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On L-series of normal varieties

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Introduction.

Let U and V be normal varieties defined over a finite field k with q elements, and assume that U is a Galois covering of V with the Galois group $\mathfrak{G}$. Under these circumstances several authors defined the L-series associated with the characters of $\mathfrak{G}$. In [**7**], Lang introduced an L-series following the original idea of Artin [**2**] and proved the density theorem. But in his definition the singular points and the branch points of V are all neglected. For his purposes it is sufficient, but for other purposes it may be inconvenient. We shall give, borrowing the ideas in [**3**], [**4**], a new definition of L-series without neglecting the singular and branch points, which is a natural generalization of Lang's one and Weil's one given in the case of curves in [**9**]. Ishida also treated L-series in a different way in [**6**]. It will be seen that our definition and the one given in [**6**] are the same one.

On the other hand Sampson and Washnitzer [**8**] obtained a functional equation of the zeta-function of the non-singular variety U under some assumption. Using the same assumption as that used in [**8**], we shall deduce a functional equation of our L-series for the Galois covering V/U when U is a non-singular variety. When U is a curve, it is obtained by Weil in [**9**]. When U is an abelian variety with the abelian Galois group $\mathfrak{G}$, the same result is obtained by Ishida in [**5**]. Thus our L-series will seem to be a satisfactory one.

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§ 1. Galois coverings defined over a finite field k.

Let $\pi: U \rightarrow V$ be a Galois covering of degree n, defined over a finite field k with q elements.[1] In the following we shall assume that U and V are normal, projective varieties of dimension r. Let $\alpha, \sigma, \tau, \cdots$, be the automor-

1) For the definition of a Galois covering of an algebraic variety, see Lang [8].

phisms of the function field E of U/k over the function field K of V/k. T_α will denote the induced correspondence of U into itself by α, which is biregular and birational. Then the Galois group $\mathfrak{G}$ is identified with the transformation group of U consisting of $T_\alpha, T_\sigma, T_\tau, \cdots$.

Let Q be a point of V algebraic over k and let $\mathfrak{O}$ be the quotient ring of Q in V/k. We shall call such a local ring in K a "*locality*" of dimension zero following Chevalley, and we shall say that Q belongs to the locality $\mathfrak{O}$ or that Q is a point of the locality $\mathfrak{O}$. A point Q' of V belongs to the locality $\mathfrak{O}$ if and only if Q' is a conjugate point of Q with respect to k. In what follows, we shall treat exclusively the localities of dimension zero. Therefore, for simplicity, we shall always understand by a locality, a locatity of dimension zero. Let $\mathfrak{p}$ be the maximal ideal of the locality $\mathfrak{O}$. Then we shall denote by $\deg \mathfrak{p}$ the number of the points which belong to $\mathfrak{O}$. Then $\deg \mathfrak{p}$ is equal to the degree $[\mathfrak{O}/\mathfrak{p} : k]$.

Let $\mathfrak{O}^*$ be the integral closure of $\mathfrak{O}$ in E and let $\mathfrak{P}_1, \cdots, \mathfrak{P}_g$ the maximal ideals of $\mathfrak{O}^*$. Then each local ring $\mathfrak{O}^*_{\mathfrak{P}_i} (i=1, \cdots, g)$ is a locality in E and at least a point of $\pi^{-1}(Q)$[2] belongs to $\mathfrak{O}^*_{\mathfrak{P}_i}$. Conversely each point of $\pi^{-1}(Q)$ belongs to one of the localities $\mathfrak{O}^*_{\mathfrak{P}_1}, \cdots, \mathfrak{O}^*_{\mathfrak{P}_g}$.

Let $\mathfrak{Z}_{\mathfrak{P}_i}$ be the splitting group of $\mathfrak{P}_i/\mathfrak{p}$ and let $\mathfrak{T}_{\mathfrak{P}_i}$ be the inertia group of $\mathfrak{P}_i/\mathfrak{p}$.[3] Then it can be seen easily that $\mathfrak{Z}_{\mathfrak{P}_i}$ consists of the elements T_α of $\mathfrak{G}$ such that T_α transforms each point of $\mathfrak{O}^*_{\mathfrak{P}_i}$ into a point of $\mathfrak{O}^*_{\mathfrak{P}_i}$ and that $\mathfrak{T}_{\mathfrak{P}_i}$ consists of the elements T_α of $\mathfrak{G}$ such that T_α fixed each point of $\mathfrak{O}^*_{\mathfrak{P}_i}$.

Since the order of $\mathfrak{T}_{\mathfrak{P}_i}$ and the index $[\mathfrak{Z}_{\mathfrak{P}_i} : \mathfrak{T}_{\mathfrak{P}_i}]$ are independent of i and depend only on $\mathfrak{p}$, we shall denote these values by $e_\mathfrak{p}$ and $f_\mathfrak{p}$ respectively. The number $e_\mathfrak{p}$ will be called *the ramification index* of $\mathfrak{P}_i/\mathfrak{p}$ and the number $f_\mathfrak{p}$ will be called *the relative degree* of $\mathfrak{P}_i/\mathfrak{p}$. Then we have the equality $n = e_\mathfrak{p} f_\mathfrak{p} g$. Since the residue group $\mathfrak{Z}_{\mathfrak{P}_i}/\mathfrak{T}_{\mathfrak{P}_i}$ is isomorphic to the Galois group of $\mathfrak{O}^*/\mathfrak{P}_i$ over $\mathfrak{O}/\mathfrak{p}$, there exists an element T_α of $\mathfrak{Z}_{\mathfrak{P}_i}$ such that $T_\alpha(P) = P^{(q^{\deg \mathfrak{p}})}$[4] for any point P of $\mathfrak{O}^*_{\mathfrak{P}_i}$. Therefore we shall understand by a *Frobenius correspondence* for $\mathfrak{P}_i/\mathfrak{p}$ such an element that has the property as above. If T_{κ_i} transforms a point of $\mathfrak{O}^*_{\mathfrak{P}_1}$ to a point of $\mathfrak{O}^*_{\mathfrak{P}_i}$ and if T_{σ_1} is a Frobenius correspondence for $\mathfrak{P}_1/\mathfrak{p}$, then the Frobenius correspondences for $\mathfrak{P}_i/\mathfrak{p}$ are the elements of the set $\mathfrak{T}_{\mathfrak{P}_i} T_{\kappa_i} T_{\sigma_1} T_{\kappa_i}^{-1} = T_{\kappa_i} \mathfrak{T}_{\mathfrak{P}_1} T_{\sigma_1} T_{\kappa_i}^{-1}$.

Let $\mathfrak{H}$ be a subgroup of $\mathfrak{G}$ and let F be the fixed subfield of E for $\mathfrak{H}$.

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2) We shall understand always $\pi(P), \pi^{-1}(Q)$, etc., in the set theoretic sense.

3) For these definitions, see Chap. I, 7 in Abhyankar [**1**].

4) If P is a point of a variety U, then $P^{(q^\mu)}$ denotes the point which is the transform of P by ω^μ, where ω is the automorphism of the universal domain Ω such that $a^\omega = a^q$ for any a in Ω.

Let W be the normalization of V in F and π'' be the natural rational mapping $W\to V$, which is everywhere regular on W. We have also a rational mapping $\pi'\colon U\to W$ such that $\pi=\pi''\pi'$. U is a Galois covering of W with $\mathfrak{H}$ as the Galois group.

Let $\bar{\mathfrak{O}}$ be the integral closure of $\mathfrak{O}$ in F, and $\mathfrak{q}_1,\cdots,\mathfrak{q}_{g'}$ be the maximal ideals in $\bar{\mathfrak{O}}$. Renumbering the $\mathfrak{P}_i$ in $\mathfrak{O}^*$, we denote by $\mathfrak{P}_{j1},\cdots,\mathfrak{P}_{jg_j}$ the maximal ideals in $\mathfrak{O}^*$ which lie over $\mathfrak{q}_j$. Let n' be the order of $\mathfrak{H}$, let $e'_{\mathfrak{q}_j}$ be the ramification index of $\mathfrak{P}_{ji}/\mathfrak{q}_j$ and let $f'_{\mathfrak{q}_j}$ be the relative degree of $\mathfrak{P}_{ji}/\mathfrak{q}_j$. Then we have the equalities $n'=e'_{\mathfrak{q}_j}f'_{\mathfrak{q}_j}g_j$ $(j=1,\cdots,g')$. If we put $[\bar{\mathfrak{O}}/\mathfrak{q}_j:\mathfrak{O}/\mathfrak{P}]=f''_{\mathfrak{q}_j}$, then we have $f_{\mathfrak{p}}=f'_{\mathfrak{q}_j}f''_{\mathfrak{q}_j}$.

Let $\bar{\mathfrak{Z}}_{\mathfrak{P}_{ji}}$ be the splitting group of $\mathfrak{P}_{ji}/\mathfrak{q}_j$ and let $\bar{\mathfrak{T}}_{\mathfrak{P}_{ji}}$ the inertia group of $\mathfrak{P}_{ji}/\mathfrak{q}_j$. Then we have $\bar{\mathfrak{Z}}_{\mathfrak{P}_{ji}}=\mathfrak{Z}_{\mathfrak{P}_{ji}}\cap\mathfrak{H}$ and $\bar{\mathfrak{T}}_{\mathfrak{P}_{ji}}=\mathfrak{T}_{\mathfrak{P}_{ji}}\cap\mathfrak{H}$. In particular, if we put $e''_{\mathfrak{q}_j}=e_{\mathfrak{p}}/e'_{\mathfrak{q}_j}$, then we have the equalities

$$(1)\qquad e''_{\mathfrak{q}_j}=[\mathfrak{T}_{\mathfrak{P}_{ji}}:\bar{\mathfrak{T}}_{\mathfrak{P}_{ji}}],\qquad e''_{\mathfrak{q}_1}f''_{\mathfrak{q}_1}+\cdots+e''_{\mathfrak{q}_{g'}}f''_{\mathfrak{q}_{g'}}=n/n'=[F:K]\,.$$

§ 2. A fundamental lemma.

The notations being as above, let us divide the group $\mathfrak{G}$ into the sum of the left cosets of a subgroup $\mathfrak{H}$ as follows:

$$\mathfrak{G}=\mathfrak{H}T_{\tau_1}+\cdots+\mathfrak{H}T_{\tau_{n''}}\,.$$

Let ψ be a character of $\mathfrak{H}$. We shall understand that the value $\psi(T_\alpha)$ is zero, when T_α does not belong to $\mathfrak{H}$. Then it is well known that the function

$$(2)\qquad \chi_\psi(T_\alpha)=\sum_{j=1}^{n''}\psi(T_{\tau_j}T_\alpha T_{\tau_j}^{-1})\qquad \text{for } T_\alpha\in\mathfrak{G}$$

is a character of $\mathfrak{G}$, and is called *the induced character* by ψ of $\mathfrak{G}$.

Let $\chi_i(i=1,2,\cdots,h)$ be the simple characters of the group $\mathfrak{G}$, where χ_1 is the principal character of $\mathfrak{G}$. Let $\mathfrak{H}^{(j)}(j=1,\cdots,s)$ be all the cyclic subgroups of the group $\mathfrak{G}$ and let $\psi_{ji}(i=1,\cdots,h_{(j)})$ be the simple characters of $\mathfrak{H}^{(j)}$, where ψ_{j1} is the principal character of $\mathfrak{H}^{(j)}$.

Then, by Artin [**2**],[5] we have the following

LEMMA 1. *Each non-principal character χ_i is expressed as a linear combination of $\chi_{\psi_{ji}}(j=1,\cdots,s;\ i=2,\cdots,h_{(j)})$ with coefficients consisting of rational numbers, where $\chi_{\psi_{ji}}$ are the induced characters by ψ_{ji} of $\mathfrak{G}$.*

The next lemma is analogous to the result[6] obtained by Artin in [**3**], in the case of algebraic number fields, and the proof will be given in the same

5) See pp. 102–103 in Artin [**2**].
6) See pp. 4–5 in Artin [**3**].

line as that of Artin's. But the lemma is fundamental in the following discussions, hence we shall write down the complete proof.

LEMMA 2. *Retaining the notations as in* §1, *let* T_σ *be a Frobenius correspondence for* $\mathfrak{P}_{11}/\mathfrak{p}$, *and let* T_{ρ_i} *be a Frobenius correspondence for* $\mathfrak{P}_{i1}/\mathfrak{q}_i$[7] $(i=1, \cdots, g')$. *Then we have*

$$\sum_{T_\alpha \in \mathfrak{T}_{\mathfrak{P}_{11}}} \chi_\psi(T_\sigma^\mu T_\alpha) = \sum_{f''_{\mathfrak{q}_i} \mid \mu} e''_{\mathfrak{q}_i} f''_{\mathfrak{q}_i} \sum_{T_\alpha \in \mathfrak{T}_{\mathfrak{P}_{i1}}} \psi(T_{\rho_i}^{\mu/f''_{\rho_i}} T_\alpha) \tag{3}$$

for any positive integer μ, *where* ψ *is a character of the subgroup* $\mathfrak{H}$ *and* χ_ψ *is the induced character by* ψ *of* $\mathfrak{G}$.

PROOF. For simplicity, in the proof we put $e_{\mathfrak{q}_i} = e''_i$ and $f''_{\mathfrak{q}_i} = f''_i$. Let T_{σ_i} be a Frobenius corre spondence for $\mathfrak{P}_{i1}/\mathfrak{p}$, and let P be a point of $\mathfrak{O}^*_{\mathfrak{P}_{i1}}$. Then we have $T_{\sigma_i}^{f''_i}(P) = P^{(q f''_i \deg \mathfrak{p})} = P^{(q \deg \mathfrak{q}_i)7)}$ and $T_{\rho_i}^{-1}(P^{(q \deg \mathfrak{q})}) = P$ and we have $T_{\rho_i}^{-1} T_{\sigma_i}^{f''_i}(P) = P$. Hence $T_{\rho_i}^{-1} T_{\sigma_i}^{f''_i}$ is in $\mathfrak{T}_{\mathfrak{P}_{i1}}$ and we have $T_{\sigma_i}^{f''_i} \mathfrak{T}_{\mathfrak{P}_{1i}} = T_{\rho_i} \mathfrak{T}_{\mathfrak{P}_{i1}}$. Since $\mathfrak{T}_{\mathfrak{P}_{i1}}$ is a normal subgroup of $\mathfrak{Z}_{\mathfrak{P}_{i1}}$ which contains T_{σ_i} and T_{ρ_i}, we have $T_{\sigma\, i}^{\nu f''_i} \mathfrak{T}_{\mathfrak{P}_{i1}} = T_{\rho_1}^{\nu} \mathfrak{T}_{\mathfrak{P}_{i1}}$. From this fact we can see that the coset $T_{\sigma_i}^{\nu f''_i} \mathfrak{T}_{\mathfrak{P}_{i1}}$ contains an element $T_{\rho_i}^{\nu}$ of $\mathfrak{H}$.

Conversely, if the coset $T_{\sigma_i}^{\nu} \mathfrak{T}_{\mathfrak{P}_{i1}}$ contains an element $T_\gamma = T_{\sigma i}^{\lambda} T_{\delta i}$ of $\mathfrak{H}$, then we have $T_\gamma(P) = P^{(q^\lambda \deg \mathfrak{p})}$. Since T_γ is in $\mathfrak{H}$, it follows that $\pi' T_\gamma(P) = \pi'(P)$. Therefore we have $\pi'(P) = \pi'(P)^{(q^\lambda \deg \mathfrak{p})}$. As $\pi'(P)$ belongs to $\bar{\mathfrak{O}}_{\mathfrak{q}_i}$, it can be seen easily that $\lambda \deg \mathfrak{p}$ is a multiple of $\deg \mathfrak{q}_i$ and that λ is a multiple of f''_i. Then $T^\lambda_{\sigma i} \mathfrak{T}_{\mathfrak{P}_{i1}} = T_{\rho_i}^{\lambda/f''_i} \mathfrak{T}_{\mathfrak{P}_{i1}}$. Therefore we have the following assertion:

(*) The intersection of $T^\lambda_{\sigma_i} \mathfrak{T}_{\mathfrak{P}_{i1}}$ with $\mathfrak{H}$ is empty if λ is not a multiple of f''_i, and it consists of the elements of $T_{\rho i}^{\lambda/f''_i} \bar{\mathfrak{T}}_{\mathfrak{P}_{i1}}$ if λ is a multiple of f''_i.

Let $T_{\kappa_{ij}}$ be an element of $\mathfrak{G}$ such that it transforms a point of $\mathfrak{O}_{\mathfrak{P}_{11}}$ to a point of $\mathfrak{O}_{\mathfrak{P}_{ij}}$.

Now we consider two cosets of the forms $\mathfrak{H} T_{\zeta_i} T_{\kappa_{i1}}$ and $\mathfrak{H} T_{\zeta'_j} T_{\kappa_{j1}}$, where T_{ζ_i} is in $\mathfrak{Z}_{\mathfrak{P}_{i1}}$ and $T_{\zeta'_j}$ is in $\mathfrak{Z}_{\mathfrak{P}_{j1}}$. If they are same, $T_\gamma = T_{\zeta'_j} T_{\kappa_{j1}} T_{\kappa_{i1}}^{-1} T_{\zeta_i}^{-1}$ must be in $\mathfrak{H}$. Then it can be seen that $i=j$ and $T_\gamma = T_{\zeta'_j} T_{\zeta_i}^{-1}$. Hence T_γ must be in $\bar{\mathfrak{Z}}_{\mathfrak{P}_{i1}}$. Since the index $[\mathfrak{Z}_{\mathfrak{P}_{i1}} : \bar{\mathfrak{Z}}_{\mathfrak{P}_{i1}}]$ is equal to $e''_i f''_i$, $\mathfrak{Z}_{\mathfrak{P}_{i1}}$ is divided into the sum of $e''_i f''_i$ cosets of $\mathfrak{Z}_{\mathfrak{P}_{i1}}$ as follows;

$$\mathfrak{Z}_{\mathfrak{P}_{i1}} = \bar{\mathfrak{Z}}_{\mathfrak{P}_{i1}} T_{\zeta_{i1}} + \bar{\mathfrak{Z}}_{\mathfrak{P}_{i1}} T_{\zeta_{i2}} + \cdots\cdots .$$

Then the cosets $\mathfrak{H} T_{\zeta_{ij}} T_{\kappa_{i1}} (i=1, 2, \cdots, g'; j=1, 2, \cdots, e''_i f''_i)$ are different each

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7) For convenience we shall understand that a Frobenius correspondence for $\mathfrak{P}_{i1}/\mathfrak{q}_i$ and $\deg \mathfrak{q}_i$ mean a Frobenius correspondence for $\mathfrak{P}_{i1}/\mathfrak{q}_i \bar{\mathfrak{O}}_{\mathfrak{q}_i}$ and $\deg \mathfrak{q}_i \bar{\mathfrak{O}}_{\mathfrak{q}_i}$.

other by the above observation. The number of those cosets is $\sum_{i=1}^{g'} e''_i f''_i$ and hence by (1) those cosets are all the cosets of $\mathfrak{H}$ in $\mathfrak{G}$. Therefore we have from (3)

$$\chi_\psi(T_\alpha) = \sum_{i=1}^{g'} \sum_{j=1}^{e''_i f''_i} \psi(T_{\zeta_{ij}} T_{\kappa_{i1}} T_\alpha T_{\kappa_{i1}}^{-1} T_{\zeta_{ij}}^{-1}) . \tag{4}$$

On the other hand, we can easily see that

$$\sum_{T_\alpha \in \mathfrak{T}_{\mathfrak{P}_{11}}} T_{\zeta_{ij}} T_{\kappa_{i1}} T_\sigma^\mu T_\alpha T_{\kappa_{i1}}^{-1} T_{\zeta_{ij}}^{-1} \tag{5}$$

$$= \sum_{T_\alpha \in \mathfrak{T}_{\mathfrak{P}_{i1}}} T_{\zeta_{ij}} T_{\sigma_1}^\mu T_\alpha T_{\zeta_{ij}}^{-1} = \sum_{T_\alpha \in \mathfrak{T}_{\mathfrak{P}_{i1}}} T_{\sigma_1}^\mu T_\alpha .$$

By (4), (5) and (*), it follows that

$$\sum_{T_\alpha \in \mathfrak{T}_{\mathfrak{P}_{11}}} \chi_\psi(T_\sigma^\mu T_\alpha) = \sum_{i=1}^{g'} \sum_{j=1}^{e''_i f''_i} \sum_{T_\alpha \in \mathfrak{T}_{\mathfrak{P}_{i1}}} \psi(T_{\sigma_i}^\mu T_\alpha)$$

$$= \sum_{i=1}^{g'} \sum_{T_\alpha \in \mathfrak{T}_{\mathfrak{P}_{i1}}} e''_i f''_i \psi(T_{\sigma_i}^\mu T_\alpha) = \sum_{f''_i \mid \mu} e''_i f''_i \sum_{T_\alpha \in \mathfrak{T}_{\mathfrak{P}_{i1}}} \psi(T_{\rho_i}^{\mu/f''_i} T_\alpha) .$$

This completes the proof.

§ 3. Definition of *L*-series.

The notations being as above, let $T_{\sigma_{ij}}$ be a Frobenius correspondence for $\mathfrak{P}_{ij}/\mathfrak{p}(i=1, 2, \cdots, g'; j=1, 2, \cdots, g_i)$ and let χ be a (not necessary simple) character of $\mathfrak{G}$. Then it is easily seen that for any positive integer μ, the values

$$\frac{1}{e_\mathfrak{p}} \sum \chi(T_{\sigma_{ij}}^\mu T_\alpha) \qquad (i=1, 2, \cdots, g'; j=1, 2, \cdots, g_i)$$

are same, depend on $\mathfrak{p}$ only and will be denoted by $\chi(\mathfrak{p}^\mu)$.

Then *L-series* $L(u, \chi, U/V)$ for the Galois covering $\pi: U \to V$, associated with a character χ is defined as follows;

$$\log L(u, \chi, U/V) = \sum_{\mu=1}^{\infty} \sum_{\mathfrak{p}} \frac{\chi(\mathfrak{p}^\mu)}{\mu} u^{\mu \deg \mathfrak{p}} , \tag{6}$$

where the sum $\sum_{\mathfrak{p}}$ are taken over the maximal ideals of all the localities in K.

From this definition, we have immediately

PROPOSITION 1. *For any two characters χ, χ' of $\mathfrak{G}$, we have*

$$L(u, \chi+\chi', U/V) = L(u, \chi, U/V)\, L(u, \varphi', U/V) .$$

Now we consider the special case when $\mathfrak{G}$ is an abelian group and when

χ is a simple character of $\mathfrak{G}$. Then since the inertia group $\mathfrak{T}_{\mathfrak{P}_{ij}}$ and a Frobenius correspondence T_σ for $\mathfrak{P}_{ij}/\mathfrak{p}$ depend only on $\mathfrak{p}$, we put $\mathfrak{T}_{\mathfrak{P}_{ij}}=\mathfrak{T}_\mathfrak{p}$ and $T_\sigma=T_{\sigma_\mathfrak{p}}$. Moreover we put $\varepsilon_\mathfrak{p}=1$ if χ induces the principal character on $\mathfrak{T}_\mathfrak{p}$, and $\varepsilon_\mathfrak{p}=0$ otherwise. Then we have

$$\chi(\mathfrak{p}^\mu)=\frac{1}{e_\mathfrak{p}}\sum_{T_\alpha\in\mathfrak{T}_\mathfrak{p}}\chi(T^\mu_{\sigma_\mathfrak{p}}T_\alpha)=\varepsilon_\mathfrak{p}\,\chi(T_{\sigma_\mathfrak{p}})^\mu,$$

and therefore

$$\log L(u,\chi,U/V)=\sum_{\mathfrak{p},\mu}\frac{\varepsilon_\mathfrak{p}\,\chi(T_{\sigma_\mathfrak{p}})}{\mu}u^{\mu\deg\mathfrak{p}}$$
$$=-\sum_\mathfrak{p}\varepsilon_\mathfrak{p}\log\left(1-\chi(T_{\sigma_\mathfrak{p}})u^{\deg\mathfrak{p}}\right).$$

Therefore we have the following

Proposition 2. *If $\mathfrak{G}$ is an abelian group and if χ is a simple character of $\mathfrak{G}$, then we have*

$$L(u,\chi,U/V)=\prod_\mathfrak{p}(1-\chi(T_{\sigma_\mathfrak{p}})u^{\deg\mathfrak{p}})^{-\varepsilon_\mathfrak{p}},$$

where $T_{\sigma_\mathfrak{p}}$ and $\varepsilon_\mathfrak{p}$ are as above. In particular, each coefficient of u in the expression of $L(u,\chi,U/V)$ as a power series of u is an integer in an algebraic number field of finite degree.

Returning to general cases, we shall obtain some results which are also analogous to the results[8] of algebraic number fields.

Proposition 3. *If ψ is a character of $\mathfrak{H}$ and if χ_ψ is the induced character by ψ of $\mathfrak{G}$, then we have*

$$L(u,\chi_\psi,U/V)=L(u,\psi,U/W)$$

where W is the normalization of V in the fixed subfield of E for $\mathfrak{H}$.

Proof. The same convention as in the proof of Lemma 2 will be retained for $e''_{\mathfrak{q}_i}$ and $f''_{\mathfrak{q}_i}$.

Dividing the both sides of (3) by $e_\mathfrak{p}$, we have

$$\chi_\psi(\mathfrak{p}^\mu)=\sum_{f''_i|\mu}f''_i\psi(\mathfrak{q}_i^{\mu/f''_i})$$

and hence

$$\log L(u,\chi_\psi,U/V)=\sum_{\mathfrak{p},\mu}\sum_{f''_i|\mu}\frac{f''_i\psi(\mathfrak{q}_i^{\mu/f''_i})}{\mu}u^{\mu\deg\mathfrak{p}}$$
$$=\sum_{\mathfrak{p},\lambda}\sum_{\mathfrak{q}_i|\mathfrak{p}}\frac{\psi(\mathfrak{q}_i^\lambda)}{\lambda}u^{\lambda\deg\mathfrak{q}_i}=\sum_{\mathfrak{q},\lambda}\frac{\psi(\mathfrak{q}^\lambda)}{\lambda}u^{\lambda\deg\mathfrak{q}}$$
$$=\log L(u,\psi,U/W).$$

8) See the formula (9) in Artin [4].

This completes the proof.

THEOREM 1. *Let $\mathfrak{H}^{(i)}(i=1,2,\cdots,s)$ be all the cyclic subgroups of $\mathfrak{G}$, and let $\psi_{ij}(j=2,3,\cdots,h_{(i)})$ be all the non-principal simple characters of $\mathfrak{H}^{(i)}$. Moreover let W_i be the normalization of V in the fixed subfield of E for $\mathfrak{H}^{(i)}$. Then we have, for each non-principal simple character χ_t of $\mathfrak{G}$,*

$$L(u,\chi_t,U/V)=\prod_{i=1}^{s}\prod_{j=2}^{h_{(i)}}L(u,\psi_{ij},U/W_i)^{r_{ij}(t)},$$

where $r_{ij}(t)$ are rational numbers depending on χ_t.

PROOF. This is a direct consequence of Lemma 1 and Proposition 3.

PROPOSITION 4. *Let $\mathfrak{H}$ be a normal subgroup of $\mathfrak{G}$ and let W be the normalization of V in the field F corresponding to $\mathfrak{H}$. Then the natural mapping $\pi'':W\to V$ is considered as a Galois covering with $\mathfrak{G}/\mathfrak{H}$ as its Galois group, and a character χ of $\mathfrak{G}/\mathfrak{H}$ is also considered as a character of $\mathfrak{G}$. In this situation we have*

$$L(u,\chi,U/V)=L(u,\chi,W/V).$$

PROOF. The notation being same as in §1, we can easily see that the inertia group $\mathfrak{T}_1{}^*$ of $\mathfrak{q}_1/\mathfrak{p}$ is the group $\mathfrak{T}_{\mathfrak{P}_{11}}\mathfrak{H}/\mathfrak{H}$ and that if T_{σ_1} is a Frobenius correspondence for $\mathfrak{P}_{11}/\mathfrak{p}$, then the class $T_{\sigma_1}{}^*=T_{\sigma_1}\mathfrak{H}$ is a Frobenius correspondence for $\mathfrak{q}_1/\mathfrak{p}$. Let $e^*_{\mathfrak{p}}$ be the order of $\mathfrak{T}_{\mathfrak{P}_{11}}\mathfrak{H}/\mathfrak{H}$ and let g^* be the order of $\mathfrak{T}_{\mathfrak{P}_{11}}\mathfrak{H}$. Then we have

$$\frac{1}{e_{\mathfrak{p}}}\sum_{T_\alpha\in\mathfrak{T}_{\mathfrak{P}_{11}}}\chi(T^{\mu}{}_{\sigma_1}T_\alpha)=\frac{1}{g^*}\sum_{T_\alpha\in\mathfrak{T}_{\mathfrak{P}_{11}}\mathfrak{H}}\chi(T^{\mu}{}_{\sigma_1}T_\alpha)$$

$$=\frac{1}{e^*_{\mathfrak{p}}}\sum_{T^*_\alpha\in T^*_1}\chi(T^{*\mu}{}_{\sigma_1}T_\alpha{}^*).$$

This relation shows that our assertion is true.

§ 4. Expression of L-series as the logarithmic derivative.

Let k_μ be, as usual, the unique extension of k of degree μ. Let $\mathfrak{p}$ be the maximal ideal of a locality $\mathfrak{O}$ in K such that $\deg\mathfrak{p}$ is a divisor of μ. If a point Q belongs to $\mathfrak{O}$. then Q is a rational point with respect to k_μ. Now let us denote by $\mathfrak{p}_\mu(Q)$ the maximal ideal of the locality $\mathfrak{O}_\mu$ in V/k_μ with the unique point Q. Let P be a point of $\pi^{-1}(Q)$ and let $\mathfrak{P}$ be the maximal ideal of the locality $\mathfrak{O}^*$ in E to which P belongs. The geometric interpretation of the inertia group $\mathfrak{T}_{\mathfrak{P}}$ and a Frobenius correspondence $\mathfrak{T}_{\sigma_{\mathfrak{P}}}$ for $\mathfrak{P}/\mathfrak{p}$ yields the following

$$\chi(\mathfrak{p}^{\mu/\deg\mathfrak{p}})=\frac{1}{e_{\mathfrak{p}}}\sum_{T_\alpha\in\mathfrak{T}_{\mathfrak{P}}}\chi(T^{\mu/\deg\mathfrak{p}}T_\alpha)=\chi(\mathfrak{p}_\mu(Q)),\tag{7}$$

where in the right hand side the field of definition is considered to be k_μ.

Let us denote by V_μ the set of the rational points on V over k_μ. From (6), it follows that

$$\frac{d}{du}\log L(u,\chi,U/V)=\sum_{\mu=1}^{\infty}\sum_{\mathfrak{p}}\chi(\mathfrak{p}^\mu)\deg\mathfrak{p}\,u^{\mu\deg\mathfrak{p}-1}$$
$$=\sum_{\lambda=1}^{\infty}\{\sum_{\deg\mathfrak{p}|\lambda}\chi(\mathfrak{p}^{\lambda/\deg\mathfrak{p}})\deg\mathfrak{p}\}u^{\lambda-1}.$$

Therefore we have from (7)

$$\frac{d}{du}\log L(u,\chi,U/V)=\sum_{\mu=1}^{\infty}\{\sum_{Q\in V_\mu}\chi(p_\mu(Q))\}u^{\mu-1}. \tag{8}$$

Now we shall express L-series by the geometric languages. Let us denote by $U_\mu(T_\alpha)$ the set of the points P on U such that $T_\alpha(P)=P^{(q^\mu)}$, and let $N_\mu(T_\alpha)$ be the number of the points which belong to $U_\mu(T_\alpha)$. Then we put, for any character χ of $\mathfrak{G}$,

$$c_\mu(\chi)=\frac{1}{n}\sum_{T_\alpha\in\mathfrak{G}}\chi(T_\alpha)N_\mu(T_\alpha)\qquad(\mu=1,2,\cdots). \tag{9}$$

Let P be a point of $U_\mu(T_\alpha)$, then if we put $Q=\pi(P)$, we have $\pi(P^{(q^\mu)})=\pi(T_\alpha(P))=\pi(P)=Q$ and hence $Q^{(q^\mu)}=Q$, since π is defined over k. This means that Q is a rational point on V over k_μ. If P belongs to $\mathfrak{O}^*$, whose maximal ideal is $\mathfrak{P}$, then we have, for any T_τ of $\mathfrak{T}_\mathfrak{P}$, $T_\alpha T_\tau(P)=T_\alpha(P)=P^{(q^\mu)}$ and hence P belongs also to $U_\mu(T_\alpha T_\tau)$ for any T_τ of $\mathfrak{T}_\mathfrak{P}$. Conversely if P belongs to $U_\mu(T_{\alpha'})$, then we have $T_{\alpha'}(P)=T_\alpha(P)=P^{(q^\mu)}$ and therefore $T_\alpha^{-1}T_{\alpha'}(P)=P$. This means that $T_\alpha^{-1}T_{\alpha'}$ belongs to $\mathfrak{T}_\mathfrak{P}$. Thus, $T_{\alpha'}$ is an element of $T_\alpha\mathfrak{T}_\mathfrak{P}$.

Now P' be a point of $\pi^{-1}(Q)$. If T_γ is an element such that $T_\gamma(P)=P'$, we have, for any $T_\tau\in\mathfrak{T}_\mathfrak{P}$, $T_\gamma T_\alpha T_\tau T_\gamma^{-1}(P')=P'^{(q^\mu)}$ and hence P' belongs to $U_\mu(T_\gamma T_\alpha T_\tau T_\gamma^{-1})$ for any $T_\tau\epsilon\mathfrak{T}_\mathfrak{P}$. It can be also seen that P' belongs to these $U_\mu(T_\gamma T_\alpha T_\tau T_\gamma^{-1})$ only.

On the other hand, by the definition of $\chi(\mathfrak{p}_\mu(Q))$, we have

$$e_\mathfrak{p}\chi(\mathfrak{p}_\mu(Q))=\sum_{T_\tau\in\mathfrak{T}_\mathfrak{P}}\chi(T_\alpha T_\tau)=\sum_{T_\tau\in\mathfrak{T}_\mathfrak{P}}\chi(T_\gamma T_\alpha T_\tau T_\gamma^{-1}).$$

Since the number of the points of $\pi^{-1}(Q)$ is $n/e_\mathfrak{p}$, it can be seen easily that the effect of the points of $\pi^{-1}(Q)$ in $nc_\mu(\chi)$ is exactly equal to $n\chi(\mathfrak{p}_\mu(Q))$. Therefore we have

$$c_\mu(\chi)=\sum_{Q\in V_\mu}\chi(\mathfrak{p}_\mu(Q)). \tag{10}$$

Thus, by (8) and (10), we have the following

THEOREM 2. *For any character χ of $\mathfrak{G}$, we have*

(11) $$\frac{d}{du}\log L(u, \chi, U/V)=\sum_{\mu=1}^{\infty}c_\mu(\chi)u^{\mu-1},$$

where $c_\mu(\chi)$ *are constants determined by* (9).

Remark. This theorem shows that our L-series is nothing else than Ishida's one defined in [**6**].

Next we shall consider the case when the covering variety U is non-singular. In this case, the number $N_\mu(T_\alpha)$ is given by the intersection numbers of $U\times U$-cycles as follows:

Let us denote by I_μ the graph of the rational mapping which maps a point P on U to the point $P^{(q^\mu)}$ on U. Moreover we shall denote by Γ_α the graph of the correspondence T_α. Then we have the following

LEMMA 3. *If* U *is non-singular, then the number* $N_\mu(T_\alpha)$ *is equal to the degree of the cycle* $I_\mu\cdot\Gamma_\alpha$ *of dimension zero on* $U\times U$ *for each* $T_\alpha\in\mathfrak{G}$.

PROOF. It is enough to show, by the criterion of multiplicity 1, that Γ_α is transversal to I_μ at each component of $I_\mu\cdot\Gamma_\alpha$. Let $P\times P^{(q^\mu)}$ be a component of $I_\mu\cdot\Gamma_\alpha$. Then it is evident that $(U\times P^{(q^\mu)})\cdot\Gamma_\alpha=P\times P^{(q^\mu)}$ and therefore Γ_α is transversal to $U\times P^{(q^\mu)}$ at $P\times P^{(q^\mu)}$. On the other hand it can be seen easily that $U\times P^{(q^\mu)}$ and I_μ have the same tangent linear variety to them at $P\times P^{(q^\mu)}$. This fact means the lemma.

Now let $\mathfrak{N}(U\times U)$ denote the group of numerical equivalence classes of cycles on $U\times U$, $\mathfrak{N}^r(U\times U)$ will stand for the subgroup consisting of classes of dimension r. Let $\mathfrak{d}_\mu$ denote the numerical equivalence class of the cycle I_μ for every positive integer μ, and let $\mathfrak{c}_\alpha$ denote the numerical equivalence class of the cycle Γ_α for every $T_\alpha\in\mathfrak{G}$. Indicating the canonical scalar product in $\mathfrak{N}^r(U\times U)$ by symbol $\langle\mathfrak{x},\mathfrak{y}\rangle$,[9] we have from Lemma 3

(12) $$N_\mu(T_\alpha)=\langle\mathfrak{d}_\mu,\mathfrak{c}_\alpha\rangle,$$

and hence

(13) $$c_\mu(\chi)=\frac{1}{n}\sum_{T_\alpha\in\mathfrak{G}}\chi(T_\alpha)\langle\mathfrak{d}_\mu,\mathfrak{c}_\alpha\rangle.$$

Thus, from Theorem 2, we have the following

COROLLARY. *If* U *is non-singular, we have*

(14) $$\frac{d}{du}\log L(u, \chi, U/V)=\sum_{\mu=1}^{\infty}\Big\{\frac{1}{n}\sum_{T_\alpha\in\mathfrak{G}}\chi(T_\alpha)\langle\mathfrak{d}_\mu,\mathfrak{c}_\alpha\rangle\Big\}u^{\mu-1}.$$

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9) If D_1, D_2 belong to $\mathfrak{x},\mathfrak{y}$, respectively, and if $D_1\cdot D_2$ is defined, then $\langle\mathfrak{x},\mathfrak{y}\rangle$ is nothing other than deg $(D_1\cdot D_2)$.

§ 5. The functional equation of L-series.[10]

In [8], Sampson and Washnitzer gave the functional equation of the zeta-function of a non-singular variety under a certain assumption which will be defined and be denoted by the hypothesis (FC) later on. In this paragraph, we shall show that their methods are also applicable to give the functional equation of L-series when the covering variety U is non-singular.

First we shall give a lemma which is a generalization of theorem 1 in [8].

LEMMA 4. *Let L be an algebraic number field of finite degree. Let $R(x)=\sum_{\mu=1}^{\infty} a_\mu x^{\mu-1}$ be a power series satisfying the following conditions:*

(i) *$R(x)$ is a rational function of x and each of its poles is the inverse of an algebraic integer.*

(ii) *Each a_μ is an integer in L.*

(iii) *If we put $R_h(x)=\sum_{\mu=1}^{\infty} a_{\mu h} x^{\mu-1}$ for $h=1, 2, \cdots$, then the function $\exp\left\{\int_0^x R_h(x)dx\right\}$ has a representation as a power series in x with coefficient consisting of integers in L.*

Then $R(x)$ has a partial fraction decomposition of the form

$$R(x)=\gamma_1/(1-\alpha_1 x)+\cdots+\gamma_s/(1-\alpha_s x)\,.$$

Proof is similar to that of Theorem 1 in [8]. Therefore we shall give brief suggestions. By the condition (i), we have

$$R(x)=\sum \gamma_j/(1-\alpha_j x)^{m_j}+P(x) \tag{15}$$

where the α_j are algebraic integers and where $P(x)$ is a polynomial with coefficients in $L'=L(\alpha_1, \cdots, \alpha_s)$. Let $\mathfrak{S}$ be the ring of the integers in L'. By conditions (ii) and (iii), we have

$$a_h^p \equiv a_{hp} \qquad (\mathrm{mod}\, p) \tag{16}$$

for all rational primes p and all rational integers h. From (15), the coefficients of x^{h-1} and x^{hp-1} in $R(x)$ are, respectively,

$$\gamma_j \alpha_j^{h-1} m_j(m_j+1)\cdots(m_j+h-2)/(h-1)!+b_{h-1}\,, \tag{17}$$

and

$$\gamma_j \alpha^{hp-1} m_j(m_j+1)\cdots(m_j+hp-2)/(hp-1)! \quad \text{for large } p, \tag{18}$$

where b_{h-1} is the coefficient of x^{h-1} in $P(x)$.

Now the relation

$$m_j(m_j+1)\cdots(m_j+hp-2)/(hp-1)! \equiv 0 \ (\mathrm{mod}\, p) \quad \text{for large } p,\ m_j \neq 1 \tag{19}$$

10) The author was communicated, after he had completed the work, that M. Ishida had also obtained the similar results in this section.

is shown in the proof of theorem 1 in [8]. Let $\mathfrak{p}$ be a prime ideal in $\mathfrak{S}$ of degree 1 such that the norm $N_{\mathfrak{P}}=p$ is sufficiently large and such that $\mathfrak{p}$ does not appear in divisors of the γ_j and the b_j. Then we can easily seen, from (16), (17), (18), (19) and Fermat's theorem, that

$$\sum_{(m_j>1)} \gamma_j \alpha_j^{h-1} m_j(m_j+1)\cdots(m_j+h-2)/(h-1)!+b_{h-1} \in \mathfrak{p}\mathfrak{S}_{\mathfrak{p}}\,. \tag{20}$$

Since this relation holds for infinitely many prime ideals in $\mathfrak{S}$, we can conclude that

$$\sum_{(m_j>1)} \gamma_j/(1-\alpha_j x)^{m_j}+P(x)=0\,.$$

This means Lemma 4.

Let the notations be same as those in § 4, and assume that U is non-singular. Let P be a generic point of U over k. Then we shall denote by Γ'_{μ} the locus of $(P^{(q^{\mu})}, P)$ over k and denote by $\mathfrak{N}(\mathfrak{d})$ the subgroup of $\mathfrak{N}^r(U\times U)$ generated by the classes $\mathfrak{d}_{\mu}$ and $\mathfrak{d}'_{\mu}(\mu=0,1,2,\cdots)$, where $\mathfrak{d}'_{\mu}$ are the classes of the divisors Γ'_{μ}.

Then, the following hypothesis plays an essential rôle to give the functional equations of L-series.

HYPOTHESIS (FC). *The group* $\mathfrak{N}(\mathfrak{d})$ *is finitely generated.*[11]

In what follows, we shall assume always the hypothesis (FC). Now we define three regular mapping ϕ, σ and τ of $U\times U$ onto itself as follows:

$$\phi(P,Q)=(P^{(q)},Q),\qquad \sigma(P,Q)=(Q,P),\qquad \tau(P,Q)=(P^{(q)},Q^{(q)})\,,$$

where P and Q are points of U. These mapping are defined over k and are related by the identities

$$\sigma\sigma=1,\qquad\qquad \sigma\phi\sigma\phi=\tau, \tag{21}$$

and more generally

$$\sigma\phi^{\nu}\sigma\phi^{\nu}=\tau^{\nu}, \tag{21$'$}$$

where 1 is the identity mapping of $U\times U$ and ϕ^{ν}, τ^{ν} are the ν-fold iterations of ϕ, τ.

It is known that each of the mappings $\phi^{\nu}, \sigma, \tau^{\nu}$ induces an endomorphism of $\mathfrak{N}^r(U\times U)$. These endomorphisms will be denoted by $\phi^{\nu *}, \sigma^*, \tau^{\nu *}$, respectively. Then we can see that $\phi^{\nu *}, \sigma^*$ and $\tau^{\nu *}$ map $\mathfrak{N}(\mathfrak{d})$ into itself and that the following equality holds

$$\tau^{\nu *}=(\tau^*)^{\nu}=q^{r\nu}\times \text{identity in } \mathfrak{N}(\mathfrak{d})\,.^{12)} \tag{22}$$

11) As to the curves, this hypothesis is true by the theorem of Néron-Severi, which shows that the group of algebraic equivalence classes of divisors on a variety has a finite base.

12) For this equality, see No. 5 in [8].

The group $\mathfrak{N}^r(U\times U)$ is free from torsion. Therefore, because of (FC), $\mathfrak{N}(\mathfrak{d})$ must be a free group of finite rank ρ. Since ϕ^* must satisfy consequently its characteristic equation, there exist rational integers $e_1, \cdots, e_\rho$ such that

$$(\phi^*)^\nu + e_1(\phi^*)^{\nu-1} + \cdots + e_\rho(\phi^*)^{\nu-\rho} = 0 \quad \text{in } \mathfrak{N}(\mathfrak{d}) \tag{23}$$

for every $\nu \geqq \rho$.

On the other hand, we can see that $\phi^{\nu *}(\mathfrak{d}_\mu) = \mathfrak{d}_{\nu+\mu}$ where $\mathfrak{d}_\lambda$ is the class of I_λ as defined in § 4.

Therefore, from (12) and (23), it follows that

$$N_\mu(T_\alpha) + e_1 N_{\mu-1}(T_\alpha) + \cdots + e_\rho N_{\mu-\rho}(T_\alpha) = 0\,, \tag{24}$$

for $\mu \geqq \rho$ and for every T_α of $\mathfrak{G}$. Therefore, from (9) we have

$$c_\mu(\chi) + e_1 c_{\mu-1}(\chi) + \cdots + e_\rho c_{\mu-\rho}(\chi) = 0\,, \tag{25}$$

for $\mu \geqq \rho$ and for any character χ of $\mathfrak{G}$.

From (25), we can conclude that the function $\frac{d}{du}\log L(u, \chi, U/V)$ is a rational function of u satisfying the condition (i) of the Lemma 4, whenever the hypothesis (FC) is true. From this fact we have the following

THEOREM 3. *Suppose that U is non-singular and that the hypothesis* (FC) *on U is true.*

Then the function $\frac{d}{du}\log L(u, \chi, U/V)$ has a partial fraction decomposition of the form

$$\gamma_{1,\chi}/(1-\alpha_1 u) + \cdots + \gamma_{m,\chi}/(1-\alpha_m u)\,, \tag{26}$$

where the $\gamma_{i,\chi}$ depend on χ, and where the α_i depend on the covering variety U only.

PROOF. We consider first the case when the Galois group $\mathfrak{G}$ is an abelian group and the character χ is a simple character of $\mathfrak{G}$. Then the theorem is a direct consequence of (8), Proposition 2 and Lemma 4, since the rationality of the function has been showed already. In the case when $\mathfrak{G}$ is any group and χ is a non principal simple character, we can reduce to the above case by Theorem 1. If χ is the principal character of G, then $L(u, \chi, U/V)$ is the zeta-function of the variety V. Therefore the condition (iii) of Lemma 4 is satisfied for $\frac{d}{du}\log L(u, \chi, U/V)$ and other conditions are evidently satisfied. Thus, we have also the theorem in this case. In general case, χ is a linear combination of simple characters of $\mathfrak{G}$ with integral coefficients. Therefore this case is a consequence of above cases. It is evident that $\alpha_1, \cdots, \alpha_m$ are the distinct roots of the equation $x^\rho + e_1 x^{\rho-1} + \cdots + e_\rho = 0$. Hence the α_i are depend only on U.

Thus the proof is completed.

Now let $\mathfrak{n}_1, \cdots, \mathfrak{n}_\rho$ be a base of $\mathfrak{N}(\mathfrak{d})$ and we put

$$(27)\qquad \begin{aligned} \phi^*(\mathfrak{n}_i) &= \sum_{j=1}^{\rho} a_{ij}\mathfrak{n}_j \\ \sigma^*(\mathfrak{n}_i) &= \sum_{j=1}^{\rho} s_{ij}\mathfrak{n}_j \end{aligned} \qquad (i=1,2,\cdots,\rho),$$

the a_{ij} and the s_{ij} being rational integers. Write $A=(a_{ij}), S=(s_{ij})$.

If $f(x)$ is the characteristic equation of A, then we have $f(x)=x_\rho+e_1x_\rho^{-1}+\cdots+e_\rho$, where the e_i are same as in (23). Then we can easily see, from (21) and (22), that

$$f(x)=e_\rho^{-1}x^\rho f(q^r/x).$$

Therefore if α_j is a root of $f(x)$, then q^r/α_j is also a root of $f(x)$ and will be denoted by α_{sj}. It is evident that $j \rightarrow sj$ designates a permutation of $1, 2, \cdots, m$ of the period 2 if $\alpha_1, \cdots, \alpha_m$ are the distinct roots of $f(x)$.

Since it can be seen easily that $\sigma^*(\mathfrak{d}_0)=\mathfrak{d}_0$ and $\sigma^*(\mathfrak{c}_\alpha)=c_{\alpha^{-1}}$ (notice $T_\alpha^{-1}=T_{\alpha^{-1}}$!!), we have

$$(28)\qquad \sum_{i=1}^{\rho} c_i s_{ij}=c_j,$$

putting $\mathfrak{d}_0=c_1\mathfrak{n}_1+\cdots+c_\rho\mathfrak{n}_\rho$.

Let the coefficients of A^ν be denoted by $a_{ij}^{(\nu)}$. Then we have from (12)

$$(29)\qquad N_\nu(T_\alpha)=\sum_{i,j} c_i a_{i,j}^{(\nu)}\langle \mathfrak{n}_j, \mathfrak{c}_\alpha\rangle$$

because of $\mathfrak{d}_\nu=\phi^{\nu*}(\mathfrak{d}_0)$. Since σ is a biregular mapping, we have $\langle \mathfrak{d}_\nu, \mathfrak{c}_\alpha\rangle = \langle \sigma^*(\mathfrak{d}_\nu), \sigma^*(\mathfrak{c}_\alpha)\rangle$. Therefore we have, using the relation $A^\nu S=q^{r\nu}SA^{-\nu}$ which is a direct consequence of (21′) and (22),

$$(30)\qquad \begin{aligned} N_\nu(T_\alpha) &= \langle \sigma^*(\mathfrak{d}_\nu), \sigma^*(\mathfrak{c}_\alpha)\rangle \\ &= \sum_{i,j,k} c_i a_{ij}^{(\nu)} s_{j,k}\langle \mathfrak{n}_k, \mathfrak{c}_{\alpha^{-1}}\rangle \\ &= q^{r\nu}\sum_{i,j,k} c_i s_{ij} a_{jk}^{(-\nu)}\langle \mathfrak{n}_k, \mathfrak{c}_{\alpha^{-1}}\rangle \\ &= q^{r\nu}\sum_{j,k} c_j a_{jk}^{(-\nu)}\langle \mathfrak{n}_k, \mathfrak{c}_{\alpha^{-1}}\rangle . \end{aligned}$$

Let us now define $N_\nu(T_\alpha)$ and $c_\nu(\chi)$ for $\nu \leqq 0$ by means of the difference equations (24) and (25) respectively. It is clear that the values so obtained for $N_{-1}(T_\alpha)$, $N_{-2}(T_\alpha)$, etc. are same as the values calculated from (29) by putting $\nu=-1,-2$, etc. and that the relation (9) is also satisfied for $\nu \leqq 0$. Then we have from (30)

$$(31)\qquad N_\nu(T_\alpha)=q^{r\nu}N_{-\nu}(T_\alpha^{-1}) \qquad \text{for } \nu=0,1,2, \text{ etc.},$$

and hence from (9)

(32) $$c_\nu(\chi)=q^{r\nu}c_{-\nu}(\bar{\chi}),$$

where $\bar{\chi}$ is the conjugate character of χ as usual.

Now we put $\beta_{j,\chi}=\gamma_{j,\chi}/\alpha_j$, where the $\gamma_{j,\chi}$ are the constants determined in (26). Then it follows that

(33) $$c_\nu(\chi)=\sum_{j=1}^{m}\beta_{j,\chi}\alpha^\nu{}_j \qquad \text{for } \nu=0, \pm 1, \pm 2, \text{ etc.}$$

This relation is trivial for $\nu\geqq 0$ and as to the case for $\nu<0$, it is enough to consider the fact that for each j, the $\alpha^\nu{}_j$ satisfy the difference equation with same coefficients as (25). Then we can see, since the α_j are distinct, that $\beta_{j,\chi}=\beta_{sj,\bar{\chi}}$.

Now we have by the Theorem 3,

$$\frac{d}{du}\log L(1/q^r u, \chi, U/V)=\sum_{j=1}^{m}\beta_{j,\chi}\alpha_j/(1-\alpha_j/q^r u)$$

$$=\sum_{j=1}^{m}\beta_{j,\chi}\alpha_j/(1-1/\alpha_{sj}u)=-\sum_{j=1}^{m}\beta_{j,\chi}\alpha_j\alpha_{sj}u/(1-\alpha_{sj}u)$$

$$=-q^r u\sum_{j=1}^{m}\beta_{sj,\bar{\chi}}/(1-\alpha_{sj}u)=-q^r u^2\sum_{j=1}^{m}\beta_{j,\bar{\chi}}\alpha_j/(1-\alpha_j u)-q_r u\sum_{j=1}^{m}\beta_{j,\chi}$$

$$=-q^r u^2\frac{d}{du}\log L(u, \bar{\chi}, U/V)-q^r u c_0(\bar{\chi}).$$

From this we have

(34) $$-\frac{1}{q^r u^2}\left\{\frac{d}{du}\log L(1/q^r u, \chi, U/V)\right\}$$

$$=\frac{d}{du}\log L(u, \bar{\chi}, U/V)+\frac{1}{u}c_0(\bar{\chi}).$$

On the other hand, by the result[13] of Ishida [**6**], we can easily see that $L(u, \chi, U/V)$ is a power series with a positive convergent radius. Now we shall consider a domain D in the complex u-plane D_0 with the property as follows: Let J be a Jordan arc whose end points are $1/\alpha_1$ and the point at infinity. Moreover $1/\alpha_2, \cdots, 1/\alpha_m$ are on J and the origin is not on J. Then D consists of the points which do not belong to J. Then, by the theorem 3, $L(u, \chi, U/V)$ defines a univalent regular function on D. This function will be also denoted by $L(u, \chi, U/V)$. Now we shall determine the functional equation of this function.

From (34) we have

(35) $$L(1/q^r u, \chi, U/V)=C_\chi u^{c_0(\bar{\chi})}L(u, \bar{\chi}, U/V),$$

where C_χ is a constant depending on χ, and where a suitable branch is chosen in $u^{c_0(\bar{\chi})}$.

13) See the corollary of the Theorem 1 in [**6**].

Since $\beta_{j,\chi}=\beta_{sj,\bar{\chi}}$ for $j=1,\cdots,m$, we have $c_0(\chi)=c_0(\bar{\chi})$. If α is a root of $f(x)$, then the complex conjugate $\bar{\alpha}$ of α is also a root of $f(x)$. If we put $\bar{\alpha}_j=\alpha_{tj}$, then $j\to tj$ designates a permutation of $1,\cdots,m$ of the period 2. Since we have $\overline{c_\mu(\chi)}=c_\mu(\chi)$ for each $\mu>0$, it can be seen, from (33), that $\beta_{j,\bar{\chi}}=\overline{\beta_{tj,\chi}}$ for each j. Therefore we can conclude that $c_0(\chi)$ is a real number for any χ.

If we replace u by $1/q^r u$ in (35), we have

$$L(u,\chi,U/V)=C_\chi(1/q^r u)^{c_0(\bar{\chi})}L(1/q_r u,\bar{\chi},U/V)\,.$$

Therefore we can see that $|C_\chi C_{\bar{\chi}}|=|q^{rc_0(\chi)}|$, since $c_0(\chi)$ is real.

Now we assume that the $\beta_{j,\chi}$ are all real numbers. From Theorem 3, we have

$$L(u,\chi,U/V)=\sum_{j=1}^{m}(1-\alpha_j u)^{-\beta_{j,\chi}}$$

if suitable branches are chosen. From this relation and (35), we have, putting $u=1$,

$$|\prod_{j=1}^{m}(1-\alpha_j/q^r)^{-\beta_{j,\chi}}|=|C_\chi||\prod_{j=1}^{m}(1-\alpha_j)^{-\beta_{j,\bar{\chi}}}|\,.$$

Hence we have, using $\beta_{j,\chi}=\beta_{sj,\bar{\chi}}$ and $\alpha_j\alpha_{sj}=q^r$,

$$|C_\chi|=\prod_{j=1}^{m}|\alpha_j{}^{\beta_{j,\chi}}|\,.$$

Moreover we have, using $\beta_{j,\chi}=\beta_{tj,\bar{\chi}}$ ($\alpha_{j,\chi}$ is real !!)

$$|C_{\bar{\chi}}|=\prod_{j=1}^{m}|\alpha_j{}^{\beta_{j,\bar{\chi}}}|=\prod_{j=1}^{m}|\alpha_j{}^{\beta_{tj,\chi}}|=\prod_{j=1}^{m}|\bar{\alpha}_{tj}{}^{\beta_{tj,\chi}}|$$
$$=\prod_{j=1}^{m}|\alpha_j{}^{\beta_{j,\chi}}|=\prod_{j=1}^{m}|\alpha_j{}^{\beta_{j,\chi}}|=|C_\chi|\,.$$

Thus we have shown that $|C_\chi|=|C_{\bar{\chi}}|=|q^{rc_0(\chi)/2}|$, if the $\beta_{j,\chi}$ are all real numbers. In conclusion we have

THEOREM 4. *Suppose that the covering variety U is non-singular, and that the hypothesis* (FC) *on U is true, then $L(u,\chi,U/V)$, considered as a function in the domain D, satisfies the following functional equation*

$$L(1/q^r u,\chi,U/V)=C_\chi u^{c_0(\chi)}L(u,\bar{\chi},U/V)\,,$$

where C_χ is a constant such that $|C_\chi C_{\bar{\chi}}|=|q^{rc_0(\chi)}|$ and where $c_0(\chi)=\frac{1}{n}\sum_{T_\alpha\in\mathfrak{G}}\chi(T_\alpha)$ $\langle\mathfrak{d}_0,\mathfrak{c}_\alpha\rangle=\sum_{j=1}^{m}\beta_{j,\chi}$.

Moreover, if the $\beta_{j,\chi}$ are all real numbers, we have

$$|C_\chi| = \prod_{j=1}^{m} |\alpha_j^{\beta_{j,\chi}}| = |q^{rc_0(\chi)/2}|.$$

Remark. If U is a curve and if χ is a non-principal simple character of $\mathfrak{G}$, the value $c_0(\chi)$ is calculated as follows: Using notations in Weil [**9**], the trace $\sigma(T_\alpha)$ of the correspondence T is equal to $2-\langle \mathfrak{d}_0, \mathfrak{c}_\alpha \rangle$ by the definition. Therefore from the orthogonality of characters we have

$$c_0(\chi) = \frac{1}{n} \sum_{T_\alpha \in \mathfrak{G}} \chi(T_\alpha) \langle \mathfrak{d}_0, \mathfrak{c}_\alpha \rangle = -\frac{1}{n} \sum_{T_\alpha \in \mathfrak{G}} \chi(T_\alpha) \sigma(T_\alpha).$$

This means that our functional equation and Weil's one in [**9**] are same, if we do not refer to the constant C_χ.

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