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Structure of rings satisfying certain polynomial identities

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A well-known theorem of Jacobson [2] asserts that if R is an associative ring with the property that, for all x in R, there exists an integer $m(x)>1$ such that $x^{m(x)}=x$, then R is isomorphic to a subdirect sum of fields. Our present object is to extend Jacobson's Theorem by determining the structure of a certain class of associative rings satisfying polynomial identities involving n elements $x_1, \cdots, x_n$ of R. In order to be able to state this generalization, we first define a *word* $w(x_1, \cdots, x_n)$ in $x_1, \cdots, x_n$ to be a product in which each factor is x_i for some $i=1, \cdots, n$. A *polynomial* $f(x_1, \cdots, x_n)$ is, then, an expression of the form $c_1w_1(x_1, \cdots, x_n)+\cdots+c_mw_m(x_1, \cdots, x_n)$, where the c_i are integers. The *degree* of x_i in the word $w(x_1, \cdots, x_n)$ is the number of times x_i appears as a factor in $w(x_1, \cdots, x_n)$. Suppose that $f(x_1, \cdots, x_n)=c_1w_1(x_1, \cdots, x_n)+\cdots+c_mw_m(x_1, \cdots, x_n)$ is a polynomial in $x_1, \cdots, x_n$. The *degree* of x_i in $f(x_1, \cdots, x_n)$ is the *smallest* value among the following: degree of x_i in $w_1(x_1, \cdots, x_n), \cdots$, degree of x_i in $w_m(x_1, \cdots, x_n)$. The following theorem is proved:

THEOREM 1. *Suppose R is an associative ring and n is a fixed positive integer. Suppose that for all elements $x_1, \cdots, x_n$ of R, there exists a polynomial $f=f_{x_1,\cdots,x_n}(x_1, \cdots, x_n)$, depending on $x_1, \cdots, x_n$, such that degree of each x_i in $f \geqq 2$, and suppose*

$$x_1 \cdots x_n=f_{x_1,\cdots,x_n}(x_1, \cdots, x_n)\,.$$

Then R is isomorphic to a subdirect sum of fields and a nilpotent ring S satisfying $S^n=(0)$.

Observe that Theorem 1 generalizes Jacobson's Theorem quoted above (take $n=1$ and $f_{x_1}(x_1)=x_1{}^{m(x_1)}$).

In preparation for the proof of Theorem 1, we proceed to establish the following lemmas. But, first, *we make the assumption that $n>1$ throughout*, since Theorem 1 is true for $n=1$ (see proof of Lemma 3).

LEMMA 1. *Suppose S is an associative subdirectly irreducible ring which does not have an identity. Suppose, moreover, that for all $x_1, \cdots, x_n$ in S, there exists a polynomial $f=f_{x_1,\cdots,x_n}(x_1, \cdots, x_n)$, depending on $x_1, \cdots, x_n$ such that*

(1) $$x_1 \cdots x_n=f_{x_1,\cdots,x_n}(x_1, \cdots, x_n)\,;\ \textit{degree of each } x_i \textit{ in } f \geqq 2\,.$$

Then (i) S *has no nonzero idempotent elements;* (ii) S *is a nil ring.*

PROOF. First, we show that all the idempotents of S are in the center. Let $e^2 = e \in S$, and let $x \in S$. By (1), there exists a polynomial $f = f_{e,e,\cdots,e,ex-exe}(e, e, \cdots, e, ex-exe)$ such that

(2) $$ee \cdots e(ex-exe) = f; \text{ degree of each argument in } f \geqq 2.$$

Now, each word in the polynomial f involves e at least twice and involves $ex-exe$ at least twice (as a factor). Thus each word of f involves $(ex-exe)^2 = 0$, or involves $(ex-exe)e = 0$, and hence $f = 0$. Therefore, by (2), $ex = exe$. A similar argument shows that $xe = exe$, and hence e is in the center. Next, we prove that $e = 0$. To this end, define A and B by

$$A = \{ex - x \mid x \in S\}, \qquad B = \{ex \mid x \in S\}.$$

Since e is in the center of S, it is easily seen that both A and B are ideals in S, and, moreover, $A \cap B = (0)$. But, since S is subdirectly irreducible, the intersection of all *nonzero* ideals in S is nonzero, and hence $A = (0)$ or $B = (0)$. Now, the possibility $A = (0)$ is ruled out since, by hypothesis, S does not have an identity. Therefore $B = (0)$, and hence $e = ee = 0$. This proves (i).

To prove (ii), let $x \in S$ and set $x_1 = \cdots = x_n = x$ in (1), we get

(3) $$x^n = x^{2n} f(x) \text{ for some polynomial } f(x) \text{ with integer coefficients}.$$

Hence $e = x^n f(x)$ is idempotent, and therefore by (i), $x^n f(x) = 0$. Thus, by (3), we obtain

(4) $$x^n = 0 \qquad \text{for all } x \text{ in } S,$$

and the lemma is proved.

LEMMA 2. *Under all the hypotheses of Lemma* 1, *we have* $S^n = (0)$.

PROOF. Let $x \in S$. Then by (4), $x^n = 0$, and hence there exists a smallest positive integer m such that

(5) $$x^m S^{n-1} = (0), \qquad S^{n-1} x^m = (0), \qquad m \text{ minimal}.$$

We now assume that $m > 1$ and obtain a contradiction. First, observe that if $N \geqq n$, then by replacing x_n by $x_n x_{n+1} \cdots x_N$ in (1), where $x_1, \cdots, x_N \in S$, we obtain a polynomial $g = g_{x_1,\cdots,x_N}(x_1, \cdots, x_N)$ such that

(6) $$x_1 \cdots x_N = g_{x_1,\cdots,x_N}(x_1, \cdots, x_N); \text{ degree of each } x_i \text{ in } g \geqq 2; \quad i = 1, \cdots, N \quad (N \geqq n).$$

Now, suppose $r_1, \cdots, r_{2mn-(m-1)} \in S$, and define

(7) $$x_1 = x^{m-1} r_1, \cdots, x_{2mn-(m-1)} = x^{m-1} r_{2mn-(m-1)}; \quad x_{2mn-m+2} = \cdots = x_{2mn} = x.$$

Then, by (6) and (7), we get (taking $N=2mn$)

(8) $$(x^{m-1}r_1)\cdots(x^{m-1}r_{2mn-(m-1)})x^{m-1}=g_{x_1,\cdots,x_{2mn}}(x_1,\cdots,x_{2mn}),$$

where "x" appears as a factor in each word $w=w(x_1,\cdots,x_{2mn})$ of g at least $2(m-1)\geqq m$, since $m\geqq 2$. (This follows since each of $x_{2mn-m+2},\cdots,x_{2mn}$ appears at least twice in w.) Now, if all of these x's appear together in w, then w involves x^m. On the other hand, if some two of these x's are separated in the word w, then w involves the product $x(x^{m-1}r_j)x$, and hence again w involves x^m. Thus w has one of the forms

(9) $$w=x^m w_1, \quad \text{or } w=w_2x^m, \quad \text{or } w=w_3x^m w_4.$$

Hence, by (5), (6), (7), (8), and by a consideration of degrees, we conclude that the word $w=w(x_1,\cdots,x_{2mn})$ satisfies the following:

(10) $$w\in x^mS^{n-1}, \text{ or } w\in S^{n-1}x^m, \text{ or } w\in S^{n-1}x^mS, \text{ or } w\in Sx^mS^{n-1}.$$

Hence, by (5) and (10), $w=0$ for *every* word w in g. Therefore, $g=0$, and hence by (8),

(11) $$(x^{m-1}S)^lx^{m-1}=(0) \qquad (l=2mn-m+1).$$

Now, returning to (1), an easy induction (which we omit) shows that, for all $x_1,\cdots,x_n$ in S, there exists a polynomial $h=h_{x_1,\cdots,x_n}(x_1,\cdots,x_n)$ such that

(12) $$x_1\cdots x_n=h_{x_1,\cdots,x_n}(x_1,\cdots,x_n); \text{ degree of each } x_i \text{ in } h\geqq l+3.$$

Let $x_1=x^{m-1}$, $x_2=r_1,\cdots,x_n=r_{n-1}$ in (12), we get

(13) $$x^{m-1}r_1\cdots r_{n-1}=h(x^{m-1},r_1,\cdots,r_{n-1}); \text{ degree of each argument in } h \geqq l+3.$$

Now, let w be any word in the polynomial h in (13). Then, either w involves $x^{m-1}x^{m-1}$ and hence w involves x^m——in which case $w=0$ by above argument, or w has the form

(14) $$w=\cdots x^{m-1}\cdots x^{m-1}\cdots x^{m-1}\cdots.$$

Since, by (13), x^{m-1} appears at least $l+3$ times in (14), we easily see that

(15) $$w\in S[(x^{m-1}S)^lx^{m-1}]S.$$

Hence, by (15) and (11), $w=0$ for *every* word w in the polynomial h in (13), and (13) thus reduces to

$$x^{m-1}r_1\cdots r_{n-1}=0, \qquad \text{for all } r_1,\cdots,r_{n-1}\in S.$$

Therefore, $x^{m-1}S^{n-1}=(0)$. A similar argument shows that $S^{n-1}x^{m-1}=(0)$. Hence, we have

(16) $$x^{m-1}S^{n-1}=(0), \qquad S^{n-1}x^{m-1}=(0).$$

This, however, contradicts the minimality of m (see (5)). This contradiction proves that $m=1$, and hence by (5), we have

$$xS^{n-1}=S^{n-1}x=(0)\,, \qquad \text{for all } x\in S\,.$$

Therefore, $S^n=(0)$, and the lemma is proved.

LEMMA 3. *Suppose S is an associative subdirectly irreducible ring with identity* 1 $(1\neq 0)$. *Suppose, moreover, that for all $x_1, \cdots, x_n$ in S, there exists a polynomial $f=f_{x_1,\cdots,x_n}(x_1, \cdots, x_n)$, depending on $x_1, \cdots, x_n$ such that*

$$x_1\cdots x_n=f_{x_1,\cdots,x_n}(x_1, \cdots, x_n)\,; \quad \textit{degree of each } x_i \textit{ in } f\geqq 2\,. \tag{17}$$

Then S is a field of prime characteristic p, and, moreover, for every x in S, there exists a positive integer $k(x)$ such that $x^{p^{k(x)}}=x$.

PROOF. Let $x\in S$. In (17), set $x_1=x$, $x_i=1$ for each $i\neq 1$. We get

$$x=x^2p_x(x)\,; \quad p_x(x) \text{ is a polynomial (with integer coefficients).} \tag{18}$$

Now, by a well-known theorem of Herstein [1], equation (18) implies that S is commutative. Moreover, it is easy to see that, in view of (18), S has no nonzero nilpotent elements. Hence [**3**; p. 130] S is a field. Again, on account of (18), the prime field of S must be $GF(p)$, p prime. Hence, by (18) again, every element x in S is algebraic over $GF(p)$, and therefore the subfield F_x of S generated by x is finite. Thus, as is well-known, $x^{p^{k(x)}}=x$ for some positive integer $k(x)$, and the lemma is proved.

We are now in a position to prove Theorem 1.

PROOF OF THEOREM 1. It is well-known [**3**; p. 129] that the ground ring R is isomorphic to a subdirect sum of subdirectly irreducible rings S_i, $i\in\Gamma$. Moreover, each such ring S_i, being a homomorphic image of R, inherits all the hypotheses imposed on the ring R (in Theorem 1). Hence, by Lemmas 2, 3, we have that each S_i is either a field with the properties described in Lemma 3, or S_i satisfies $S_i^n=(0)$. Now, it is easily seen that we can collect all the nilpotent rings S_i together and thus obtain a nilpotent ring S satisfying the conclusion of Theorem 1. This proves the theorem.

We conclude with the following

REMARK. The following two examples show that the restrictions on the degrees in Theorem 1 and Lemmas 1, 2 cannot be weakened.

EXAMPLES. Let R_1, R_2 be given by

$$R_1=\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\1&0\end{pmatrix}, \begin{pmatrix}1&1\\1&1\end{pmatrix} \,\middle|\, 0,\ 1\in GF(2)\right\},$$

$$R_2=\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&1\end{pmatrix}, \begin{pmatrix}1&1\\0&0\end{pmatrix}, \begin{pmatrix}1&1\\1&1\end{pmatrix} \,\middle|\, 0,\ 1\in GF(2)\right\}.$$

It is easy to check that, for *all* x, y in R_1,

$$xy = x^m y^n + x^q y + (xy)^n, \qquad \text{for } all \text{ positive integers } m,\ n,\ q.$$

However, the subdirectly irreducible ring R_1 does not satisfy the conclusions of any of Theorem 1, Lemma 1, or Lemma 2. Similarly, we have, for *all* x, y in R_2,

$$xy = x^m y^n + xy^q + (xy)^m, \qquad \text{for } all \text{ positive integers } m,\ n,\ q.$$

Again, the subdirectly irreducible ring R_2 does not satisfy the conclusions of any of Theorem 1, Lemma 1, or Lemma 2.

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