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Some properties of the Stone-Čech compactification.

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In this note, we shall investigate some topological and uniform properties of Tychonoff space X (completely regular T_1-space) in connection with the properties of the Stone-Čech compactification βX.

The existence of compactification, the complete regularity and the uniformizability are equivalent each other, so that the Stone-Čech compactification may reasonably be expected to play an important role in the theory of uniform spaces. The consideration of uniformity $\mathcal{U}=\{V_\alpha\}$ in $\beta X\times\beta X$ leads us to consider the set $\boldsymbol{R}=\bigcap_\alpha \tilde{V}_\alpha$, where $\tilde{V}_\alpha$ denotes the interior of the closure of V_α taken in $\beta X\times\beta X$. The set $\boldsymbol{R}$ defined above will be called throughout as the radical of uniform space $(X, \mathcal{U})$. We shall show that the radical determines topologically the completion $\hat{X}$ of $(X, \mathcal{U})$. In fact, $\hat{X}$ is obtained as a quotient space $\bar{X}/\mathcal{R}$ (with the quotient topology), where $\bar{X}=\{p\in\beta X; (p, p)\in\boldsymbol{R}\}$ and $\mathcal{R}$ is the relation on $\bar{X}$ defined by the radical $\boldsymbol{R}$. The completeness will be characterized in terms of the radical as follows: $(X, \mathcal{U})$ is complete if and only if $\boldsymbol{R}=\Delta_X$. As a direct consequence of this, we shall obtain a necessary and sufficient condition for an entirely normal space to be topologically complete (Theorem 2.2). (We call the space X entirely normal if the family of all neighborhoods of the diagonal of $X\times X$ forms a uniformity for X.) The condition is stated as a property of points contained in $\beta X-X$ (points at infinity). A slightly stronger condition will be examined as well, and the relationship between entire normality and paracompactness will be made clear in a simple form (Theorem 2.3).

The idea to treat the completion of uniform space in connection with the compactification is due to H. Nakano [**11**]. We shall be concerned with the completion of uniform space in §3 and discuss some topological properties of the completion of uniform space in terms of the radical.

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§ 1. Preliminary.

In the first place, we shall state some lemmas concerning regularly open sets, which will be used in the following arguments. For the sake of convenience, we shall use the following notations. Let X be a subspace of a topological space Y, and A a set in X, then the closure of A taken in X (or in Y) will be denoted by $\mathrm{Cl}_X(A)$ (respectively, $\mathrm{Cl}_Y(A)$). Similarly, the interior of A will be denoted by $\mathrm{Int}_X(A)$ or $\mathrm{Int}_Y(A)$ according as it is taken in X or in Y. A set F in a topological space X is said to be *regularly open* if the interior of the closure of F is identical with F, that is, $F=\mathrm{Int}_X(\mathrm{Cl}_X(F))$. Evidently, it is an open set.

LEMMA 1.1. *Let X be a topological space and A a regularly open set in X. Let B be an open set which is not contained in A. Then there is an open set C contained in B such that $C\frown A=\phi$.*

PROOF. If $B\subset \mathrm{Cl}_X(A)$, then $B\subset \mathrm{Int}_X(\mathrm{Cl}_X(A))=A$. Therefore we have $B\not\subset \mathrm{Cl}_X(A)$, and $C=B\frown[\mathrm{Cl}_X(A)]^C$ is obviously a desired one.

LEMMA 1.2. *Let X be a dense subspace of a topological space Y and A a set in X. Then* $\mathrm{Int}_X(\mathrm{Cl}_X(A))=\mathrm{Int}_Y(\mathrm{Cl}_Y(A))\frown X$.

PROOF. The inclusion $\mathrm{Int}_X(\mathrm{Cl}_X(A))\supset \mathrm{Int}_Y(\mathrm{Cl}_Y(A))\frown X$ is obvious. Therefore we have only to prove the reversed inclusion. If p is any point of $\mathrm{Int}_X(\mathrm{Cl}_X(A))$, then there is in Y an open set $U(p)$ containing p such that $U(p)\frown X\subset \mathrm{Cl}_X(A)=\mathrm{Cl}_Y(A)\frown X$. It follows that $U(p)\subset \mathrm{Cl}_Y(A)$ and p is therefore contained in $\mathrm{Int}_Y(\mathrm{Cl}_Y(A))\frown X$. For, if $U(p)\not\subset \mathrm{Cl}_Y(A)$, then $U(p)\frown[\mathrm{Cl}_Y(A)]^C$ is a non-void open set in Y and since X is dense in Y it must contain a point of X, which contradicts the above fact that $U(p)\frown X\subset \mathrm{Cl}_Y(A)\frown X$. Thus we have $\mathrm{Int}_X(\mathrm{Cl}_X(A))\subset \mathrm{Int}_Y(\mathrm{Cl}_Y(A))\frown X$, and the proof is completed.

LEMMA 1.3. *Let X be a dense subspace of a topological space Y.*

(a) *If A is regularly open in Y, then $A\frown X$ is also regularly open in X.*

(b) *B $(\subset X)$ is regularly open in X if and only if* $B=\mathrm{Int}_Y(\mathrm{Cl}_Y(B))\frown X$.

(c) *If A is regularly open and B is open in Y and if $A\frown X\supset B\frown X$, then $A\supset B$. Therefore two regularly open sets A, B in Y are identical if and only if $A\frown X=B\frown X$.*

PROOF. In view of Lemma 1.2, statement (a) follows easily by a direct computation: $A\frown X\subset \mathrm{Int}_X(\mathrm{Cl}_X(A\frown X))=\mathrm{Int}_Y(\mathrm{Cl}_Y(A\frown X))\frown X\subset \mathrm{Int}_Y(\mathrm{Cl}_Y(A))\frown X$ $=A\frown X$, hence $\mathrm{Int}_X(\mathrm{Cl}_X(A\frown X))=A\frown X$. Statement (b) is also an immediate consequence of Lemma 1.2. We now establish the statement (c). Suppose that $A\not\supset B$, then there will be an open set $C\subset B$ such that $C\frown A=\phi$, by Lemma 1.1. Since X is dense in Y, it follows that there is a point $p\in X$ such that $p\notin A$ and $p\in B$, which contradicts the assumption that $A\frown X\supset B\frown X$. Thus we have $A\supset B$. Moreover, if both A and B are regularly open and $A\frown X=B\frown X$, then we have $A\supset B$ and $B\supset A$ and hence $A=B$. On the other hand, it is evident that $A=B$ implies $A\cap X=B\cap X$. The proof is completed.

Throughout the sequel, we shall limit ourselves to consider the Tychonoff spaces (completely regular T_1-spaces). Let X be a Tychonoff space. A compactification BX of X is a compact Hausdorff space containing a dense subspace homeomorphic with X. The Stone-Čech compactification βX is the space of all maximal ideals in the ring $C(X)$ of all real-valued continuous functions on X, whose topology base is given by the family of open sets $\mathcal{V}=\{\mathcal{V}_f; f\in C(X)\}$, where $\mathcal{V}_f$ denotes the set of all maximal ideals which do not contain f.[1] It is characterized among the compactifications of X by the fact that every bounded continuous function $f\in C^*(X)$ has a unique continuous extension over βX,[2] where $C^*(X)$ denotes the ring of bounded real-valued continuous functions on X. The clucial properties of the Stone-Čech compactification are provided by the following theorems.

THEOREM 1.1 (*Čech*). *Any compactification BX of X is the image of βX under a* (*unique*) *continuous map φ such that $X'=\varphi(X)$ is homeomorphic to X and $\varphi(\beta X-X)=BX-X'$.*

For the proof, see [**2**, p. 831].

THEOREM 1.2 (*Stone*). *If f is any continuous map of a Tychonoff space X into a compact Hausdorff space Y, then f has a* (*unique*) *continuous extension f^* which carries βX into Y.*

For the proof, see [**9**, p. 153]. (cf. [**16**, Theorem 88.])

There exists an important subspace νX of βX, which is called sometimes real compactification of X. It is a subspace of βX consisting of all real ideals in $C(X)$, which is defined to be a maximal ideal $\mathfrak{M}$ such that the quotient field $C(X)/\mathfrak{M}$ is isomorphic to the real number field. νX is characterized by the following properties[1]: (1) $X\subset \nu X\subset \beta X$; (2) every continuous function on X has a continuous extension over νX; (3) for each point $p\in\beta X-\nu X$ there is a continuous function $f\in C(X)$ such that f can not be continuously extended over the point p (cf. [**8**, p. 90]).

Upon applying the above theorem (Theorem 1.2) to $f\in C(X)$ and any compactification BR of real number space R, we have a continuous extension[3] f^* of f over βX. Let X_f be the set of points $p\in\beta X$ such that $f^*(p)\in R$, then X_f is the maximal subspace of βX over which f can be continuously extended. It is easy to see that X_f is open (dense) in βX and that $\nu X=\bigcap_{f\in C(X)} X_f$. A space such that $\nu X=X$ is called a Q-space[4]. As is well

1) Cf. [**8**].
2) Cf. [**2**].
3) BR-valued extension (f^* is a function on βX to BR).
4) Cf. [**8**], [**14**].

known, there is another definition of Q-space due to L. Nachbin, which may be stated as follows: X is a Q-space if and only if it is complete relative to the weakest uniformity for X with respect to which every continuous function is uniformly continuous. The equivalence of these two definitions will be established in the next section. At this moment, we prove the following

PROPOSITION 1.1. *Let* $V_f=\{(p,q)\in X\times X;\ |f(p)-f(q)|<1\}$ *and let* $\tilde{V}_f$ *be the interior of the closure of* V_f *taken in* $\beta X\times\beta X$. *Then* $X_f=\{p\in\beta X;\ (p,p)\in\tilde{V}_f\}$. *Therefore we have* $\varDelta_{\nu X}=\bigcap_{f\in C(X)}\tilde{V}_f$, *where* $\varDelta_{\nu X}=\{(p,p)\in\beta X\times\beta X;\ p\in\nu X\}$.

PROOF. Let f^0 be the extension of f over X_f and let $V_f{}^0=\{(p,q)\in X_f\times X_f;\ |f^0(p)-f^0(q)|<1\}$. Since $\tilde{V}_f$ is regularly open in $\beta X\times\beta X$, $\tilde{V}_f\frown(X_f\times X_f)$ is also regularly open in $X_f\times X_f$ by Lemma 1.3, (a). It is evident that $[\tilde{V}_f\frown(X_f\times X_f)]\frown(X\times X)=\tilde{V}_f\frown(X\times X)\supset V_f=V_f{}^0\frown(X\times X)$, and therefore we have $\tilde{V}_f\frown(X_f\times X_f)\supset V_f{}^0$ by Lemma 1.3, (c). Put $\varDelta_{X_f}=\{(p,p)\in\beta X\times\beta X;\ p\in X_f\}$, then clearly $\varDelta_{X_f}\subset V_f{}^0$ and we have $\varDelta_{X_f}\subset\tilde{V}_f$. On the other hand, if $(p,q)\in\beta X\times\beta X$ is contained in $\tilde{V}_f$, then there is in $\beta X\times\beta X$ an open neighborhood of (p,q) of the form $U(p)\times W(q)$ such that $U(p)\times W(q)\subset\tilde{V}_f$. Let x be a point of $U(p)\frown X$ and put $f(x)=a$, then $|f(y)-a|\leqq 1$ for each $y\in U(p)\frown X$ and therefore f must be bounded on $U(p)\frown X$. It follows that $p\in X_f$, for if $p\notin X_f$, then for each $U(p)\frown X$ and for each positive integer n there is $z\in U(p)\frown X$ such that $f(z)>n$. Similarly, we have $q\in X_f$, and therefore $(p,q)\in X_f\times X_f$ for each $(p,q)\in\tilde{V}_f$. Thus, we have $\tilde{V}_f\subset X_f\times X_f$ and hence $\tilde{V}_f\frown\varDelta_{\beta X}\subset(X_f\times X_f)\frown\varDelta_{\beta X}=\varDelta_{X_f}$. Consequently $\varDelta_{X_f}=\tilde{V}_f\frown\varDelta_{\beta X}$, and it follows that $X_f=\{p\in\beta X;\ (p,p)\in\tilde{V}_f\}$. Finally, it is easy to see that $\bigcap_{f\in C^*(X)}\tilde{V}_f=\varDelta_{\beta X}$, and the last statement follows immediately from the fact that $\nu X=\bigcap_{f\in C(X)}X_f$.

§ 2. Characterization of complete uniform spaces and some topological spaces.

Let X be a Tychonoff space. Let $\{V_\alpha\}$ and $\{V_\beta'\}$ be two equivalent uniformities for X. Then for each V_α there is a V_β' contained in V_α, and it follows that $\tilde{V}_\beta'\subset\tilde{V}_\alpha$ for some V_β', where $\tilde{V}_\alpha$ and $\tilde{V}_\beta'$ are the interiors of the closures of V_α and V_β' respectively taken in $\beta X\times\beta X$. Similarly, for each V_β' there is a V_γ such that $\tilde{V}_\gamma\subset\tilde{V}_\beta'$. Therefore $\boldsymbol{R}=\bigcap_\alpha\tilde{V}_\alpha$ is identical with $\boldsymbol{R}'=\bigcap_\beta\tilde{V}_\beta'$, and consequently the set $\boldsymbol{R}$ is determined by the uniform structure for X. The set $\boldsymbol{R}$ defined above will be called throughout this research the *radical* of uniform space $(X,\{V_\alpha\})$. In this section we treat the characterization of the completeness in terms of the radical. A necessary and sufficient condition for the completeness will be given in Theorem

2.1. However, the proof of the sufficiency of the condition requires somewhat intricate considerations, so it will be given in the next section. We shall use the notations and the basic results concerning Cauchy filters that are used in A. Weil's monograph [**18**].

PROPOSITION 2.1. *Let* $\{C_\alpha\}$ *be a Cauchy filter of a complete uniform space* $(X, \{V_\alpha\})$ *and let* $\bar{C}_\alpha$ *denote the closure of* C_α *in a compactification* BX *of* X. *Then* $\bigcap_\alpha \bar{C}_\alpha$ *is a point of* X.

PROOF. If $(X, \{V_\alpha\})$ is complete, then there is a point $p \in X$ such that the Cauchy filter $\{p\}$ is equivalent to $\{C_\alpha\}$. It follows that $p \in \bigcap_\alpha \bar{C}_\alpha$ and that for each V_α there is a C_α such that $C_\alpha \subset V_\alpha(p)$, in view of the definition of equivalent Cauchy filters. Let q $(\neq p)$ be any point of BX, then there is obviously a neighborhood $U(p)$ of p such that $\mathrm{Cl}_{\beta X}(U(p)) \not\ni q$. On the other hand, it is clear that $V_\alpha(p) \subset U(p) \cap X \subset U(p)$ for some V_α, and it follows that q is not contained in $\bar{C}_\alpha$ for some α. This proves the proposition.

PROPOSITION 2.2. *Let* X *be a dense subspace of a Tychonoff space* Y *and let* $\{V_\alpha\}$ *be a uniformity for* X, *where each* V_α *is assumed to be symmetric and regularly open in* $X \times X$. *Let* $\tilde{V}_\alpha$ *be the interior of the closure of* V_α *taken in* $Y \times Y$. *Then* $\tilde{V}_\alpha \circ \tilde{V}_\alpha \subset \widetilde{V_\alpha \circ V_\alpha}$, *and therefore* $V_\beta \circ V_\beta \subset V_\alpha$ *implies that* $\tilde{V}_\beta \circ \tilde{V}_\beta \subset \tilde{V}_\alpha$.

PROOF. Suppose that $\tilde{V}_\alpha \circ \tilde{V}_\alpha \not\subset \widetilde{V_\alpha \circ V_\alpha}$. Then there is by Lemma 1.1 an open set $C \subset \tilde{V}_\alpha \circ \tilde{V}_\alpha$ such that $C \cap \widetilde{V_\alpha \circ V_\alpha} = \phi$, since $\widetilde{V_\alpha \circ V_\alpha}$ is regularly open and $\tilde{V}_\alpha \circ \tilde{V}_\alpha$ is open in $Y \times Y$. Let (p, q) be a point of $C \cap (X \times X)$, then $(p, r') \in \tilde{V}_\alpha$ and $(r', q) \in \tilde{V}_\alpha$ for some point $r' \in Y$. Since X is dense in Y and since $\tilde{V}_\alpha$ is open in $Y \times Y$, there is a point $r \in X$ such that $(p, r) \in \tilde{V}_\alpha$ and $(r, q) \in \tilde{V}_\alpha$. It follows that $(p, r) \in \tilde{V}_\alpha \cap (X \times X) = V_\alpha$, $(r, q) \in \tilde{V}_\alpha \cap (X \times X) = V_\alpha$ by Lemma 1.3, (b), and therefore $(p, q) \in V_\alpha \circ V_\alpha$. But this contradicts the fact that $C \cap \widetilde{V_\alpha \circ V_\alpha} = \phi$, and therefore we have $\tilde{V}_\alpha \circ \tilde{V}_\alpha \subset \widetilde{V_\alpha \circ V_\alpha}$. The last statement is an immediate consequence of this fact.

As the radical is determined by the uniform structure for X, we may assume throughout, without loss of generality, that each member V_α of a uniformity $\mathcal{U} = \{V_\alpha\}$ is symmetric and regularly open.

THEOREM 2.1. *A uniform space* $(X, \{V_\alpha\})$ *is complete if and only if the radical is identical with the diagonal* Δ_X. *That is,* $\bigcap \tilde{V}_\alpha = \boldsymbol{R} = \Delta_X$, *where* $\Delta_X = \{(p, p) \in \beta X \times \beta X; p \in X\}$.

PROOF. (Necessity.) As it is evident that $\Delta_X \subset \boldsymbol{R}$, it is only necessary to show that $\Delta_X \supset \boldsymbol{R}$. If $(p, q) \in \boldsymbol{R}$, then $(p, q) \in \tilde{V}_\alpha$ for each $\tilde{V}_\alpha$, and since $\tilde{V}_\alpha$ is open there are open neighborhoods $U_\alpha(p)$, $W_\alpha(q)$ of p and q respectively such that $U_\alpha(p) \times W_\alpha(q) \subset \tilde{V}_\alpha$. By virtue of Proposition 2.2, there is for each V_β a V_α such that $\tilde{V}_\alpha \circ \tilde{V}_\alpha \subset \tilde{V}_\beta$. It follows that $U_\alpha(p) \times U_\alpha(p) \subset \tilde{V}_\alpha \circ \tilde{V}_\alpha \subset \tilde{V}_\beta$

and $W_\alpha(q)\times W_\alpha(q)\subset \tilde V_\alpha\circ \tilde V_\alpha\subset \tilde V_\beta$, and therefore $[(U_\alpha(p)\cup W_\alpha(q))\cap X]\times[(U_\alpha(p)\cup W_\alpha(q))\cap X)]\subset(\tilde V_\alpha\circ \tilde V_\alpha)\cap(X\times X)\subset \tilde V_\beta\cap(X\times X)=V_\beta$. This implies that $\{C_\alpha\}=\{(U_\alpha(p)\cup W_\alpha(q))\cap X\}$ is a Cauchy filter relative to the uniformity $\{V_\alpha\}$. By virtue of Proposition 2.1, it follows that $p=q\in X$, since $p\in\bigcap_\alpha \bar C_\alpha$ and $q\in\bigcap_\alpha \bar C_\alpha$, and therefore $\varDelta_X\supset \boldsymbol{R}$. Thus we have $\varDelta_X=\boldsymbol{R}$, and the necessity of the condition is proved.

Proof of the sufficiency will be given in the next section.

COROLLARY 1. *A complete metric space is a G_δ in βX.*

PROOF. Let $\{V_n\}$ be the metric uniformity for X and let $X_n=\{p\in\beta X;\ (p,p)\in \tilde V_n\}$. Then X_n is open and dense in βX and it follows from the above theorem that $X=\{p\in\beta X;\ (p,p)\in\bigcap_{n=1}^\infty \tilde V_n\}=\bigcap_{n=1}^\infty\{p\in X;\ (p,p)\in\tilde V_n\}=\bigcap_{n=1}^\infty X_n$. Hence X is a G_δ in βX.

REMARK. It is worth while to notice that each locally compact space X is open and dense in βX,[5] and that each complete metric space X is an intersection of a countable number of open dense subsets of βX as we have just observed. These are deservedly the situtation that Baire's theorem should hold; namely, each countable intersection of open dense subset of X is itself dense in X.

COROLLARY 2. *X is a Q-space if and only if it is complete relative to the weakest uniform structure with respect to which every continuous function is uniformly continuous.*

PROOF. The uniform structure stated in this proposition is given by the uniformity generated by $\{V_{f,n}\}_{f\in C(X),\, n\in I}$, where $V_{f,n}=\{(p,q)\in X\times X;\ |f(p)-f(q)|<1/2^n\}$. The proof may easily be completed by Theorem 2.1 and Proposition 1.1.

A space X is said to be *topologically complete* if there is a uniformity $\mathcal{U}$ such that $(X,\mathcal{U})$ is complete. We now discuss some topological properties which are closely related to topological completeness. If the family of all neighborhoods of the diagonal of $X\times X$ forms a uniformity for X, then we shall say that the space X is *entirely normal.* It is well known that each paracompact space is entirely normal[6] and that each entirely normal space is collectionwise normal[7] hence is normal. J. Kelley [**9**] suggests the possibility of characterizing paracompactness by the entire normality plus another condition similar to topological completeness, and an answer was given by H. Corson [**4**]. We are now able to give another result on this problem. First, we observe the relationship between entire normality and

5) See [**7**].
6) See [**6**].
7) See [**3**].

topological completeness.

THEOREM 2.2. *An entirely normal space X is topologically complete if and only if there is for each point $p \in \beta X - X$ a regularly open set $\tilde{V}$ in $\beta X \times \beta X$ containing Δ_X such that $\tilde{V} \not\ni (p, p)$.*

PROOF. By virtue of Theorem 2.1, it follows that X is topologically complete if and only if there is a uniformity $\{V_\alpha\} = \mathcal{U}$ such that $\bigcap_\alpha \tilde{V}_\alpha = \Delta_X$, and therefore the necessity of the condition is clear. Suppose conversely that for each $p \in \beta X - X$ there is $\tilde{V}_P \supset \Delta_X$ such that $\tilde{V}_P \not\ni (p, p)$. Then $(\bigcap_{P \in \beta X - X} V_P) \cap \Delta_{\beta X} = \Delta_X$ and it follows immediately that X is complete relative to the universal uniform structure, by Theorem 2.1.

EXAMPLE. There is an example of entirely normal space which is not topologically complete. Let Ω_0 be the set of all ordinals less than the first uncountable ordinal Ω, and let τ be the order topology for Ω_0. Then the topological space (Ω_0, τ) is entirely normal as may easily be seen from the fact that $\tilde{V} \supset \Delta_{\beta X}$ for each neighborhood V of the diagonal of $X \times X$. Similarly, it is clear that (Ω_0, τ) is not topologically complete (see [**5**]).

The following theorem establishes a relationship between paracompactness and entire normality.

THEOREM 2.3. *A space X is paracompact if and only if it is entirely normal and there is for each compact set G in $\beta X - X$ a regularly open set $\tilde{V}$ containing Δ_X such that $\tilde{V} \cap \Delta_G = \phi$, where $\Delta_G = \{(p, p) \in \beta X \times \beta X;\, p \in G\}$.*

PROOF. (Necessity.) To prove the necessity, we have only to construct a regularly open set $\tilde{V}$ in $\beta X \times \beta X$ containing Δ_X such that $\tilde{V} \cap \Delta_G = \phi$. For each point $p \in X$, there is in βX an open neighborhood $U(p)$ of p such that $\mathrm{Cl}_{\beta X}(U(p)) \cap G = \phi$. Consider a covering $\{U'(p)\}$, where $U'(p) = U(p) \cap X$, and take a locally finite refinement $\{U_\lambda\}$ of $\{U'(p)\}$. Let $\sum \varphi_\lambda = 1$ be a locally finite partition of unity subordinate to the refinement $\{U_\lambda\}$ and put $d(p, q) = \sum |\varphi_\lambda(p) - \varphi_\lambda(q)|$. Then $d(p, q)$ defines a pseudo-metric for X. Put $V = \{(p, q) \in X \times X;\, d(p, q) < 1/2\}$ and let $\tilde{V}$ be the interior of the closure of V taken in $\beta X \times \beta X$. We shall show that $(z, z) \notin \tilde{V}$ for each $z \in G$, which will complete the proof. Suppose that there is $z \in G$ such that $(z, z) \in \tilde{V}$, then $W(z) \times W(z) \subset \tilde{V}$ for some neighborhood $W(z)$ of z. Let p be a point of $W(z) \cap X$, then there exists only a finite number of φ_λ's, say $\varphi_1, \cdots, \varphi_n$, which do not vanish at p. We put $H_\lambda = \{x \in X;\, \varphi_\lambda(x) > 0\}$. Since $q \notin \bigcup_{k=1}^n H_k$ implies that $d(p, q) \geqq 1$, it follows that $W(z) \cap X \subset \bigcup_{k=1}^n H_k$ and hence z is contained in $\mathrm{Cl}_{\beta X}(\bigcup_{k=1}^n H_k)$. But H_k is clearly contained in some $U(p)$, since $\{U_\lambda\}$ is a refinement of $\{U'(p)\}$, and therefore $\mathrm{Cl}_{\beta X}(\bigcup_{k=1}^n H_k)$ is disjoint from G. Thus, we have a contradiction, and the necessity is then proved.

(Sufficiency.) Let $\{U_\nu\}$ be any open covering of X. For each U_ν there is U_ν^* which is open in βX such that $U_\nu^* \cap X = U_\nu$. Put $F_\nu = (U_\nu^*)^C$ and put $F = \bigcap_\nu F_\nu$, then F is a compact set which is contained in $\beta X - X$. There is a regularly open set $\tilde{V}$ containing Δ_X such that $\tilde{V} \cap \Delta_F = \phi$, in view of our assumption. Put $V_1 = \tilde{V} \cap (X \times X)$, then since X is assumed to be entirely normal there is a V_2 such that $V_2 \circ V_2 \subset V_1$. It follows that there is a countable family $\{V_n\}$ such that $V_n \circ V_n \subset V_{n-1}$. We now consider the space (X, τ) topologized by the uniformity $\{V_n\}$. Obviously, (X, τ) is pseudo-metrizable hence is paracompact.[8] Let d be the pseudo-metric such that $d(p, q) = 1$ whenever $(p, q) \notin V_1$, and consider a covering $\{G(p)\}$ of X, where $G(p) = \{q \in X;\ d(p, q) < 1/2^3\}$. Let $\{W_\lambda\}$ be a locally finite open refinement of $\{G(p)\}$, then $\mathrm{Cl}_{\beta X}(W_\lambda) \cap F = \phi$ as we now verify. Since the original topology for X is stronger than τ, $\{W_\lambda\}$ is necessarily an open locally finite refinement with respect to the original topology. By the same reason, $d_P(q) = d(p, q)$ is a bounded continuous function on X with respect to the original topology and hence it has a continuous extension d_P^* over βX, in view of Theorem 1.2. Suppose that $\mathrm{Cl}_{\beta X}(W_\lambda) \cap F \neq \phi$ for some W_λ, and let r be a point of $\mathrm{Cl}_{\beta X}(W_\lambda) \cap F$. Since $W_\lambda \subset G(p)$ for some $p \in X$, there is for each neighborhood $U(r)$ of r a point $q \in U(r) \cap X$ such that $d_P(q) < 1/2^3$. It follows that $d_P^*(r) \leqq 1/2^2$ and therefore $W(r) \cap X \subset V_2^*(p)$ for some neighborhood $W(r)$ of r, where $V_2^* = \{(p, q) \in X \times X;\ d(p, q) < 1/2\}$. It is clear that $V_2^* \circ V_2^* \subset V_1$. Therefore $(W(r) \cap X) \times (W(r) \cap X) \subset V_2^* \circ V_2^* \subset V_1 \subset \tilde{V}$, and we have $W(r) \times W(r) \subset \tilde{V}$. This implies that $(r, r) \in \tilde{V}$ and hence $\tilde{V} \cap \Delta_F \neq \phi$, which is a contradiction. Consequently, we have a locally finite covering $\{W_\lambda\}$ of X consisting of open sets W_λ in X such that $\mathrm{Cl}_{\beta X}(W_\lambda) \cap F = \phi$. Returning to the covering $\{U_\nu\}$ of X, we find that $\{U_\nu^*\}$ covers $\overline{W}_\lambda = \mathrm{Cl}_{\beta X}(W_\lambda)$, for $\{U_\nu^*\}$ covers $\beta X - F$ and $\overline{W}_\lambda \subset \beta X - F$. Since $\overline{W}_\lambda$ is compact, there is a finite number of U_ν^*, say $U_{\nu,1}^*, \cdots, U_{\nu,n}^*$ which cover $\overline{W}_\lambda$. Putting $W_\lambda \cap U_{\nu,k}^* = H_{\lambda,k}$, and constructing a finite open sets $H_{\lambda,k}$, for each λ in this way, we have a locally finite refinement $\{H_{\lambda,k}\}$ of $\{U_\nu\}$. The proof is completed.

From the proof of the preceding theorem, we have:

Corollary.[9] *A space X is paracompact if and only if for each compact set $F \subset \beta X - X$, there is an "entourage" V such that $\tilde{V} \cap \Delta_F = \phi$.*

We now give a characterization of the topological completeness.

Theorem 2.4.[10] *A space X is topologically complete if and only if for each point $p \in \beta X - X$, there is a locally finite partition of unity $\sum \varphi_\lambda = 1$ such that $0 \leqq \varphi_\lambda \leqq 1$ and $\varphi_\lambda^*(p) = 0$ for each λ, where $\varphi_\lambda \in C(X)$ and φ_λ^* denotes the*

8) See [**9**, p. 160] or [**14**].

9) J. Nagata's result [**10**, Corollary] may be derived from this corollary.

10) Cf. [**17**].

continuous extension of φ_λ over βX.

PROOF. From the proof of the sufficiency of the preceding theorem, it follows that if X is topologically complete, then for each point $p \in \beta X - X$, there is a covering $\{W_\lambda\}$ such that $\mathrm{Cl}_{\beta X}(W_\lambda) \not\ni p$. It should be noticed here that the space (X, τ) mentioned above is paracompact and that $\{W_\lambda\}$ is a locally finite covering of (X, τ). It follows that there is a locally finite partition of unity $\sum \varphi_\lambda = 1$ subordinate to $\{W_\lambda\}$. Since the original topology for X is stronger than τ, φ_λ is continuous and $\sum \varphi_\lambda = 1$ is a locally finite partition of unity with respect to the original topology for X. The necessity is then proved. Conversely, if the condition of the present theorem is satisfied, then there is for each point $p \in \beta X - X$ an "entourage" V such that $\tilde{V} \not\ni (p, p)$. (Let $V = \{(x, y) \in X \times X;\ \sum_\lambda |\varphi_\lambda(x) - \varphi_\lambda(y)| < 1/2\}$, then $\tilde{V} \not\ni (p, p)$ as may easily be seen from the proof of the necessity of the above theorem.) It follows that X is topologically complete, in view of Theorem 2.1.

A space X is said to be *pseudo-compact* if and only if $C(X) = C^*(X)$; in other words, every continuous function on X is bounded.

COROLLARY. *Every pseudo-compact topologically complete space is compact.*

PROOF. Suppose that X is topologically complete and is not compact, then there will be a point $p \in \beta X - X$. We have by Theorem 2.4 a locally finite partition of unity $\sum \varphi_\lambda = 1$ such that $\varphi_\lambda^*(p) = 0$ for each λ. It is clear that the number of φ_λ's is infinite, since $\sum_{k=1}^n \varphi_k = 1$ implies that $\sum_{k=1}^n \varphi_k^* = 1$ on βX which is impossible. Choose an enumerable infinite number of φ_λ's, say $\varphi_1, \cdots, \varphi_k, \cdots$, and put $h = \sum_{k=1}^\infty a_k \varphi_k$, where a_k is a constant such that there is $p \in X$ for which $a_k \varphi_k(p) > k$. Then h is evidently an unbounded continuous function on X, and therefore X can not be pseudo-compact. It follows that every pseudo-compact topologically complete space must be compact.

We now give a characterization of the pseudo-compactness in terms of the uniformity.

THEOREM 2.5.[11] *A space X is pseudo-compact if and only if each uniformity for X is totally bounded.*

PROOF. Notice first that if X is a dense subspace of Y and if X is pseudo-compact, then Y must be pseudo-compact. Let $\mathcal{U}$ be any uniformity for X and let $\hat{X}$ be the completion of $(X, \mathcal{U})$. Then $\hat{X}$ is obviously a pseudo-compact topologically complete space. Therefore $\hat{X}$ must be compact in view of the preceding corollary. It follows that $\mathcal{U}$ is totally bounded. Conversely, if each uniformity for X is totally bounded, then we have $\nu X = \beta X$ by corollary 2 of Theorem 2.1. It is clear that $\nu X = \beta X$ implies

11) This is a generalization of P. Samuel's result [**12**, Theorem XV].

$C(X)=C^*(X)$, since $C(X)=C(\nu X)$ and $C^*(X)=C(\beta X)$. This completes the proof.

§ 3. Completion of uniform space.

Let $(X,\{V_\alpha\})$ be a uniform space and let $(\hat{X}\{\hat{V}_\alpha\})$ be its completion. Let βX and $\beta\hat{X}$ be the Stone-Čech compactification of X and $\hat{X}$ respectively. Then $\beta\hat{X}$ is also a compactification of X, and therefore there is by Theorem 1.1 a continuous map φ of βX onto $\beta\hat{X}$ such that φ induces a homeomorphism on X and $\varphi(\beta X-X)=\beta\hat{X}-X'$, where $X'=\varphi(X)$. Putting $\Phi(p,q)=(\varphi(p),\varphi(q))$, we have a continuous map Φ of $\beta X\times\beta X$ onto $\beta\hat{X}\times\beta\hat{X}$ such that Φ induces a homeomorphism on $X\times X$ and $\Phi^{-1}(X'\times X')=X\times X$. We put $\varphi(X)=X'$ and $\Phi(X\times X)=X'\times X'$, and the image of V_α with respect to the map Φ will be denoted by V_α'. It will be assumed throughout that each $\hat{V}_\alpha$ is symmetric and regularly open in $\hat{X}\times\hat{X}$ and also that $V_\alpha'=\hat{V}_\alpha\cap(X'\times X')$. Then, we have:

Lemma 3.1. *The interior of the closure of V_α' taken in $\beta\hat{X}\times\beta\hat{X}$ is identical with that of $\hat{V}_\alpha$.*

Proof. To prove that $\mathrm{Int}_{\beta\hat{X}\times\beta\hat{X}}(\mathrm{Cl}_{\beta\hat{X}\times\beta\hat{X}}(V_\alpha'))=\mathrm{Int}_{\beta\hat{X}\times\beta\hat{X}}(\mathrm{Cl}_{\beta\hat{X}\times\beta\hat{X}}(\hat{V}_\alpha))$, it is only necessary to show that the restrictions on $X'\times X'$ of both sides of this equality is identical, by virtue of Lemma 1.3, (c). Since $\hat{V}_\alpha$ is assumed to be regularly open and since $V_\alpha'=\hat{V}_\alpha\cap(X'\times X')$, it follows that V_α' is regularly open in $X'\times X'$ by Lemma 1.3, (a), and therefore $V_\alpha'=\mathrm{Int}_{\beta\hat{X}\times\beta\hat{X}}(\mathrm{Cl}_{\beta\hat{X}\times\beta\hat{X}}(V_\alpha'))\cap(X'\times X')$ by Lemma 1.3, (b). Similarly, we have $\mathrm{Int}_{\beta\hat{X}\times\beta\hat{X}}(\mathrm{Cl}_{\beta\hat{X}\times\beta\hat{X}}(\hat{V}_\alpha))\cap(X'\times X')=\mathrm{Int}_{\beta\hat{X}\times\beta\hat{X}}(\mathrm{Cl}_{\beta\hat{X}\times\beta\hat{X}}(\hat{V}_\alpha))\cap(\hat{X}\times\hat{X})\cap(X'\times X')=\hat{V}_\alpha\cap(X'\times X')=V_\alpha'$, and the proof is completed.

Proposition 3.1. *Let $\tilde{V}_\alpha$ and V_α^* be the interiors of the closures of V_α and $\hat{V}_\alpha$ taken in $\beta X\times\beta X$ and $\beta\hat{X}\times\beta\hat{X}$ respectively. Then, $\tilde{V}_\alpha\supset\Phi^{-1}(V_\alpha^*)$.*

Proof. It is evident that $\tilde{V}_\alpha$ is regularly open and $\Phi^{-1}(V_\alpha^*)$ is open in $\beta X\times\beta X$. By virtue of Lemma 1.3, (c), it is sufficient to show that $\tilde{V}_\alpha\cap(X\times X)\supset\Phi^{-1}(V_\alpha^*)\cap(X\times X)$. Since Φ induces a homeomorphism on $X\times X$, it follows from Lemma 3.1 that $\Phi^{-1}(V_\alpha^*)\cap(X\times X)=\Phi^{-1}(V_\alpha^*)\cap\Phi^{-1}(X'\times X')\subset\Phi^{-1}((V_\alpha^*)\cap(X'\times X'))=\Phi^{-1}(V_\alpha')=V_\alpha=\tilde{V}_\alpha\cap(X\times X)$. The proof is completed.

The following proposition shows that the radical $\boldsymbol{R}$ is identical with the complete inverse image of $\Delta_{\hat{X}}$ (in $\beta\hat{X}\times\beta\hat{X}$) with respect to the map Φ.

Proposition 3.2. *A point $(p,q)\in\beta X\times\beta X$ is contained in $\boldsymbol{R}$ if and only if $\varphi(p)=\varphi(q)\in\hat{X}$.*

Proof. If $(p,q)\in\boldsymbol{R}$, then $(p,q)\in\tilde{V}_\alpha$ for each α, and there are open neighborhoods $U_\alpha(p)$, $W_\alpha(q)$ of p and q respectively such that $U_\alpha(p)\times W_\alpha(q)\subset\tilde{V}_\alpha$, since $\tilde{V}_\alpha$ is open. The similar argument done in the proof of Theorem 2.1 yields the fact that $\{C_\alpha\}=\{(U_\alpha(p)\cup W_\alpha(q))\cap X\}$ is a Cauchy filter relative to the uniformity $\{V_\alpha\}$ for X. Therefore $\{\varphi(C_\alpha)\}=\{C_\alpha'\}$ is also a Cauchy filter relative to the uniformity $\{\hat{V}_\alpha\}$ for $\hat{X}$. Since $\varphi: \beta X \to \beta\hat{X}$ is continuous and since $p\in \mathrm{Cl}_{\beta X}(C_\alpha)$, $q\in \mathrm{Cl}_{\beta X}(C_\alpha)$ both $\varphi(p)$ and $\varphi(q)$ are contained in $\mathrm{Cl}_{\beta\hat{X}}(C_\alpha')$, and it follows that $\varphi(p)=\varphi(q)\in\hat{X}$ by Proposition 2.1. Conversely, if $\varphi(p)=\varphi(q)\in\hat{X}$, then $(\varphi(p),\varphi(q))$ is contained in V_α^* and hence $(p,q)\in \Phi^{-1}(\varphi(p),\varphi(q))\subset\Phi^{-1}(V_\alpha^*)\subset\tilde{V}_\alpha$ for each α, by Proposition 3.1. It follows that $(p,q)\in \boldsymbol{R}$, and the proof is completed.

We are now able to complete the proof of Theorem 2.1.

PROOF OF THEOREM 2.1. (Sufficiency.) Suppose that $(X,\{V_\alpha\})$ is not complete, then there will be a point $q'\in\hat{X}$ which is not contained in $X'=\varphi(X)$. Let q be a point in βX such that $\varphi(q)=q'$, then $(q,q)\in\boldsymbol{R}$ by Proposition 3.2, and since $q\notin X$ it follows that $\boldsymbol{R}\neq\Delta_X$, which contradicts the assumption of the theorem. Thus, the sufficiency of the condition of Theorem 2.1 is proved.

We now give a new construction of the completion of a uniform space, which has close connection with that of H. Nakano [**11**]. To this end, we prepare a lemma which concerns with the closed relations.

LEMMA 3.2. *Let $\mathscr{R}$ be a closed relation on Y, and let X be a subspace of Y such that $\varphi^{-1}(\varphi(X))=X$, where φ denotes the canonical map of Y onto $Y/\mathscr{R}$. Let $\mathscr{R}_X$ be the restriction of $\mathscr{R}$ on X. Then the quotient space $X/\mathscr{R}_X$ is homeomorphic with $\varphi(X)$ and $\mathscr{R}_X$ is a closed relation on X.*

For the proof, see [**1**, p. 85, Proposition 2].

THEOREM 3.1. *Let $\hat{X}$ be the completion of a uniform space $(X,\{V_\alpha\})$ and let $\bar{X}$ be the subspace of βX consiting of the point $p\in\beta X$ such that $(p,p)\in\boldsymbol{R}$, where $\boldsymbol{R}$ denotes the radical of $(X,\{V_\alpha\})$. Then $\boldsymbol{R}$ defines a closed relation $\mathscr{R}$ on $\bar{X}$, and the completion $\hat{X}$ is homeomorphic with the quotient space $\bar{X}/\mathscr{R}$.*

PROOF. First, we observe that $\boldsymbol{R}$ defines a relation $\mathscr{R}$ on $\bar{X}$. According to Proposition 2.2, there is for each V_α a V_β such that $\tilde{V}_\beta\circ\tilde{V}_\beta\subset\tilde{V}_\alpha$. It follows that $\boldsymbol{R}\circ\boldsymbol{R}=\boldsymbol{R}$ and therefore $\boldsymbol{R}$ defines a relation on $\bar{X}$, since $\boldsymbol{R}$ is obviously symmetric and $\boldsymbol{R}\supset\Delta_{\bar{X}}$ (cf. [**9**, p. 9]). Next, the map φ of βX onto $\beta\hat{X}$ defines a closed relation $\mathscr{R}^*$ on βX and φ is precisely the canonical map of βX onto $\beta X/\mathscr{R}^*$ $(=\beta\hat{X})$. On the other hand, Proposition 3.2 shows that $p\in\bar{X}$ if and only if $\varphi(p)\in\hat{X}$. Therefore $\bar{X}=\varphi^{-1}(\hat{X})$ and consequently we have $\varphi^{-1}(\varphi(\bar{X}))=\bar{X}$. Finally, it follows from Proposition 3.2 that the relation $\mathscr{R}$ on $\bar{X}$ defined by the radical $\boldsymbol{R}$ is identical with the restriction on $\bar{X}$ of the relation $\mathscr{R}^*$. Now, the proof may easily be completed by Lemma 3.2.

REMARK 1. By virtues of Proposition 2.2 and Lemma 3.1, we can see that the family $\{\hat{V}_\alpha\}$, where $\hat{V}_\alpha=\mathrm{Int}_{\hat{X}\times\hat{X}}(\mathrm{Cl}_{\hat{X}\times\hat{X}}(V_\alpha'))$, defines a uniformity for $\hat{X}$, and therefore $(\hat{X},\{\hat{V}_\alpha\})$ is precisely the completion of $(X,\{V_\alpha\})$.

REMARK 2. It is easy to see that the restriction of φ on $\bar{X}$ is a closed map. Therefore, if $\bar{X}$ is normal (or paracompact), then $\hat{X}$ is also normal (respectively, paracompact[12]) as may easily be seen.

Corollary. *Let $\mathcal{U}$ and $\mathcal{U}'$ be two uniformities for X. Then, the completions of X relative to these two uniformities are homeomorphic each other if and only if the radicals are homeomorphic.*

Proof. By virtue of Proposition 3.2, it follows that $\boldsymbol{R}$ is determined by the map φ as follows: $\boldsymbol{R}=\{(p,q)\in\beta X\times\beta X;\ \varphi(p)=\varphi(q)\in\hat{X}\}$, and the necessity of the condition is then clear. The converse follows immediately from the preceding theorem.

We now discuss the possibility of extending continuous functions on X over the completion $\hat{X}$ in terms of the radical.

Theorem 3.2. *Let $\hat{X}$ be the completion of a uniform space $(X,\mathcal{U})$. Then, a continuous function $f\in C(X)$ has a continuous extension over $\hat{X}$ if and only if $\tilde{V}_{f,n}$ contains the radical $\boldsymbol{R}$ for each n, where $\tilde{V}_{f,n}$ is the interior of the closure of $V_{f,n}$ taken in $\beta X\times\beta X$ and $V_{f,n}=\{(p,q)\in X\times X;\ |f(p)-f(q)|<1/2^n\}$.*

Proof. Suppose that f has a continuous extension f over $\hat{X}$. Then, since $\bar{X}=\varphi^{-1}(\hat{X})$ (by Proposition 3.2), $\bar{f}=f\circ\varphi$ is a continuous function on $\bar{X}$. Let f^0 be the extension of f over X_f, then clearly $\bar{X}\subset X_f$ and we have $\bar{f}(p)=f\circ\varphi(p)=f^0(p)$ for each $p\in\bar{X}$. Reviewing the proof of Proposition 1.1, we can see that $(p,q)\in\tilde{V}_{f,n}$ if $(p,q)\in X_f\times X_f$ and $|f^0(p)-f^0(q)|<1/2^n$. If $(p,q)\in\boldsymbol{R}$, then $\varphi(p)=\varphi(q)\in\hat{X}$ by Proposition 3.2 and hence $f^0(p)=f^0(q)$, which implies that $(p,q)\in\tilde{V}_{f,n}$. Thus, we have $\boldsymbol{R}\subset\tilde{V}_{f,n}$ for each n. Conversely, if $\boldsymbol{R}\subset\tilde{V}_{f,n}$ for each n, then $\bar{X}\subset X_f$ by Proposition 1.1, and therefore $f^0(p)=f^0(q)$ for each point $(p,q)\in\boldsymbol{R}$. It follows that there is a function f on $\hat{X}$ such that $f\circ\varphi=\bar{f}$ $(=f^0)$ on $\bar{X}$. Since the restriction of φ on $\bar{X}$ is identical with the canonical map of $\bar{X}$ onto $\bar{X}/\mathcal{R}=\hat{X}$ and since $\bar{f}$ is continuous, it follows that f is a continuous function on $\hat{X}$[13]. It is clear that f is the desired extension of f, and the proof is completed.

Corollary 1. *Every uniformly continuous function has a continuous extension over $\hat{X}$, and the extension is also uniformly continuous.*

Proof. If f is uniformly continuous, then $V_{f,n}\supset V_\alpha$ for some V_α and therefore we have $\tilde{V}_{f,n}\supset\tilde{V}_\alpha\supset\boldsymbol{R}$. This shows that f has a continuous extension over $\hat{X}$. Let $\hat{f}$ be the extension of f over $\hat{X}$, and put $\hat{V}_{f,n}=\{(p,q)\in\hat{X}\times\hat{X};\ |\hat{f}(p)-\hat{f}(q)|\leqq 1/2^n\}$. Then $\hat{V}_{f,n}\supset\mathrm{Int}_{\hat{X}\times\hat{X}}(\mathrm{Cl}_{\hat{X}\times\hat{X}}(V_{f,n}))$ by Lemma 1.3, (b), and since $\tilde{V}_{f,n}\supset\tilde{V}_\alpha$ implies that $\mathrm{Int}_{\hat{X}\times\hat{X}}(\mathrm{Cl}_{\hat{X}\times\hat{X}}(V_{f,n}))\supset\mathrm{Int}_{\hat{X}\times\hat{X}}(\mathrm{Cl}_{\hat{X}\times\hat{X}}(V_\alpha))=\hat{V}_\alpha$, f is uniformly continuous. (cf. Remark 1 of Theorem 3.1.)

Corollary 2. *Let $\hat{X}$ be the completion of a uniform space $(X,\mathcal{U})$. Then, every bounded continuous function on X has a continuous extension over $\hat{X}$ if*

12) See [12].
13) See [1, p. 75, Théorème 1].

and only if the radical is contained in the diagonal $\Delta_{\beta X}$ of $\beta X\times\beta X$.

PROOF. It is easy to see that $\bigcap_{f\in C^*(X)}(\bigcap_{n=1}^{\infty} V_{f,n})=\Delta_{\beta X}$, and therefore the present corollary follows immediately from Theorem 3.2.

COROLLARY 3. *Let $\hat{X}$ be the completion of a uniform space $(X,\mathcal{U})$. Then every continuous function on X has a continuous extension over $\hat{X}$ if and only if $\boldsymbol{R}\subset\Delta_{\nu X}$, where $\boldsymbol{R}$ is the radical of $(X,\mathcal{U})$ and $\Delta_{\nu X}=\Delta_{\beta X}\cap(\nu X\times\nu X)$.*

PROOF. In view of Proposition 1.1, it follows that $\Delta_{\nu X}=\bigcap_{f\in C(X)}(\bigcap_{n=1}^{\infty} V_{f,n})$, and the proof may easily be completed by Theorem 3.2.

Finally, we observe another property of the radical of uniform space. We have seen that there is a continuous map φ of βX onto $\beta\hat{X}$ and that φ defines a closed relation $\mathcal{R}^*$ on βX. Recall that the set defined by the closed relation on a compact space E is closed in $E\times E$ (see, [**1**, p. 97, Proposition 8]). The set $\boldsymbol{R}^*$ defined by $\mathcal{R}^*$ is closed in $\beta X\times\beta X$, and we have $\boldsymbol{R}^*\cap(\bar{X}\times\bar{X})=\boldsymbol{R}$ by Proposition 3.2. It might be expected that the set $\boldsymbol{R}^*$ should be characterized by the radical $\boldsymbol{R}$. The following proposition establishes a relationship between $\boldsymbol{R}^*$ and the radical $\boldsymbol{R}$.

PROPOSITION 3.3. *$\boldsymbol{R}^*$ is minimal with respect to the following properties:*

(a) $\boldsymbol{R}^*\cap(\bar{X}\times\bar{X})=\boldsymbol{R}$,

(b) *$\boldsymbol{R}^*$ is the set defined by the relation on βX. That is, it satisfies the following conditions*: (1) $\boldsymbol{R}^*\supset\Delta_{\beta X}$; (2) *$\boldsymbol{R}^*$ is symmetric*; (3) $\boldsymbol{R}^*\circ\boldsymbol{R}^*=\boldsymbol{R}^*$.

PROOF. If $\boldsymbol{R}'$ is any set in $\beta X\times\beta X$ satisfying the above conditions, then $\boldsymbol{R}^0=\boldsymbol{R}'\cap\boldsymbol{R}^*$ satisfies these conditions as well. Let φ^0 denotes the canonical map of βX onto $\beta X/\mathcal{R}^0$, where $\mathcal{R}^0$ is the relation on βX defined by $\boldsymbol{R}^0$. It is easy to see that $\varphi^{0-1}(\varphi^0(\bar{X}))=\bar{X}$, and it follows from Lemma 3.2 that $\beta X/\mathcal{R}^0$ is a compact Hausdorff space containing $\beta\bar{X}/\mathcal{R}=\beta\hat{X}$ as a dense subspace. Therefore $\beta X/\mathcal{R}^0$ is a compactification of $\hat{X}$. On the other hand, we have $\beta X/\mathcal{R}^*=(\beta X/\mathcal{R}^0)/(\mathcal{R}^*/\mathcal{R}^0)$, in view of the definition of $\boldsymbol{R}^0$ (see, [**1**, p. 78, Proposition 3]). It therefore follows that $\beta X/\mathcal{R}^0=\beta X/\mathcal{R}^*$, by Theorem 1.1, and consequently we have $\boldsymbol{R}^0=\boldsymbol{R}^*$. This implies that $\boldsymbol{R}'\supset\boldsymbol{R}^*$, and the proof is completed.

It is not known to the writer whether the closure $\overline{\boldsymbol{R}}$ of the radical $\boldsymbol{R}$ taken in $\beta X\times\beta X$ is identical with $\boldsymbol{R}^*$ or not.

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