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A cohomology for Lie algebras

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1. Introduction.

Dixmier [1] has proposed a cohomology for Lie rings (that is, Lie algebras over the ring of integers). In this paper we propose a cohomology for Lie algebras over a ring in which the element 2 is invertible. First we construct a complex over a Lie algebra and then define a cohomology. We then show that the 0-cohomology module is isomorphic to the submodule of invariant elements of the module of coefficients, the 1-cohomology module is the module of crossed homomorphisms of the Lie algebra into the module of coefficients modulo the principal homomorphisms, and the 2-cohomology module is in one-one correspondence with the set of equivalence classes of special (or singular) extensions of the Lie algebra with the module of coefficients as kernel. While trying to interpret the 3-cohomology module the task of showing that every element of it is indeed an obstruction becomes too difficult and it has not been possible to accomplish it.

There is a great similarity between the constructions and proofs given in this paper and those given in [2], but they do need working out since the structure of a Lie algebra, thanks to the Jacobi identity, is not as simple as that of an associative algebra and one cannot be sure of the truth of a theorem without a comprehensive proof. Those definitions which have not been given here formally can be obtained from [2] with obvious changes (e. g. for an associative algebra substitute a Lie algebra).

2. Definition of cohomology.

Let K be a commutative ring with unit element 1 ($\neq 0$) such that there exists an element $k \in K$ for which $2k=1$. Throughout this paper we shall consider Lie algebras over the ring K. A differential graded Lie algebra over the ring K is a graded K-module $U=\sum_{n\geqq 0} U_n$ together with (i) a K-homomorphism $U \underset{K}{\otimes} U \to U$ given by $u_i \otimes u_j \to [u_i, u_j]$, where $u_i \in U_i$, $u_j \in U_j$ and $[u_i, u_j] \in U_{i+j}$, satisfying the following relations:

(2.1) $\qquad [u, u]=0, \qquad$ where $u \in U$ is homogeneous element of *even* degree;

(2.2) $$[u_i, u_j]=(-1)^{ij+1}[u_j, u_i], \text{ where } u_i \in U_i,\ u_j \in U_j;$$

(2.3) $$(-1)^{ki}[u_i, [u_j, u_k]]+(-1)^{ij}[u_j, [u_k, u_i]]+(-1)^{jk}[u_k, [u_i, u_j]]=0,$$

where $u_i \in U_i$, $u_j \in U_j$, $u_k \in U_k$; and (ii) K-homomorphism $d: U \to U$ such that

(2.4) $$dd=0,\ d(U_n) \subset U_{n-1},\ d[u_i, u_j]=[du_i, u_j]+(-1)^i[u_i, du_j],$$

where $u_i \in U_i$, $u_j \in U_j$. We denote the restriction of d to U_n by d_n. (Actually since there exists an element $k \in K$ such that $2k=1$ the relation (2.1) follows from (2.2) but we shall find it convenient to retain it separately.)

A (left) U-representation of U is a K-module M together with a K-homomorphism $U \underset{K}{\otimes} M \to M$ given by $u \otimes m \to u \cdot m$, where $u \in U$, $m \in M$ such that

$$u_i \cdot (u_j \cdot m)-(-1)^{ij} u_j \cdot (u_i \cdot m)=[u_i, u_j] \cdot m,$$

where $u_i \in U_i$, $u_j \in U_j$ and $m \in M$. For brevity we call M a (left) U-module.

Let $\mathfrak{g}$ be a Lie algebra. We shall construct a differential graded Lie algebra $U=\sum_{n \geqq 0} U_n$ and a homomorphism of differential graded Lie algebras $\varepsilon: U \to \mathfrak{g}$ (the differential and the grading in $\mathfrak{g}$ being trivial) such that

(i) the sequence of K-modules $\cdots \to U_n \xrightarrow{d_n} U_{n-1} \to \cdots \to U_1 \xrightarrow{d_1} U_0 \xrightarrow{\varepsilon} \mathfrak{g} \to 0$ is exact, and

(ii) there is a map $\sigma: \mathfrak{g} \to U_0$ for which $\sigma([x, y])=[\sigma(x), \sigma(y)]$, where $x, y \in \mathfrak{g}$ and $\varepsilon\sigma=$ identity map.

Let X_0 be a set in one-to-one correspondence with $\mathfrak{g}$ and let a multiplication be defined in X_0 such that the product of any two elements in X_0 is the element in X_0 which corresponds to the product of their images in $\mathfrak{g}$. Let $K(X_0)$ be the K-free module with X_0 as base. The multiplication in X_0 induces on $K(X_0)$ the structure of a non-associative algebra. The one-to-one correspondence $X_0 \to \mathfrak{g}$ induces a K-homomorphism of non-associative algebras $\bar{\varepsilon}: K(X_0) \to \mathfrak{g}$. The inverse map $\mathfrak{g} \to X_0$ gives a map $\bar{\sigma}: \mathfrak{g} \to K(X_0)$ such that $\bar{\varepsilon}\bar{\sigma}=$ identity map. We define sets $X_1, \cdots, X_n, \cdots$ by induction over n. Suppose we have defined the sets $X_0, X_1, \cdots, X_n$ and an exact sequence of K-modules

$$K(X_n) \xrightarrow{\bar{d}_n} K(X_{n-1}) \to \cdots \to K(X_1) \xrightarrow{\bar{d}_1} K(X_0) \xrightarrow{\bar{\varepsilon}} \mathfrak{g} \to 0$$

such that (i) $K(X_p)$ is a K-free module with X_p as base $(0 \leqq p \leqq n)$ and (ii) X_p is a set in one-to-one correspondence with the kernel N_{p-1} of the K-homomorphism $\bar{d}_{p-1}: K(X_{p-1}) \to K(X_{p-2})$ for $2 \leqq p \leqq n$, while X_1 is a set in one-to-one correspondence with the kernel N_0 of the K-homomorphism $\bar{\varepsilon}: K(X_0) \to \mathfrak{g}$. Let X_{n+1} be a set in one-to-one correspondence with the kernel N_n of the K-homomorphism $\bar{d}_n: K(X_n) \to K(X_{n-1})$. Let $K(X_{n+1})$ be the K-free module with X_{n+1} as base. The kernel N_n being a K-submodule of $K(X_n)$ the bijective map

$X_{n+1}\to N_n$ induces a K-homomorphism $K(X_{n+1})\to N_n$ which when composed with the inclusion map $N_n\to K(X_n)$ gives a K-homomorphism $\bar{d}_{n+1}: K(X_{n+1})\to K(X_n)$ such that the sequence

$$K(X_{n+1})\xrightarrow{\bar{d}_{n+1}} K(X_n)\xrightarrow{\bar{d}_n} K(X_{n-1})\to\cdots\to K(X_1)\xrightarrow{\bar{d}_1} K(X_0)\xrightarrow{\bar{\varepsilon}}\mathfrak{g}\to 0$$

is exact.

The direct sum $\sum_{n\geqq 0} K(X_n)$ is a K-free differential graded module. We shall define inductively maps

$$X_i\times X_j\to X_{i+j}\qquad (i\geqq 0,\ j\geqq 0)$$

(the image of (x_i, x_j) being denoted by $[x_i, x_j]$) which when extended by K-linearity give to $\sum_{n\geqq 0} K(X_n)$ the structure of a K-free non-associative differential graded algebra. For $i=0$, $j=0$ the map $X_0\times X_0\to X_0$ has already been defined. Suppose that the maps have been defined for $i+j\leqq n$ such that

$$\bar{d}_{i+j}[x_i, x_j]=[\bar{d}_i x_i, x_j]+(-1)^i[x_i, \bar{d}_j x_j]\,. \tag{2.5}$$

We take $\bar{d}_0=0$. In order to define the map for $i+j=n+1$, consider the expression

$$[\bar{d}_i x_i, x_j]+(-1)^i[x_i, \bar{d}_j x_j]\in K(X_n)\,.$$

It is annulled by $\bar{d}_n$ and so belongs to N_n. The element in X_{n+1} which corresponds to it under the one-to-one correspondence $X_{n+1}\to N_n$ is defined to be the product $[x_i, x_j]$. By this definition the relation (2.5) is true for $i+j=n+1$. We observe that $K(X_n)$ is not only a K-free module but also a $K(X_0)$-module.

Let X be the sum set $\sum_{n\geqq 0} X_n$. Then $\sum_{n\geqq 0} K(X_n)=K(X)$, the K-free module with X as base; indeed it is a K-free differential graded non-associative algebra. Let $\mathfrak{p}$ be the two-sided ideal generated by the following elements

$\bar{\sigma}(0)$, $[x_{2p}, x_{2p}]$, $[x_i, x_j]+(-1)^{ij}[x_j, x_i]$, and $(-1)^{ki}[x_i, [x_j, x_k]]+(-1)^{ij}[x_j, [x_k, x_i]]+(-1)^{jk}[x_k, [x_i, x_j]]$, where $x_{2p}\in X_{2p}(p\geqq 0)$, $x_i\in X_i$, $x_j\in X_j$, $x_k\in X_k$. The quotient algebra $U=K(X)/\mathfrak{p}$ is a differential graded Lie algebra. If U_n denotes the image of $K(X_n)$ under the canonical map $K(X)\to K(X)/\mathfrak{p}$, we have $U=\sum_{n\geqq 0} U_n$ with maps $d_n: U_n\to U_{n-1}$ $(n\geqq 1)$, $d_0=0$ induced by $\bar{d}_n$ $(n\geqq 0)$. The homomorphism $\bar{\varepsilon}: K(X_0)\to\mathfrak{g}$ yields a Lie algebra homomorphism $\varepsilon: U_0\to\mathfrak{g}$ and the map $\bar{\sigma}:\mathfrak{g}\to K(X_0)$ gives a map $\sigma:\mathfrak{g}\to U_0$ which is such that $\sigma([x, y])=[\sigma(x), \sigma(y)]$ for $x, y\in\mathfrak{g}$ and $\varepsilon\sigma=$ identity map. We can also define maps $s_0:\mathrm{Ker}\,\varepsilon\to U_1$ and $s_{n-1}:\mathrm{Ker}\,d_{n-1}\to U_n$ $(n>1)$ with the help of the bijective maps $X_n\to N_{n-1}$ $(n\geqq 1)$ such that $d_1 s_0$ and $d_n s_{n-1}$ are identity maps.

Let us define with Dixmier [**1**, p. 63] the algebra $G(U)$ of the graded K-module U. We recall that $G(U)$ is the (associative) quotient algebra of the tensor algebra (over K) of U by the two-sided ideal generated by the elements of the form

$$u\otimes v+(-1)^{\alpha\beta}v\otimes u, \quad \text{where} \quad u\in U_\alpha,\ v\in U_\beta;$$

and $w\otimes w$, where w is a homogeneous element of *even* degree in U. Every element of $G(U)$ is a K-linear combination of the elements of the form $\langle u_1|\cdots|u_n\rangle$, $u_i\in U_{\alpha_i}$, $1\leqq i\leqq n$, where $\langle u_1|\cdots|u_n\rangle$ denotes the image of $u_1\otimes\cdots\otimes u_n$ in $G(U)$. The image of the unit element 1 of K in $G(U)$ is denoted by $\langle\,\rangle$. In particular $\langle u\rangle$ denotes the image in $G(U)$ of the homogeneous element u of U. Indeed U can also be identified with its image in $G(U)$. We say that the element $\langle u_1|\cdots|u_n\rangle$ is of *degree* $\alpha_1+\cdots+\alpha_n$ and *order* n. We define the *total degree* of $\langle u_1|\cdots|u_n\rangle$ in $G(U)$ to be the sum of the degree and the order, namely, $n+\alpha_1+\cdots+\alpha_n$. We note that $G(U)$ possesses a unit element, namely, $\langle\,\rangle$ which is taken to be of zero degree and zero order. If u (resp. v) is a homogeneous element of $G(U)$ of degree α (resp. β) and order α' (resp. β') we have

$$\langle v|u\rangle=(-1)^{\alpha\beta+\alpha'\beta'}\langle u|v\rangle.$$

If U^+ denotes the sum of U_n for n even and U^- denotes the sum of U_n for n odd, then

$$G(U)=E(U^+)\underset{K}{\otimes}S(U^-),$$

where $E(U^+)=G(U^+)$ is the exterior algebra of the K-module U^+ and $S(U^-)=G(U^-)$ is the symmetric algebra of the K-module U^-.

Let M be a (left) $\mathfrak{g}$-module. The K-linear combination of the elements of the form $\langle u_1|\cdots|u_n\rangle$, $u_i\in U_{\alpha_i}$, $i=1,\cdots,n$ form a sub-K-module of $G(U)$ which we denote by $U_{\alpha_1,\cdots,\alpha_n}$. For $n=0$ we take K instead of $U_{\alpha_1,\cdots,\alpha_n}$. Let

$$\mathrm{Hom}_K(G(U),M)=\sum_{(\alpha_1,\cdots,\alpha_n)}\mathrm{Hom}_K(U_{\alpha_1,\cdots,\alpha_n},M)$$

the sum being taken over all finite monotonic increasing sequences of non-negative integers $(\alpha_1,\cdots,\alpha_n)$ including the case $n=0$. The degree, the order and the total degree in $G(U)$ induce degree, order and total degree in $\mathrm{Hom}_K(G(U),M)$. We define a differential δ in $\mathrm{Hom}_K(G(U),M)$ such that for $f\in\mathrm{Hom}_K(G(U),M)$ we have

$$\begin{aligned}(2.6)\qquad \delta f\langle u_1|\cdots|u_n\rangle=(-1)^{n+1}\Big[&\sum_{i=1}^{n}(-1)^{\alpha_1+\cdots+\alpha_{i-1}}f\langle u_1|\cdots|du_i|\cdots|u_n\rangle\\ &+\sum_{i=1}^{n}(-1)^{i-1}(\varepsilon u_i)f\langle u_i|\cdots|\hat{u}_i|\cdots|u_n\rangle\\ &-\sum_{1\leqq i<j\leqq n}(-1)^{\varepsilon_{ij}}f\langle[u_i,u_j]|u_1|\cdots|\hat{u}_i|\cdots|\hat{u}_j|\cdots|u_n\rangle\Big],\end{aligned}$$

where $\varepsilon_{ij}=\sum_{p<q}(\alpha_p\alpha_q+1)$, $p\in\{1,\cdots,i-1,i+1,\cdots,j-1\}$, $q\in\{i,j\}$ and $\hat{u}_i$ means that u_i has to be omitted.

It can be verified that $\delta\delta f=0$. Indeed we can write $\delta=\delta_1+\delta_2$, where

$$\delta_1 f\langle u_1|\cdots|u_n\rangle=(-1)^{n+1}\sum_{i=1}^{n}(-1)^{\alpha_1+\cdots+\alpha_{i-1}}f\langle u_1|\cdots|du_i|\cdots|u_n\rangle,$$

$$\delta_2 f\langle u_1|\cdots|u_n\rangle=(-1)^{n+1}\Big[\sum_{i=1}^{n}(-1)^{i-1}(\varepsilon u_i)f\langle u_1|\cdots|\hat{u}_i|\cdots|u_n\rangle$$

$$-\sum_{1\leq i<j\leq n}(-1)^{\varepsilon_{ij}}f\langle[u_i,u_j]|u_1|\cdots|\hat{u}_i|\cdots|\hat{u}_j|\cdots|u_n\rangle\Big]$$

and verify that $\delta_1^2=0$, $\delta_2^2=0$ and $\delta_1\delta_2=-\delta_2\delta_1$.

DEFINITION. The graded cohomology module $H^*(\mathrm{Hom}_K(G(U),M))$ is called the cohomology module of the Lie algebra $\mathfrak{g}$ with coefficients in the $\mathfrak{g}$-module M. We write

$$H^n(\mathfrak{g},M)=H^n(\mathrm{Hom}_K(G(U),M)).$$

3. Interpretations of $H^0(\mathfrak{g},M)$ and $H^1(\mathfrak{g},M)$.

We write $\sigma(x)=(x)$ for $x\in\mathfrak{g}$ and $s_1(n)=(n)$ for $n\in\mathrm{Ker}\,d_1$, etc.. An element $n\in\mathrm{Ker}\,\varepsilon$ is of the form $\sum_i k_i(x_i)$, where $k_i\in K$, $x_i\in\mathfrak{g}$ and $\sum_i k_i x_i=0$. An element $n\in\mathrm{Ker}\,d_1$ is of the form $\sum_j k_j(n_j)$, where $k_j\in K$, $n_j\in\mathrm{Ker}\,\varepsilon$ and $\sum_j k_j n_j=0$.

A 0-cochain is an element of $\mathrm{Hom}_K(K,M)$ and so may be identified with an element of M. If $f\in M$, then $\delta f\in\mathrm{Hom}_K(U,M)$ and

$$\delta f\langle(x)\rangle=x\cdot f,\quad\text{where}\quad x\in\mathfrak{g}\quad\text{and}\quad(x)\in U_0. \tag{3.1}$$

To avoid cumbersome notation we shall write $\langle x\rangle$ instead of $\langle(x)\rangle$, $\langle x_1|x_2\rangle$ instead of $\langle(x_1)|(x_2)\rangle$ etc.. If f is a 0-cocycle, we have $x\cdot f=0$ for every $x\in\mathfrak{g}$. A 0-coboundary being the zero element of M it follows that $H^0(\mathfrak{g},M)$ is isomorphic to the sub-K-module of M consisting of the *invariant* elements of M.

A 1-cochain is an element $f\in\mathrm{Hom}_K(U_0,M)$ and $\delta f\in\mathrm{Hom}_K(U_{0,0},M)+\mathrm{Hom}_K(U_1,M)$ such that

$$\delta f\langle x_1|x_2\rangle=-x_1 f\langle x_2\rangle+x_2 f\langle x_1\rangle+f\langle[x_1,x_2]\rangle, \tag{3.2}$$

$$\delta f\langle n\rangle=\sum_i k_i f\langle x_i\rangle, \tag{3.3}$$

where $x_1,x_2,x_i\in\mathfrak{g}$, $k_i\in K$ and $\sum_i k_i x_i=0$. It should be noted that we have made use of the relation $[\sigma x_1,\sigma x_2]=\sigma[x_1,x_2]$ in expressing the coboundary δf over $\langle x_1|x_2\rangle$. If f is a 1-cocycle and if $\varphi:\mathfrak{g}\to M$ is the restriction of

$f: U_0 \to M$ to $\mathfrak{g}$, we have

(i) $$\varphi([x_1, x_2]) = x_1\varphi(x_2) - x_2\varphi(x_1)$$

and

(ii) $$\sum_i k_i x_i = 0 \Rightarrow \sum_i k_i \varphi(x_i) = 0 .$$

Moreover, if $f = \delta g$ where $g \in M$, then

$$\varphi(x) = xg, \quad \text{where} \quad x \in \mathfrak{g} .$$

Hence $H^1(\mathfrak{g}, M)$ is the K-module of the crossed homomorphisms of $\mathfrak{g}$ into M reduced modulo the principal homomorphisms.

4. Interpretation of $H^2(\mathfrak{g}, M)$.

A 2-cochain is an element $f \in \mathrm{Hom}_K(U_{0,0}, M) + \mathrm{Hom}_K(U_1, M)$. Then $\delta f \in \mathrm{Hom}_K(U_{0,0,0}, M) + \mathrm{Hom}_K(U_{0,1}, M) + \mathrm{Hom}_K(U_2, M)$. We have

(4.1) $$\delta f\langle x_1 | x_2 | x_3\rangle = x_1 f\langle x_2 | x_3\rangle - x_2 f\langle x_1 | x_3\rangle + x_3 f\langle x_1 | x_2\rangle - f\langle [x_1, x_2] | x_3\rangle + f\langle [x_1, x_3] | x_2\rangle - f\langle [x_2, x_3] | x_1\rangle ,$$

(4.2) $$\delta f\langle x | n\rangle = -\sum_i k_i f\langle x | x_i\rangle - x f\langle n\rangle + f\langle [x, n]\rangle ;$$

(4.3) $$\delta f\langle n'\rangle = \sum_j k'_j f\langle n_j\rangle ,$$

where $x_1, x_2, x_3, x \in \mathfrak{g}$, $n = \sum_i k_i(x_i)$, $k_i \in k$, $x_i \in \mathfrak{g}$ such that $\sum_i k_i x_i = 0$ and $n' = \sum_j k'_j(n_j)$, $k'_j \in k$, $n_j \in \ker \varepsilon$ such that $\sum_j k'_j n_j = 0$.

If f is a 2-cocycle, it determines two maps

$$\gamma_1 : \mathfrak{g} \times \mathfrak{g} \to M$$
$$\gamma_2 : N_0 \to M$$

satisfying the following identities.

(4.4) $$\gamma_1(x, x) = 0$$

(4.5) $$\gamma_1(x_1, x_2) = -\gamma_1(x_2, x_1) ,$$

(4.6) $$x_1\gamma_1(x_2, x_3) - x_2\gamma_1(x_1, x_3) + x_3\gamma_1(x_1, x_2) - \gamma_1([x_1, x_2], x_3) + \gamma_1([x_1, x_3], x_2) - \gamma_1([x_2, x_3], x_1) = 0 ,$$

(4.7) $$\sum_i k_i \gamma_1(x, x_i) = -x\gamma_2(n) + \gamma_2([x, n]) ,$$

(4.8) $$\sum_j k'_j \gamma_2(n_j) = 0 ,$$

where x_1, x_2, x_3, x, x_i, n and n_j are as before.

Let $\mathcal{E}_f$ be the set of all pairs (m, x), where $m \in M$, $x \in \mathfrak{g}$. We define addition, multiplication and scalar multiplication by elements of K as follows:

$$(4.9) \qquad (m_1, x_1)+(m_2, x_2)=(m_1+m_2+\gamma_2(x_1, x_2), x_1+x_2)\,;$$

$$(4.10) \qquad [(m_1, x_1), (m_2, x_2)]=(x_1m_2-x_2m_1+\gamma_1(x_1, x_2), [x_1, x_2])\,;$$

$$(4.11) \qquad k(m, x)=(km+\gamma_2(k, x), kx)\,,$$

where by $\gamma_2(x_1, x_2)$ we mean $\gamma_2((x_1+x_2)-(x_1)-(x_2))$ and by $\gamma_2(k, x)$ we mean $\gamma_2((kx)-k(x))$; $x_1, x_2, x \in \mathfrak{g}$, $m \in M$, $k \in K$. After proving the associative law for the addition defined above the relations (4.9) and (4.11) can be combined into a single relation

$$(4.12) \qquad \sum_i k_i(m_i, x_i)=(\sum_i k_im_i+\gamma_2(n), \sum_i k_ix_i)\,,$$

where $k_i \in K$, $m_i \in M$, $x_i \in \mathfrak{g}$ and $n=(\sum_i k_ix_i)-\sum_i k_i(x_i) \in N_0$.

We shall show that with these operations $\mathcal{E}_f$ is a Lie algebra. We have to verify the following relations.

1. $\xi+\eta=\eta+\xi$,
2. $(\xi+\eta)+\zeta=\xi+(\eta+\zeta)$,
3. $[\xi, \eta+\zeta]=[\xi, \eta]+[\xi, \zeta]$,
4. $[\xi+\eta, \zeta]=[\xi, \zeta]+[\eta, \zeta]$,
5. $[\xi, \xi]=0$,
6. $[\xi, [\eta, \zeta]]+[\eta, [\zeta, \xi]]+[\zeta, [\xi, \eta]]=0$,
7. $[k\xi, \eta]=k[\xi, \eta]$,
8. $[\xi, k\eta]=k[\xi, \eta]$,
9. $k_1(k_2\xi)=(k_1k_2)\xi$,
10. $k(\xi+\eta)=k\xi+k\eta$,
11. $(k_1+k_2)\xi=k_1\xi+k_2\xi$,

where $\xi, \eta, \zeta \in \mathcal{E}_f$ and $k, k_1, k_2 \in K$.

Let $\xi=(m_1, x_1)$, $\eta=(m_2, x_2)$ and $\zeta=(m_3, x_3)$, where $m_1, m_2, m_3 \in M$ and $x_1, x_2, x_3 \in \mathfrak{g}$. The relation $\xi+\eta=\eta+\xi$ is trivially verified. To verify (2) we have

$$\begin{aligned}\{(m_1, x_1)+(m_2, x_2)\}+(m_3, x_3)&=(m_1+m_2+\gamma_2(x_1, x_2), x_1+x_2)+(m_3, x_3)\\ &=(m_1+m_2+m_3+\gamma_2(x_1, x_2)+\gamma_2(x_1+x_2+x_3), x_1+x_2+x_3)\end{aligned}$$

and

$$\begin{aligned}(m_1, x_1)+\{(m_2, x_2)+(m_3, x_3)\}&=(m_1, x_1)+(m_2+m_3+\gamma_2(x_2, x_3), x_2+x_3)\\ &=(m_1+m_2+m_3+\gamma_2(x_2, x_3)+\gamma_2(x_1, x_2+x_3), x_1+x_2+x_3)\,.\end{aligned}$$

We have to show that

$$\gamma_2((x_1+x_2)-(x_1)-(x_2))+\gamma_2((x_1+x_2+x_3)-(x_1+x_2)-(x_3))$$
$$=\gamma_2((x_2+x_3)-(x_2)-(x_3))+\gamma_2((x_1+x_2+x_3)-(x_1)-(x_2+x_3))$$

But this follows from (4.8) by taking

$$n_1=(x_1+x_2)-(x_1)-(x_2)\,,\quad n_2=(x_1+x_2+x_3)-(x_1+x_2)-(x_3)\,,$$
$$n_3=(x_2+x_3)-(x_2)-(x_3)\,,\quad n_4=(x_1+x_2+x_3)-(x_1)-(x_2+x_3)$$

and $k_1'=1$, $k_2'=1$, $k_3'=-1$, $k_4'=-1$.

To verify (3) we have

$$[(m_1, x_1), (m_2, x_2)+(m_3, x_3)]=[(m_1, x_1), (m_2+m_3+\gamma_2(x_2, x_3), x_2+x_3)]$$
$$=(x_1m_2+x_1m_3+x_1\gamma_2(x_2, x_3)-x_2m_1-x_3m_1+\gamma_1(x_1, x_2+x_3), [x_1, x_2+x_3])$$

and

$$[(m_1, x_1), (m_2, x_2)]+[(m_1, x_1), (m_3, x_3)]$$
$$=(x_1m_2-x_2m_1+\gamma_1(x_1, x_2), [x_1, x_2])+(x_1m_3-x_3m_1+\gamma_1(x_1, x_3), [x_1, x_3])$$
$$=(x_1m_2-x_2m_1+x_1m_3-x_3m_1+\gamma_1(x_1, x_2)+\gamma_1(x_1, x_3)$$
$$+\gamma_2([x_1, x_2], [x_1, x_3]), [x_1, x_2]+[x_1, x_3])\,.$$

We have to show that

$$x_1\gamma_2(x_2, x_3)+\gamma_1(x_1, x_2+x_3)=\gamma_1(x_1, x_2)+\gamma_1(x_1, x_3)+\gamma_2([x_1, x_2], [x_1, x_3])$$

or what is the same thing

$$\gamma_1(x_1, x_2+x_3)-\gamma_1(x_1, x_2)-\gamma_1(x_1, x_3)=-x_1\gamma_2(x_2, x_3)+\gamma_2([x_1, x_2], [x_1, x_3])\,.$$

But this follows from (4.7) by taking $x=x_1$ and $n=(x_2+x_3)-(x_2)-(x_3)$. The relation (4) can be verified in a similar manner.

The relation (5) follows from the fact that $\gamma_1(x, x)=f\langle x|x\rangle=0$. To verify Jacobi's identity we calculate

$$[(m_1, x_1), [(m_2, x_2), (m_3, x_3)]]=[(m_1, x_1), (x_2m_3-x_3m_2+\gamma_1(x_2, x_3), [x_2, x_3])]$$
$$=(x_1x_2m_3-x_1x_3m_2-[x_2, x_3]m_1+x_1\gamma_1(x_2, x_3)$$
$$+\gamma_1(x_1, [x_2, x_3]), [x_1, [x_2, x_3]])\,.$$

Permuting circularly and adding we see that Jacobi's identity is satisfied if

$$x_1\gamma_1(x_2, x_3)+x_2\gamma_1(x_3, x_1)+x_3\gamma_1(x_1, x_2)+\gamma_1(x_1, [x_2, x_3])+$$
$$\gamma_1(x_2, [x_3, x_1])+\gamma_1(x_3, [x_1, x_2])+\gamma_2(n)=0\,,$$

where

$$n=(0)-([x_1, [x_2, x_3]])-([x_2, [x_3, x_1]])-([x_3, [x_1, x_2]])=0$$

since $(0)=\sigma(0)\in\mathfrak{p}$ and

$$([x_1,[x_2,x_3]])+([x_2,[x_3,x_1]])+([x_3,[x_1,x_2]])$$
$$=[\sigma x_1,[\sigma x_2,\sigma x_3]]+[\sigma x_2,[\sigma x_3,\sigma x_1]]+[\sigma x_3,[\sigma x_1,\sigma x_2]]\in\mathfrak{p}\,.$$

This means $\gamma_2(n)=0$. Also

$$x_1\gamma_1(x_2,x_3)+x_2\gamma_1(x_3,x_1)+x_3\gamma_1(x_1,x_2)+\gamma_1(x_1,[x_2,x_3])+\gamma_1(x_2,[x_3,x_1])$$
$$+\gamma_1(x_3,[x_1,x_2])=x_1\gamma_1(x_2,x_3)-x_2\gamma_1(x_1,x_3)+x_3\gamma_1(x_1,x_2)-\gamma_1([x_1,x_2],x_3)$$
$$+\gamma_1([x_1,x_3],x_2)-\gamma_1([x_1,x_2],x_3)=0 \text{ by virtue of (4.6).}$$

To verify (7) we note that

$$[k(m_1,x_1),(m_2,x_2)]=[(km_1+\gamma_2(k,x_1),kx_1),(m_2,x_2)]$$
$$=(kx_1m_2-kx_2m_1-x_2\gamma_2(k,x_1)+\gamma_1(kx_1,x_2),[kx_1,x_2])$$

and

$$k[(m_1,x_1),(m_2,x_2)]=k(x_1m_2-x_2m_1+\gamma_1(x_1,x_2),[x_1,x_2])$$
$$=(kx_1m_2-kx_2m_1+k\gamma_1(x_1,x_2)+\gamma_2(k,[x_1,x_2]),k[x_1,x_2])\,.$$

So we have to show that

$$-x_2\gamma_2(k,x_1)+\gamma_1(kx_1,x_2)=k\gamma_1(x_1,x_2)+\gamma_2(k,[x_1,x_2])$$

or what is the same thing

$$\gamma_1(kx_1,x_2)-k\gamma_1(x_1,x_2)=x_2\gamma_2(k,x_1)+\gamma_2(k,[x_1,x_2])$$

that is

$$-k\gamma_1(x_2,x_1)+\gamma_1(x_2,kx_1)=-x_2\gamma_2((kx_1)-k(x_1))+\gamma_2(-([kx_1,x_2])+k([x_1,x_2]))\,.$$

This is a consequence of (4.7) by taking $x=x_2$ and $n=(kx_1)-k(x_1)$.

The relation (8) can be verified in a similar manner.

The relations (9), (10) and (11) can be verified in a straight-forward fashion. We have shown in this way that $\mathcal{E}_f$ is a Lie algebra, the element (0, 0) being the zero of $\mathcal{E}_f$. If we define $\alpha: M\to\mathcal{E}_f$ and $\beta:\mathcal{E}_f\to\mathfrak{g}$ by $\alpha(m)=(m,0)$ and $\beta(m,x)=x$, we have an exact sequence of Lie algebras

$$0\to M\xrightarrow{\alpha}\mathcal{E}_f\xrightarrow{\beta}\mathfrak{g}\to 0\,,$$

where M has the trivial multiplicative structure. We observe that

$$[(m_1,x_1),(m,0)]=(x_1m,0)$$

showing that the exact sequence induces on M the given $\mathfrak{g}$-module structure.

Let f' be a 2-cocycle which is cohomologous to f. This means $f'=f+\delta g$, where g is a 1-cochain. Let $\mathcal{E}_{f'}$ be the Lie algebra determined by the 2-cocycle f'. Since g is a 1-cochain, it gives a map $\psi:\mathfrak{g}\to M$, which is the restriction of g to $\mathfrak{g}$. We define a map $\phi:\mathcal{E}_f\to\mathcal{E}_{f'}$ by putting

$$\phi(m, x) = (m+\psi(x), x)$$

where $m \in M$, $x \in \mathfrak{g}$. Then

$$\phi(\sum_i k_i(m_i, x_i)) = \phi(\sum_i k_i m_i + \gamma_2(n), \sum_i k_i x_i)$$
$$= (\sum_i k_i m_i + \gamma_2(n) + \psi(\sum_i k_i x_i), \sum_i k_i x_i),$$

and

$$\sum_i k_i \phi(m_i x_i) = \sum_i k_i(m_i + \psi(x_i), x_i) = (\sum_i k_i m_i + \sum_i k_i \psi(x) + \gamma_2'(n), \sum_i k_i x_i),$$

where

$$n = (\sum_i k_i x_i) - \sum_i k_i(x_i) \in N_0,\ m_i \in M,\ x \in \mathfrak{g}.$$

But $\gamma_2'(n) - \gamma_2(n) = \delta g(n) = \psi(\sum_i k_i x_i) - \sum_i k_i \psi(x_i)$ by virtue of (3.3). Therefore

$$\phi(\sum_i k_i(m_i, x_i)) = \sum_i k_i \phi(m_i, x_i),$$

where $m_i \in M$, $x_i \in \mathfrak{g}$. Again,

$$\phi[(m_1, x_1), (m_2, x_2)] = \phi(x_1 m_2 - x_2 m_1 + \gamma_1(x_1, x_2), [x_1, x_2])$$
$$= (x_1 m_2 - x_2 m_1 + \gamma_1(x_1, x_2) + \psi([x_1, x_2]), [x_1, x_2]),$$

while

$$[\phi(m_1, x_1), \phi(m_2, x_2)] = [(m_1 + \psi(x_1), x_1), (m_2 + \psi(x_2), x_2)]$$
$$= (x_1 m_2 - x_2 m_1 + x_1 \psi(x_2) - x_2 \psi(x_1) + \gamma_1'(x_1, x_2), [x_1, x_2]),$$

where $m_1, m_2 \in M$ and $x_1, x_2 \in \mathfrak{g}$. Since by (3.2)

$$\gamma_1'(x_1, x_2) - \gamma_1(x_1, x_2) = \delta g\langle x_1 | x_2 \rangle = -x_1 \psi(x_2) + x_2 \psi(x_1) + \psi([x_1, x_2]),$$

it follows that

$$\phi[(m_1, x_1), (m_2, x_2)] = [\phi(m_1, x_1), \phi(m_2, x_2)].$$

We have now shown that ϕ is a homomorphism of Lie algebras. It is easy to verify that ϕ is bijective.

Conversely, suppose

$$0 \to M \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathfrak{g} \to 0$$

is an exact sequence of Lie algebras, where M is an abelian Lie algebra. Let $\rho: \mathfrak{g} \to \mathcal{E}$ be a map such that $\beta\rho =$ identity map, and $\rho(-x) = -\rho(x)$ where $x \in \mathfrak{g}$. This is possible since there exists an element $k \in K$ for which $2k = 1$. Let us define two maps

$$\gamma_1: \mathfrak{g} \times \mathfrak{g} \to M,$$

and

$$\gamma_2: N_0 \to M$$

by the relations

$$\gamma_1(x_1, x_2) = \rho([x_1, x_2]) - [\rho(x_1), \rho(x_2)]\,, \tag{4.13}$$

and

$$\gamma_2(n) = \sum_i k_i \rho(x_i)\,, \tag{4.14}$$

where $x_1, x_2, x_i \in \mathfrak{g}$, $k_i \in K$ and $n = \sum_i k_i(x_i) \in N_0$ such that $\sum_i k_i x_i = 0$. We observe that the relations (4.4) and (4.5) are satisfied in view of the choice of ρ. Also

$$\begin{aligned} x_1\gamma_1(x_2, x_3) - x_2\gamma_1(x_1, x_3) + x_3\gamma_1(x_1, x_2) - \gamma_1([x_1, x_2], x_3) + \gamma_1([x_1, x_3], x_2) \\ -\gamma_1([x_2, x_3], x_1) = [\rho(x_1), \rho([x_2, x_3]) - [\rho(x_2), \rho(x_3)]] \\ -[\rho(x_2), \rho([x_1, x_3]) - [\rho(x_1), \rho(x_3)]] + [\rho(x_3), \rho([x_1, x_2]) - [\rho(x_1), \rho(x_2)]] \\ -\rho([[x_1, x_2], x_3]) + [\rho([x_1, x_2]), \rho(x_3)] \\ +\rho([[x_1, x_3], x_2]) - [\rho([x_1, x_3]), \rho(x_2)] - \rho([[x_2, x_3], x_1]) \\ +[\rho([x_2, x_3]), \rho(x_1)] = \rho([x_1, [x_2, x_3]]) + \rho([x_2, [x_3, x_1]]) \\ +\rho([x_3, [x_1, x_2]]) = \gamma_2(m)\,, \end{aligned}$$

where

$$m = ([x_1, [x_2, x_3]]) + ([x_2, [x_3, x_1]]) + ([x_3, [x_1, x_2]])$$

with $x_1, x_2, x_3 \in \mathfrak{g}$. Since $m \in \mathfrak{p}$, $\gamma_2(m) = 0$. Therefore the relation (4.6) is satisfied.

Again,

$$\begin{aligned} &\sum_i k_i\gamma_1(x, x_i) + x\gamma_2(n) - \gamma_2([x, n]) \\ &\quad = \sum_i k_i\rho([x, x_i]) - \sum_i k_i[\rho(x), \rho(x_i)] + [\rho(x), \sum_i k_i\rho(x_i)] \\ &\quad\quad - \sum_i k_i\rho([x, x_i]) = 0\,, \end{aligned}$$

where $n = \sum_i k_i(x_i)$ such that $\sum_i k_i x_i = 0$, $k_i \in K$, $x_i \in \mathfrak{g}$. Therefore the relation (4.7) is satisfied. The relation (4.8) is trivially satisfied.

After the usual arguments we have

THEOREM 1. *There exists a natural one-to-one correspondence between the two-dimensional cohomology module $H^2(\mathfrak{g}, M)$ and the set of equivalence classes of the special extensions of $\mathfrak{g}$ with kernel M which induce over M the given $\mathfrak{g}$-module structure.*

5. On $H^3(\mathfrak{g}, M)$.

Let $\mathfrak{h}$ be a Lie algebra, let $D(\mathfrak{h})$ denote the Lie algebra of derivations of $\mathfrak{h}$ and let $I(\mathfrak{h})$ denote the ideal of $D(\mathfrak{h})$ consisting of the inner derivations of $\mathfrak{h}$. Consider the homomorphism of Lie algebras $\mu:\mathfrak{h}\to D(\mathfrak{h})$ which maps every element of $\mathfrak{h}$ into the inner derivation of $\mathfrak{h}$ induced by it. The kernel of this homomorphism is the centre $C_{\mathfrak{h}}$ of the Lie algebra $\mathfrak{h}$ and the image is $I(\mathfrak{h})$. So we have an exact sequence of Lie algebras

$$0\to C_{\mathfrak{h}}\to\mathfrak{h}\xrightarrow{\mu} D(\mathfrak{h})\to D(\mathfrak{h})/I(\mathfrak{h})\to 0\,. \tag{5.1}$$

We call $D(\mathfrak{h})/I(\mathfrak{h})$ the Lie algebra of exterior derivations of $\mathfrak{h}$. The centre $C_{\mathfrak{h}}$ is a $D(\mathfrak{h})/I(\mathfrak{h})$-module for the operation $\bar{D}c=Dc$, where $c\in C_{\mathfrak{h}}$, $\bar{D}\in D(\mathfrak{h})/I(\mathfrak{h})$ and D is an element of $D(\mathfrak{h})$ belonging to the coset $\bar{D}$.

Consider an exact sequence of Lie algebras

$$0\to\mathfrak{h}\xrightarrow{\alpha}\mathcal{E}\xrightarrow{\beta}\mathfrak{g}\to 0\,. \tag{5.2}$$

Since $\alpha\mathfrak{h}$ is an ideal of $\mathcal{E}$, the map $e\to ade$, where ade denotes the inner derivation of $\mathcal{E}$ induced by the element e of $\mathcal{E}$ gives a Lie algebra homomorphism $\nu:\mathcal{E}\to D(\mathfrak{h})$. Since $\alpha\mathfrak{h}$ is mapped into $I(\mathfrak{h})$, ν induces a Lie algebra homomorphism

$$\theta:\mathfrak{g}\to D(\mathfrak{h})/I(\mathfrak{h})\,. \tag{5.3}$$

Conversely, suppose we are given Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ and a homomorphism of Lie algebras $\theta:\mathfrak{g}\to D(\mathfrak{h})/I(\mathfrak{h})$. Does there exist a Lie algebra $\mathcal{E}$ and an exact sequence of Lie algebras of the type (5.2) such that the induced homomorphism (5.3) is the same as the given homomorphism θ? We note that θ gives to $C_{\mathfrak{h}}$ a $\mathfrak{g}$-module structure. We propose to associate with θ an element of $H^3(\mathfrak{g}, C_{\mathfrak{h}})$ called the *obstruction* of θ and we shall answer the question in terms of the obstruction of θ.

Let $\sigma:\mathfrak{g}\to D(\mathfrak{h})$ be a map such that $\sigma(x)$ is an element of the coset $\theta(x)$, where $x\in\mathfrak{g}$ and $\sigma(-x)=-\sigma(x)$. Since θ is a homomorphism of Lie algebras, we have

$$\sigma([x_1, x_2])-[\sigma(x_1), \sigma(x_2)]=\mu\gamma_1(x_1, x_2)\,, \tag{5.4}$$

$$\sum_i k_i\sigma(x_i)=\mu\gamma_2(n)\,, \tag{5.5}$$

where $x_1, x_2, x_i\in\mathfrak{g}$, $k_i\in K$, $n=\sum_i k_i(x_i)\in N_0$ so that $\sum_i k_ix_i=0$, and $\mu\gamma_1(x_1, x_2)$ and $\mu\gamma_2(n)$ are the inner derivations of $\mathfrak{h}$ induced by the elements $\gamma_1(x_1, x_2)$ and $\gamma_2(n)$ of $\mathfrak{h}$. The elements $\gamma_1(x_1, x_2)$ and $\gamma_2(n)$ are not well-determined but the inner derivations $\mu\gamma_1(x_1, x_2)$ and $\mu\gamma_2(n)$ are well-determined.

We define a 3-cochain of $\mathfrak{g}$ with values in $C_{\mathfrak{h}}$ by the relations

(5.6) $$f\langle x_1|x_2|x_3\rangle=\sigma(x_1)\gamma_1(x_2, x_3)-\sigma(x_2)\gamma_1(x_1, x_3)+\sigma(x_3)\gamma_1(x_1, x_2)$$
$$-\gamma_1([x_1, x_2], x_3)+\gamma_1([x_1, x_3], x_2)-\gamma_1([x_2, x_3], x_1),$$

(5.7) $$f\langle x|n\rangle=-\sum_i k_i\gamma_1(x, x_i)-\sigma(x)\gamma_2(n)+\gamma_2([x, n]),$$

(5.8) $$f\langle n'\rangle=\sum_j k'_j\gamma_2(n_j),$$

where $x_1, x_2, x_3\in\mathfrak{g}$, $n=\sum_i k_i(x_i)\in N_0$ so that $\sum_i k_i x_i=0$ and $n'=\sum_j k'_j(n_j)\in N_1$ so that $\sum_j k'_j n_j=0$, $n_j\in\ker\varepsilon$.

The second member of each of the above three relations belongs to $C_{\mathfrak{h}}$, because if we apply μ to each one of them and calculate their values we get zero. We call f an *obstruction* of θ.

PROPOSITION 1. *An obstruction f of θ is a* 3*-cocycle and any two obstructions of θ are cohomologous. If f is an obstruction of θ, then a* 3*-cocycle which is cohomologous to f is also an obstruction.*

PROOF. The maps γ_1 and γ_2 define a "2-cochain" h of $\mathfrak{g}$ with values in $\mathfrak{h}$, but with this difference that $\mathfrak{h}$ is not a $\mathfrak{g}$-module. Also the relations (5.6), (5.7) and (5.8) are similar to the relations (4.1), (4.2) and (4.3) respectively and we may write $f=\delta h$ bearing in mind that h is a "2-cochain" of $\mathfrak{g}$ with values in $\mathfrak{h}$, which is not a $\mathfrak{g}$-module. If $\mathfrak{h}$ were a $\mathfrak{g}$-module we could at once infer that $\delta f=\delta\delta h=0$; but since we do not have

$$\sigma([x_1, x_2])=[\sigma(x_1), \sigma(x_2)] \quad\text{and}\quad \sigma(\sum_i k_i x_i)=\sum_i k_i\sigma(x_i),$$

where $x_1, x_2, x_i\in\mathfrak{g}$, we shall have to verify that in the expressions for δf the terms which involve

$$\sigma([x_1, x_2])-[\sigma(x_1), \sigma(x_2)] \quad\text{and}\quad \sigma(\sum_i k_i x_i)-\sum_i k_i\sigma(x_i)$$

cancel out, the other terms getting cancelled as in the identity $\delta\delta=0$ for 2-cochains.

We observe that

$$\delta f\in \mathrm{Hom}_K(U_{0,0,0,0}, C_{\mathfrak{h}})+\mathrm{Hom}_K(U_{0,0,1}, C_{\mathfrak{h}})+\mathrm{Hom}_K(U_{0,2}, C_{\mathfrak{h}})$$
$$+\mathrm{Hom}_K(U_{1,1}, C_{\mathfrak{h}})+\mathrm{Hom}_K(U_3, C_{\mathfrak{h}}).$$

It is a matter of straightforward verification that

$$\delta f\langle x_1|x_2|x_3|x_4\rangle=0,\ \delta f\langle x_1|x_2|n\rangle=0,\ \delta f\langle x_1|n'\rangle=0,\ \delta f\langle n_1|n_2\rangle=0,\ \delta f\langle n''\rangle=0,$$

where $x_1, x_2, x_3, x_4\in\mathfrak{g}$, $n, n_1, n_2\in N_0$, $n'\in N_1$, $n''\in N_2$. Hence f is a 3-cocycle.

In order to show that two obstructions of θ are cohomologous we note that f depends upon the choice of σ and $h=(\gamma_1, \gamma_2)$. First we shall show that if we choose a second map $\sigma': \mathfrak{g} \to D(\mathfrak{h})$ such that $\sigma'(x)$ is an element of the coset $\theta(x)$, where $x \in \mathfrak{g}$ and $\sigma'(-x)=-\sigma'(x)$, we can choose h in such a way that f remains the same. Indeed $\sigma'-\sigma$ has its values in $\mu\mathfrak{h}$ since $\sigma'(x)$ and $\sigma(x)$ belong to the same coset $\theta(x)$, where $x \in \mathfrak{g}$. Let us write

$$\sigma'(x)=\sigma(x)+\mu\tau(x),$$

where $x \in \mathfrak{g}$ and $\tau(x) \in \mathfrak{h}$. Then using (5.4) and (5.5) we have

$$\sigma'([x_1, x_2])-[\sigma'(x_1), \sigma'(x_2)]=\mu\gamma_1(x_1, x_2)+\mu\{\tau([x_1, x_2]) \\ -[\tau(x_1), \sigma(x_2)]-[\sigma(x_1), \tau(x_2)]-[\tau(x_1), \tau(x_2)]\}$$

and

$$\sum_i k_i\sigma'(x_i)=\mu\gamma_2(n)+\mu(\sum_i k_i\tau(x_i)),$$

where $n=\sum_i k_i(x_i) \in N_0$. We choose

$$\gamma_1'(x_1, x_2)=\gamma_1(x_1, x_2)+\tau([x_1, x_2])-[\tau(x_1), \sigma(x_2)] \\ -[\sigma(x_1), \tau(x_2)]-[\tau(x_1), \tau(x_2)],$$

$$\gamma_2'(n)=\gamma_2(n)+\sum_i k_i\tau(x_i).$$

If f' is the 3-cocycle determined by σ' and (γ_1', γ_2'), then straightforward calculations of $f'\langle x_1|x_2|x_3\rangle$, $f'\langle x|n\rangle$ and $f'(n')$ show that $f'=f$.

If, however, we keep σ fixed and choose $h'=(\gamma_1', \gamma_2')$ instead of $h=(\gamma_1, \gamma_2)$ such that $\mu h'=\mu h$, then $h'-h=g$ has values in $C_\mathfrak{h}$ and so is a 2-cochain of $\mathfrak{g}$ with values in $C_\mathfrak{h}$. If f' is the 3-cocycle determined by h' (and σ), then

$$f'=\delta h'=\delta(h+g)=f+\delta g$$

showing that the two obstructions f and f' are cohomologous.

Finally, given an obstruction f determined by σ and h and a 3-cocycle f' cohomologous to f we have $f'=f+\delta g$, where g is a 2-cochain with values in $C_\mathfrak{h}$. Choose $h'=h+g$. This choice is permissible since $\mu h'=\mu h+\mu g=\mu h$. Then $f'=f+\delta g=\delta h+\delta g=\delta(h+g)=\delta h'$ showing that f' is also an obstruction. This proves the proposition completely.

The cohomology class ξ_θ of $H^3(\mathfrak{g}, C_\mathfrak{h})$ determined by f is called the *obstruction* of θ. We are now in a position to answer the question raised at the beginning of this section.

THEOREM 2. *A homomorphism $\theta: \mathfrak{g} \to D(\mathfrak{h})/I(\mathfrak{h})$ is induced by an extension of $\mathfrak{g}$ with kernel $\mathfrak{h}$ if and only if the obstruction $\xi_\theta=0$.*

PROOF. Let

$$0 \to \mathfrak{h} \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathfrak{g} \to 0$$

be an extension which induces θ. Let $\rho: \mathfrak{g} \to \mathcal{E}$ be a map such that $\rho(-x) = -\rho(x)$, where $x \in \mathfrak{g}$, such that $\beta\rho =$ identity. We take $\sigma: \mathfrak{g} \to D(\mathfrak{h})$ by composing ρ with $\nu: \mathcal{E} \to D(\mathfrak{h})$.

Then we can choose γ_1 and γ_2 such that

$$\gamma_1(x_1, x_2) = \rho([x_1, x_2]) - [\rho(x_1), \rho(x_2)]$$

$$\gamma_2(n) = \sum_i k_i \rho(x_i),$$

where $x_1, x_2 \in \mathfrak{g}$, $n \in \operatorname{Ker} \varepsilon$. We note that the restriction of ν to $\mathfrak{h}$ is μ. If we now substitute these values of γ_1 and γ_2 in (5.6), (5.7) and (5.8), we find $f = 0$.

Conversely, suppose $\xi_\theta = 0$. Then by virtue of Proposition 1 we can choose σ and $h = (\gamma_1, \gamma_2)$ such that $f = 0$. Consider the set $\mathcal{E}$ consisting of element of the form (a, x) where $a \in \mathfrak{h}$ and $x \in \mathfrak{g}$ and define the operations as follows

$$\sum_i k_i(a_i, x_i) = (\sum_i k_i a_i + \gamma_2(n), \sum_i k_i x_i),$$

$$[(a_1, x_1), (a_2, x_2)] = (a_1 a_2 + \sigma(x_1) a_2 - \sigma(x_2) a_1 + \gamma_1(x_1, x_2), [x_1, x_2]),$$

where $x_1, x_2, x_i \in \mathfrak{g}$, $k_i \in K$, $n = (\sum_i k_i x_i) - \sum_i k_i(x_i) \in N_0$. It can be easily verified that these operations satisfy the eleven identities of a Lie algebra, since $f = 0$. We observe that $(0, 0)$ is the zero of the Lie algebra $\mathcal{E}$ and that θ is induced by the extension

$$0 \to \mathfrak{h} \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathfrak{g} \to 0,$$

given by $\alpha(a) = (a, 0)$ and $\beta(a, x) = x$, where $a \in \mathfrak{h}$, $x \in \mathfrak{g}$.

REMARK. In order to give a complete interpretation of $H^3(\mathfrak{g}, M)$ it remains to prove the following theorem: Let $\mathfrak{g}$ be a Lie algebra and let M be a $\mathfrak{g}$-module. Let f be a 3-cocycle of $\mathfrak{g}$ with values in M. Then there exists a Lie algebra $\mathfrak{h}$ having M as centre and a homomorphism $\theta: \mathfrak{g} \to D(\mathfrak{h})/I(\mathfrak{h})$ which induces on M the given $\mathfrak{g}$-module structure such that f is an obstruction of θ.

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References

[1] J. Dixmier, Homologie des anneaux de Lie, Ann. Sci. École Norm. Sup., 74 (1957), 25–83.

[2] U. Shukla, Cohomologie des algèbres associatives, Ann. Sci. École Norm. Sup., 78 (1961), 163–209.