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On coverings of algebraic varieties

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Let U and V be algebraic varieties, and $f: U \rightarrow V$ a Galois covering of degree n, defined over a field k; let A and A_0 be Albanese varieties attached to U and V respectively. Then, in the preceding paper [3], we have proved, among several other results, the following two statements:

1) Suppose that V is embedded in some projective space. Let C be a generic hyperplane section curve on V over k and $W=f^{-1}(C)$ the inverse image of C on U; let J and J_0 be Jacobian varieties attached to (the normalization of) W and C respectively. Then the curve W generates A and we have the inequality

$$(*) \qquad \dim J - \dim A \geqq \dim J_0 - \dim A_0 .$$

2) Suppose that U and V are complete and non-singular. Then, under the assumption that the degree n is prime to the characteristic of the universal domain, the equality $\dim \mathfrak{D}_0(U) = \dim \mathfrak{D}_0(A)$ implies the equality $\dim \mathfrak{D}_0(V) = \dim \mathfrak{D}_0(A_0)$.[1]

In the present paper, we shall generalize these results to an arbitrary (i. e. not necessarily Galois) covering $f: U \rightarrow V$. Moreover, the result 2) will be replaced by a better one, i. e. the inequality

$$(**) \qquad \dim \mathfrak{D}_0(U) - \dim \mathfrak{D}_0(A) \geqq \dim \mathfrak{D}_0(V) - \dim \mathfrak{D}_0(A_0) .$$

Here we note that the numbers on the both sides of $(*)$ and $(**)$ are non-negative (cf. Lang [4] and Igusa [1]) and that the assumption on the degree n in $(**)$ is essential as easily seen in Igusa [2]. It seems to be worth noting that the inequalities $(*)$ and $(**)$ may be rewritten in the following forms:

$$(*)' \qquad \dim J - \dim J_0 \geqq \dim A - \dim A_0 .$$

$$(**)' \qquad \dim \mathfrak{D}_0(U) - \dim \mathfrak{D}_0(V) \geqq \dim \mathfrak{D}_0(A) - \dim \mathfrak{D}_0(A_0) .$$

The numbers on the both sides of $(*)'$ and $(**)'$ are also non-negative. As in [3], using the formula of Hurwitz on the genera of curves, we can deduce from $(*)'$ an estimation of the irregularity of the covering variety U of V. In addition to these two inequalities, we shall prove, for this arbitrary covering $f: U \rightarrow V$, some analogous results to the main theorems in [3].

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1) For a complete, non-singular variety W, we donote by $\mathfrak{D}_0(W)$ the space of the linear differential forms of the first kind on W.

1. Preliminaries.

Let $f: U \to V$ be a covering of degree n, defined over an algebraically closed field k. Then the function field $k(U)$ of U over k may be considered as a separable extension over $k(V)$ of degree n. Let K^* be the smallest Galois extension over $k(V)$ containing $k(U)$, which is clearly a regular extension over k. We denote by G and H the Galois groups of $K^*/k(V)$ and $K^*/k(U)$ respectively. Now let U^* be the normalization of V in K^*. Then we have the Galois coverings

$$f^*: U^* \to V \qquad \text{and} \qquad f': U^* \to U,$$

defined over k, and we have

$$f^* = f \circ f'. \tag{1}$$

We denote also by the same letters G and H the Galois groups of these coverings respectively, which consist of everywhere biregular, birational transformations T^*_σ of U^* into itself defined over k (cf. [3]). We set

$$n' = [U^*: U] = (H:1),$$
$$n^* = n'n = [U^*: V] = (G:1),$$

and decompose G into the cosets of H as follows:

$$G = \sum_{i=1}^{n} HT^*_{\rho_i}.$$

Now we list here some results in [3], which we shall need in the following arguments, without proof. Let A^* be an Albanese variety attached to U^* and α^* a canonical mapping of U^* into A^*, both defined over k, such that there exists a simple point p^* on U^* with $\alpha^*(p^*) = 0$. Then each element T^*_σ of G determines an automorphism η^*_σ of A^* and a constant point a^*_σ of A^*, both rational over k, such that

$$\alpha^* \circ T^*_\sigma(u^*) = \eta^*_\sigma \circ \alpha^*(u^*) + a^*_\sigma, \tag{2}$$

where u^* is a generic point of U^* over k. The mapping $T^*_\sigma \to \eta^*_\sigma$ is a group homomorphism.

The main theorem in [3] asserts that there exist Albanese varieties A and A_0 attached to U and V respectively, defined over k, which are quotient abelian varieties of A^* and have the following properties: Let μ' and μ^* be the canonical separable homomorphisms of A^* onto A and A_0 respectively. Then canonical mappings α and α_0 of U and V into A and A_0 may be taken to satisfy the relations

$$\left.\begin{aligned} \alpha \circ f' &= \mu' \circ \alpha^* \\ \alpha_0 \circ f^* &= \mu^* \circ \alpha^* \end{aligned}\right. \quad \text{on } U^*, \tag{3}$$

respectively. We set $C_\sigma^* = (\eta_\sigma^* - \delta_{A^*})(A^*)$[2] and let C^* be the abelian subvariety of A^*, generated by all C_σ^* for all T_σ^* in G. Then the kernel C_G^* of μ^* is the algebraic subgroup of A^* defined over k, which is the union of C^* and all its translations by a_σ^* for all T_σ^* in G. The kernel C_H^* of μ' is defined for H, in a simillar way as C_G^* for G. Since C_G^* contains C_H^* and μ' is canonical, there exists a homomorphism μ of A onto A_0, defined over k, such that we have

$$\mu^* = \mu \circ \mu' \qquad \text{on } A^*. \tag{4}$$

Since μ^* is separable and μ' is surjective, the homomorphism μ is also separable. Moreover, by (1), (3) and (4), we have

$$\alpha_0 \circ f \circ f' = \mu \circ \mu' \circ \alpha^* = \mu \circ \alpha \circ f'$$

and so, as f' is surjective, we have

$$\alpha_0 \circ f = \mu \circ \alpha \qquad \text{on } U. \tag{5}$$

Then it is easily verified that the abelian variety $A_0 = A^*/C_G^*$ is also the quotient abelian variety of A with respect to the algebraic subgroup $\mu'(C_G^*)$ and the homomorphism μ defined in (4) is the canonical separable homomorphism of A onto A_0 (cf. Rosenlicht [**5**]). Moreover, we have seen that a canonical mapping α_0 of V into A_0 may be taken to satisfy (5).

The following formulas will be used in the next section.

$$\begin{aligned} &\mu^* \circ \eta_{\sigma_i}^* = \mu^* \text{ on } A^*, \quad \mu^*(a_\sigma^*) = 0 \text{ for all } T_\sigma^* \text{ in } G. \\ &\mu' \circ \eta_\tau^* = \mu' \text{ on } A^*, \quad \mu'(a_\tau^*) = 0 \text{ for all } T_\tau^* \text{ in } H. \end{aligned} \tag{6}$$

$$\eta_{\sigma_1}^*(a_{\sigma_2}^*) = a_{\sigma_1\sigma_2}^* - a_{\sigma_1}^* \text{ for all } T_{\sigma_1}^*,\ T_{\sigma_2}^* \text{ in } G. \tag{7}$$

2. The endomorphism ρ.

First we prove the existence of an endomorphism of A, which plays an important role in the proof of the inequality (**).

LEMMA. *There exists an endomorphism ρ of A, defined over k, such that we have*

$$\rho \circ \mu' = \mu' \circ \sum_{i=1}^n \eta_{\rho_i}^* \qquad \textit{on } A^*. \tag{8}$$

PROOF. Since μ' is the canonical homomorphism, we have only to prove that the kernel of the homomorphism $\mu' \circ \sum_{i=1}^n \eta_{\rho_i}^*$ of A^* into A contains the kernel C_H^* of μ'. First we fix an element T_τ^* in H. Then, for $i = 1, \cdots, n$, each element $T_{\rho_i}^* \circ T_\tau^*$ belongs to one and only one coset $HT_{\rho_j}^*$. Clearly the mapping $i \to j = s(i)$ defines a permutation of the set $\{1, \cdots, n\}$. Hence we can write

2) For an abelian variety B, we denote by δ_B the identity automorphism of B.

$$\mu' \circ \sum_{i=1}^{n} \eta^*_{\rho_i} \circ (\eta^*_\tau - \delta_{A^*}) = \mu' \circ \sum_i (\eta^*_{\rho_i \tau} - \eta^*_{\rho_i})$$
$$= \mu' \circ \sum_i (\eta^*_{\tau_i} \eta^*_{\rho_{s(i)}} - \eta^*_{\rho_i})$$

with some $T^*_{\tau_i}$ in H and so, by (6),

$$= \mu' \circ \sum_i (\eta^*_{\rho_{s(i)}} - \eta^*_{\rho_i}) = 0\,,$$

i.e. we have

$$(\mu' \circ \sum_{i=1}^{n} \eta^*_{\rho_i})(\eta^*_\tau - \delta_{A^*})(A^*) = 0\,.$$

On the other hand, by (7), we have

$$(\mu' \circ \sum_{i=1}^{n} \eta^*_{\rho_i})(a^*_\tau) = \mu'(\sum_i (a^*_{\rho_i \tau} - a^*_{\rho_i})) = \mu'(\sum_i (a^*_{\tau_i \rho_{s(i)}} - a^*_{\rho_i})$$

with some $T^*_{\tau_i}$ in H. Then, also by (7) and (6), we have

$$\mu'(a^*_{\tau_i \rho_{s(i)}}) = \mu'(a^*_{\tau_i} + \eta^*_{\tau_i}(a^*_{\rho_{s(i)}})) = \mu'(a^*_{\rho_{s(i)}})$$

and so

$$(\mu' \circ \sum_{i=1}^{n} \eta^*_{\rho_i})(a^*_\tau) = \mu'(\sum_i (a^*_{\rho_{s(i)}} - a^*_{\rho_i})) = 0\,.$$

Therefore we have $(\mu' \circ \sum_{i=1}^{n} \eta^*_{\rho_i})(C^*_H) = 0$.

The endomorphism ρ satisfies the relation

(9) $$\mu \circ \rho = n\mu \qquad \text{on } A\,.$$

In fact, by (8), (4) and (6), we have

$$\mu \circ \rho \circ \mu' = \mu \circ \mu' \circ \sum_{i=1}^{n} \eta^*_{\rho_i} = \mu^* \circ \sum_i \eta^*_{\rho_i} = n\mu^* = n\mu \circ \mu'.$$

Then, as μ' is surjective, we have (9).

Now we prove that the abelian subvariety $\rho(A)$ of A is isogenous to A_0, an Albanese variety attached to V. We have, by (6), $n'\mu' = \mu' \circ \sum_H \eta^*_\tau$[3] and so, by (8),

$$n'\rho \circ \mu' = n'\mu' \circ \sum_{i=1}^{n} \eta^*_{\rho_i} = \mu' \circ \sum_H \eta^*_\tau \circ \sum_i \eta^*_{\rho_i} = \mu' \circ \sum_G \eta^*_\sigma\,.$$

Since the intersection $C^*_G \cap (\sum_G \eta^*_\sigma)(A^*)$ is a finite subgroup of A^* (cf. [**3**]) and the kernel C^*_H of μ' is contained in C^*_G, μ' induces a homomorphism of $(\sum_G \eta^*_\sigma)(A^*)$ onto $n'\rho(\mu'(A^*))$ with a finite kernel. As we have $\mu'(A^*) = A$, $\rho(A)$ is isogenous to $(\sum_G \eta^*_\sigma)(A^*)$, which is isogenous to A_0 (cf. Th. 2 of [**3**]).

3) The signs $\sum_H$ and $\sum_G$ mean the sums ranged over all the elements of H and G respectively.

Next we assume that the degree n is prime to the characteristic of the universal domain. Let a be any point of the intersection $\rho(A)\cap(\rho-n\delta_A)(A)$. Then we have $a=\rho(a')=(\rho-n\delta_A)(a'')$ with some a', a'' in A. Operating μ on this relation, we have, by (9), $\mu(na')=n\mu(a')=0$, i. e. na' belongs to the kernel of μ. So $na=\rho(na')$ belongs to $(\rho\circ\mu')(C_G^*)$, which is also written as $(\mu'\circ\sum_{i=1}^{n}\eta_{\rho_i}^*)$ (C_G^*) by (8). However, by the similar argument as in the proof of Lemma, we can show that $(\mu'\circ\sum_i\eta_{\rho_i}^*)(C_G^*)=0$, because we have not used there the fact that T_τ^* is in H. Hence we have $na=0$, i. e. $\rho(A)\cap(\rho-n\delta_A)(A)$ is a finite subgroup of A. Since, clearly, $\rho(A)$ and $(\rho-n\delta_A)(A)$ generate A, A is isogenous to the direct product $\rho(A)\times(\rho-n\delta_A)(A)$. Let x be a generic point of A over k. Then the mapping $\varphi(x)=\rho(x)\times(\rho-n\delta_A)(x)$ defines an isogeny of A onto $\rho(A)\times(\rho-n\delta_A)(A)$ and, conversely, the mapping $\varphi'(\rho(x)\times(\rho-n\delta_A)(x))=\rho(x)-(\rho-n\delta_A)(x)=nx$ defines also an isogeny of $\rho(A)\times(\rho-n\delta_A)(A)$ onto A. Since we have $\varphi'\circ\varphi=n\delta_A$ and n is assumed to be prime to the characteristic of the universal domain, φ and φ' are separable. Let $\tilde{\mu}$ be the canonical separable homomorphism of $\rho(A)\times(\rho-n\delta_A)(A)$ onto $\rho(A)$ with the kernel $0\times(\rho-n\delta_A)(A)$ (cf. Rosenlicht [5]). Then, as we have $(\rho\circ\mu')(C_G^*)=0$ as stated above, $\varphi(\mu'(C_G^*))$ is contained in $0\times(\rho-n\delta_A)(A)$ and so we have $(\tilde{\mu}\circ\varphi)(\mu'(C_G^*))=0$. Since μ is canonical, there exists an isogeny ψ of A_0 onto $\rho(A)$ such that $\tilde{\mu}\circ\varphi=\psi\circ\mu$. Since $\tilde{\mu}$ and φ are separable and μ is surjective, ψ is also separable. Conversely we have, by (9), $(\mu\circ\varphi')(0\times(\rho-n\delta_A)(A))=\mu((\rho-n\delta_A)(A))=0$. Hence, by the similar arguments, we can prove the existence of a separable isogeny of $\rho(A)$ onto A_0.

Then, together with the result in **1**, we have the following

THEOREM 1. *Let the notations be as explained above. Then the quotient abelian variety $A_0=A/\mu'(C_G^*)$ is an Albanese variety attached to V and a canonical mapping α_0 of V into A_0 may be taken to satisfy the relation: $\alpha_0\circ f=\mu\circ\alpha$, where μ is the canonical homomorphism of A onto A_0. On the other hand, $\rho(A)$ is isogenous to A_0, where ρ is the endomorphism of A defined in* (8). *Moreover, if the degree n is prime to the characteristic of the universal domain, then there exist separable isogenies between $\rho(A)$ and A_0.*

3. The inequality (*).

In this section, we suppose that V is embedded in some projective space. Let C be a generic hyperplane section curve on V over k; let $W=f^{-1}(C)$ and $W^*=f^{*-1}(C)$ be the inverse images of C on U and U^* respectively, which are irreducible curves. The curves C, W and W^* are defined over a regular extension K of k; let $\bar{K}$ be the algebraic closure of K. Let W' and $W^{*\prime}$ be complete, non-singular curves, which are birationally equivalent to W and W^* over $\bar{K}$ respectively. Then, in a natural way, we can define the Galois coverings

$$g^*: W^{*\prime} \to C \quad \text{and} \quad g': W^{*\prime} \to W',$$

defined over $\bar{K}$ and with the Galois groups isomorphic to G and H respectively (cf. [**3**]).

Let J^* be a Jacobian variety attached to $W^{*\prime}$. Then, by Lang [**4**] as seen in [**3**], $W^{*\prime}$ generates A^* and so there exists a homomorphism λ^* of J^* onto A^*. For each element T^*_σ in G, there correspond the automorphisms ξ^*_σ and η^*_σ of J^* and A^*, respectively, by the relations of type (2). These automorphisms satisfy the following relations:

$$\lambda^* \circ \xi^*_\sigma = \eta^*_\sigma \circ \lambda^* \quad \text{on } J^*. \tag{10}$$

$$\begin{aligned} &(\sum_H \xi^*_\tau)(J^*) \sim J, \quad (\sum_G \xi^*_\sigma)(J^*) \sim J_0, \\ &(\sum_H \eta^*_\tau)(A^*) \sim A, \quad (\sum_G \eta^*_\sigma)(A^*) \sim A_0, \text{[4]} \end{aligned} \tag{11}$$

where J and J_0 are Jacobian varieties attached to W' and C respectively (cf. [**3**]). Then, by (10), λ^* induces, in a natural way, the homomorphisms λ of $(\sum_H \xi^*_\tau)(J^*)$ onto $(\sum_H \eta^*_\tau)(A^*)$ and λ_0 of $(\sum_G \xi^*_\sigma)(J^*)$ onto $(\sum_G \eta^*_\sigma)(A^*)$. Since we have

$$\sum_G \xi^*_\sigma = (\sum_H \xi^*_\tau)(\sum_{i=1}^n \xi^*_{\rho_i}),$$

$(\sum_G \xi^*_\sigma)(J^*)$ is contained in $(\sum_H \xi^*_\tau)(J^*)$ and so the kernel of λ_0 is contained in that of λ. On the other hand, as λ and λ_0 are surjective, the dimensions of the kernels of λ and λ_0 are equal to $\dim J - \dim A$ and $\dim J_0 - \dim A_0$, by (11), respectively. Hence we have the following

THEOREM 2. *Let the notations be as explained above. Then we have the inequality*

$$\dim J - \dim J_0 \geqq \dim A - \dim A_0.$$

Let Z be a k-closed algebraic subset of V, containing all the points on V which ramify in the covering $f: U \to V$. Then, since W' is unramified over every point of $C - C \cap Z$ (cf. [**3**]), we have easily the following corollary by Theorem 2 and the formula of Hurwitz.

COROLLARY. *If the dimension of Z is less than* $\dim V - 1$, *then we have the inequality*

$$\dim A \leqq \dim A_0 + (n-1)(\dim J_0 - 1).$$

Here we note that the dimension of J_0 does not depend on the choice of

4) The sign $\sim$ means the isogenous relation between abelian varieties.

the generic curve C but depends only on V.

REMARK. By Theorem 2, there are the following two possibilities as for the relations between the numbers $\dim J - \dim A$ and $\dim J_0 - \dim A_0$:

(a) $$\dim J - \dim A = \dim J_0 - \dim A_0 .$$

(b) $$\dim J - \dim A > \dim J_0 - \dim A_0 .$$

We can give examples of the above two cases respectively.

The example of (a): Consider the case where U and V are algebraic curves. Or, consider an unramified covering of a normal algebraic surface of degree 3 in the projective space of dimension 3 (cf. §4 of [**3**]).

The example of (b): Let X be a normal variety with the irregularity larger than 1. Let s and t be rational integers larger than 1 and let $U = X(s)(t)$ and $V = X(st)$ be the t-fold symmetric product of the s-fold symmetric product of X and the st-fold symmetric product of X respectively. Then, taking their normalizations, we have a covering $f: U \to V$ of degree larger than 1. Using the above notations, we have clearly $\dim A = \dim A_0$. On the other hand, the genus of C is not less than the irregularity of V and so it is larger than 1. Then, by the formula of Hurwitz, any covering curve of C has the genus larger than that of C, i.e. we have $\dim J > \dim J_0$.

4. The inequality (**).

In this section, we suppose that U and V are complete and non-singular. (But the non-singularity of U^* is not necessary.) Let θ be an element of $\mathfrak{D}_0(A)$ such that there exists an element ω_0 of $\mathfrak{D}_0(V)$ and $\delta\alpha(\theta) = \delta f(\omega_0)$. Then we have

(12) $$\delta\rho(\theta) = n\theta .$$

In fact, by (3), (1), (2) and the definition of f^*, we have

$$\begin{aligned}\delta\alpha^* \circ \delta\mu'(\theta) &= \delta f' \circ \delta\alpha(\theta) = \delta f' \circ \delta f(\omega_0) = \delta f^*(\omega_0) = \delta T^*_{\rho_i} \circ \delta f^*(\omega_0) \\ &= \delta T^*_{\rho_i} \circ \delta\alpha^* \circ \delta\mu'(\theta) = \delta\alpha^* \circ \delta\eta^*_{\rho_i} \circ \delta\mu'(\theta) .\end{aligned}$$

Here we used the fact that, as $\delta\mu'(\theta)$ is of the first kind on A^*, it is invariant by the translation of $a^*_{\rho_i}$. Since $\delta\alpha^*$ is injective by Igusa [**1**], we have

$$\delta\mu'(\theta) = \delta\eta^*_{\rho_i} \circ \delta\mu'(\theta)$$

and so, by (8),

$$n\delta\mu'(\theta) = \delta\Big(\sum_{i=1}^{n} \eta^*_{\rho_i}\Big) \circ \delta\mu'(\theta) = \delta\mu' \circ \delta\rho(\theta) .$$

Since μ' is separable and surjective, $\delta\mu'$ is injective (cf. Igusa [**1**]) and so we have (12).

Now we assume that the degree n is prime to the characteristic of the universal domain. Then we have

$$\delta\mu(\mathfrak{D}_0(A_0)) = \delta\rho(\mathfrak{D}_0(A)) . \tag{13}$$

In fact, by (9) and the assumption on n, we have

$$\delta\mu(\mathfrak{D}_0(A_0)) = \delta\rho \circ \delta\mu(\mathfrak{D}_0(A_0)) \subset \delta\rho(\mathfrak{D}_0(A)) .$$

Moreover, as μ is separable and surjective, $\delta\mu$ is injective and so we have

$$\dim \delta\mu(\mathfrak{D}_0(A_0)) = \dim \mathfrak{D}_0(A_0) = \dim A_0 ,$$

which is equal to $\dim \rho(A)$ by Theorem 1. Now we consider the endomorphism ρ of A as a homomorphism ρ' of A onto another abelian varieiy $\rho(A)$. Denoting by ι the injection of $\rho(A)$ into A, we have $\iota \circ \rho' = \rho$ and so $\delta\rho = \delta\rho' \circ \delta\iota$. Then we have

$$\delta\rho(\mathfrak{D}_0(A)) = \delta\rho' \circ \delta\iota(\mathfrak{D}_0(A)) \subset \delta\rho'(\mathfrak{D}_0(\rho(A)))$$

and so

$$\dim \delta\rho(\mathfrak{D}_0(A)) \leqq \dim \delta\rho'(\mathfrak{D}_0(\rho(A))) \leqq \dim \mathfrak{D}_0(\rho(A)) = \dim \rho(A) = \dim A_0 .$$

Therefore, the linear space $\delta\rho(\mathfrak{D}_0(A))$ of dimension $\leqq \dim A_0$ contains the subspace $\delta\mu(\mathfrak{D}_0(A_0))$ of dimension $= \dim A_0$ and so we must have (13).

THEOREM 3. *We assume that the degree n is prime to the characteristic of the universal domain. If, for an element ω_0 in $\mathfrak{D}_0(V)$, $\delta f(\omega_0)$ belongs to $\delta\alpha(\mathfrak{D}_0(A))$, then there exists an element θ_0 of $\mathfrak{D}_0(A_0)$ such that we have*

$$\omega_0 = \delta\alpha_0(\theta_0) .$$

PROOF. Let θ be an element of $\mathfrak{D}_0(A)$ such that $\delta f(\omega_0) = \delta\alpha(\theta)$. From the assumption on the degree n, $\frac{1}{n} \cdot \theta$ belongs to $\mathfrak{D}_0(A)$, and so, by (12), we have

$$\delta\rho\left(\frac{1}{n} \cdot \theta\right) = \frac{1}{n} \cdot \delta\rho(\theta) = \frac{1}{n} \cdot n\theta = \theta ,$$

i.e. θ is contained in $\delta\rho(\mathfrak{D}_0(A))$. Then, by (13), there exists an element θ_0 of $\mathfrak{D}_0(A_0)$ such that we have $\delta\mu(\theta_0) = \theta$. Hence, by (5), we have

$$\delta f(\omega_0) = \delta\alpha(\theta) = \delta\alpha \circ \delta\mu(\theta_0) = \delta f \circ \delta\alpha_0(\theta_0) .$$

Since f is separable and surjective, δf is injective and so the statement of our theorem is proved.

Theorem 3 implies that, if an element ω_0 of $\mathfrak{D}_0(V)$ does not belong to the subspace $\delta\alpha_0(\mathfrak{D}_0(A_0))$, then also $\delta f(\omega_0)$ does not belong to the subspace $\delta\alpha(\mathfrak{D}_0(A))$ of $\mathfrak{D}_0(U)$. Since δf, $\delta\alpha$ and $\delta\alpha_0$ are injective, we have the following

COROLLARY. *Under the same assumption on n as in Theorem 3, there holds the inequality*

$$\dim \mathfrak{D}_0(U) - \dim \mathfrak{D}_0(A) \geqq \dim \mathfrak{D}_0(V) - \dim \mathfrak{D}_0(A_0) .$$

Especially, if $\dim \mathfrak{D}_0(U) = \dim \mathfrak{D}_0(A)$, *then we have the equality* $\dim \mathfrak{D}_0(V) = \dim \mathfrak{D}_0(A_0)$.

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