in addition meet the requirements of Theorem* 3.1, *then $\lim_{k\to\infty} u_k(t)=u(t)$ uniformly with respect to $t\in[0, T]$.*

PROOF. It is immediate that $\{A_k(t)\}$ and $\{A(t)\}$ may be assumed to satisfy R with the same L. Theorem 2.11 guarantees that each $u_k(t)$ and $u(t)$ may be represented as product integrals and hence Theorem 3.1 yields the uniform convergence of $u_k(t)$ to $u(t)$.

Our final result shows that under certain restrictions the convergence of the approximate Cauchy problems guarantees the existence of the solution to a particular Cauchy problem.

THEOREM 3.6. *For each integer k, let $\{A_k(t): t\in[0, T]\}$ be a family of accretive operators satisfying condition C with L independent of k. Let $u_k(t)$ be a strong solution to the Cauchy problem $u_k'(t)+A_k(t)u_k(t)=0$, $u_k(0)=x$. Let $\{A(t): t\in[0, T]\}$ be a family of accretive operators also satisfying C. Suppose* *the conditions of Theorem* 3.1 *are met and the following are true.*

(1) $A_k(t): D(A_k(0)) \to X$ *is continuous.*

(2) $D(A(t))$ *is closed.*

(3) *If* $\{x_k\} \subseteqq D(A_k(0))$ *and* $x_k \to x \in D(A(0))$ *then* $A_k(t)x_k \to A(t)x$. *Then the Cauchy problem* $u'(t)+A(t)u(t)=0$, $u(0)=x$ *has a strong solution.*

PROOF. Theorem 2.11 gives us the product integral representation of the solutions $u_k(t)=\lim_{n\to\infty}\prod_1^n (I+(t/n)A_k(it/n))^{-1}x$. The convergence of $u_k(t)$ to $u(t)=\lim_{n\to\infty}\prod_1^n (I+(t/n)A(it/n))^{-1}x$ is a consequence of Theorem 3.1 and $u(t)\in D(A(t))$ because $D(A(t))$ is closed. $\|A_k(t)u_k(t)\|=\lim_{n\to\infty}\|A_k(t)\prod_1^n (I+(t/n)A_k(it/n))^{-1}x\| \leqq M$; the existence of M is guaranteed by Lemma 3.2.

Since $A(t)u(t)=\lim_{k\to\infty} A_k(t)u_k(t)$ and $\|A(t)u(t)\|\leqq M$ a.e. t, $A(t)u(t)$ is Bochner integrable. If we apply the Lesbesque-Bochner Bounded Convergence Theorem to the equation $u_k(t)=u_k(0)-\int_0^t A_k(s)u_k(s)ds$ we obtain, $u(t)=u(0)-\int_0^t A(s)u(s)ds$. Hence $u'(t)+A(t)u(t)=0$ for a.e. $t\in[0, T]$.

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