Journal of the Mathematical Society of Japan Vol. 11, No. 1, January, 1959

Compact homogeneous spaces and the first Betti number.

By Tadashi NAGANO

(Received April 25, 1958)

1. Introduction. The main purpose of this note is to prove:

THEOREM 1. *Let M be an n-dimensional homogeneous space G/H under a compact connected Lie group G. Then we have*

$$\dim S(p)+B_1=n,$$

where $S(p)$ is the orbit of an arbitrary point p in M under the maximal (connected) semi-simple subgroup S of G and B_1 denotes the first Betti number of M.

Note that H is not assumed to be connected. In the sequel we shall preserve these hypotheses and notations.

COROLLARY 1. *If G is semi-simple, then $B_1=0$* (T. Frankel [**3**]). The converse is not true (even if G is effective), but we have

COROLLARY 2. *If $B_1=0$, G contains a semi-simple subgroup which is transitive on M* (H. C. Wang [**10**]).

COROLLARY 3. *If $n\leqq B_1$, then M is homeomorphic to the torus and, furthermore if G is effective, G is an n-dimensional toral group* (D. Montgomery and H. Samelson [**6**] and A. Borel [**1**]).

COROLLARY 4. *Any finite covering space of M has the same first Betti number as M.*

In course of the proof of the above theorem, we shall establish:

THEOREM 2. *M admits a G-invariant Riemannian metric such that for a vector field u the following three conditions are equivalent:* 1) *u is parallel,* 2) *u is harmonic, and* 3) *u belongs to the center C^L of G^L of G and u is orthogonal to $S(p)$ at p.*

COROLLARY 5. *A vector field $u(\neq 0)$ on the homogeneous space M is parallel with respect to some G-invariant Riemannian metric if and only if u belongs to the centralizer of G^L in the Killing algebra of M with some G-invariant Riemannian metric and $u(p)$ is not tangent to $S(p)$.*

COROLLARY 6. *Let h be a vector field on M harmonic with respect to a G-invariant Riemannian metric g. Then h is parallel with respect to some G-invariant metric, if and only if h belongs to the Lie algebra K^L of a compact Lie transformation group K of M. If in particular h is Killing with respect to some metric, h is parallel with respect to some (other) metric.*

If a vector field u satisfies 1) in Theorem 2, clearly there exists, for any point in M, a hypersurface N containing p such that u is a non-zero normal vector of N at each point of N. Conversely if a vector field u is a Killing vector field on a compact Riemannian space $M=G/H$ and there exists, for any point p in M, a hypersurface as above, then u satisfies 1), as is seen from [**7**].

Another converse of Theorem 2 is also true: if 1) and 2) are equivalent, then the G-invariant Riemannian metric is necessarily the one characterized in the proof, i.e. they are equivalent to 3), or, in other words, there exists a connected abelian group T in the center of G such that the tangent space of $T(p)$ is the orthogonal complement of that of $S(p)$ with respect to the metric. This fact can be verified by means of Corollary 1 and Theorems 3.3 and 4.4 in Kostant [**4**] or a theorem in [**13**]. Therefore it will not be proved in this paper.

We shall also prove the

THEOREM 3. *If $G/H=M$ is a symmetric space, then the following three conditions are equivalent:* 1) *a vector field u is parallel,* 2) *u is harmonic, and* 4) *u belongs to the center C^L of G^L.*

This theorem generalizes and sharpens a theorem of M. Matsumoto [**5**]. If G/H is not symmetric, it is possible that the conclusion of Theorem 2 is false for any G-invariant Riemannian metric.

THEOREM 4. *If $G/H=M$ is a symmetric space and if the symmetries belong to G, then the $(2k+1)$-th Betti number vanishes for $k=0,1,2,\cdots$, and so, furthermore if M is orientable,* $\dim M$ *is even.*

Acknowledgements. The author wishes to express his hearty thanks to Professor Kentaro Yano for his instruction and encouragements. He also appreciates valuable suggestions given by my friend M. Ise.

2. Two lemmas. We have only to prove the propositions in the introduction for the case where G is effective. Clearly there exists a connected abelian subgroup T (compact or not) in the center of G, such that $S\cdot T$ is transitive on M and we have $\dim S(p)+\dim T=n$ as well as $\dim T=\dim T(p)$.[1] Let α and β be the distributions which maps a point p in M to the tangent space at p of $S(p)$ and that of $T(p)$ respectively. α and β are invariant under G, for S and T are normal subgroups of G. Hence there exists a G-invariant Riemannian metric with respect to which $\alpha(p)$ is the orthocomplement of $\beta(p)$

1) We denote by $\mathfrak{g}, \mathfrak{s}, \mathfrak{h}$ and $\mathfrak{c}$ the Lie algebras of G, S, H and the center of G respectively. The Lie algebra $\mathfrak{t}$ of T is defined by the condition that $\mathfrak{c}$ is the direct sum of $\mathfrak{t}$ and $\mathfrak{c}\cap(\mathfrak{s}+\mathfrak{h})$. We have $\dim\mathfrak{t}+\dim(\mathfrak{s}+\mathfrak{h})/\mathfrak{h}=\dim\mathfrak{g}/\mathfrak{h}$. Hence an orbit under the subgroup $S\cdot T$ contains a neighborhood. Since G is compact, G (therefore $S\cdot T$) can be assumed to be an isometry group. An open orbit under an isometry group is closed because it is complete. Thus $S\cdot T$ is transitive on G/H.

in the tangent space of M at any point p. We fix this metric throughout in this section and the next.

(2.1) *Any vector field u in the Lie algebra T^L is parallel.*

The Killing vector fields which are in G^L and orthogonal to u at a point p, $u(p)$ being assumed to be different from zero, form a vector subspace U^L of G^L; $\dim U^L = \dim G^L - 1$. Since U^L contains S^L, U^L is an ideal in G^L. Therefore u is orthogonal to each vector field in U^L at any point;

$$u_\alpha w^\alpha = 0 \qquad \text{for any } w \text{ in } U^L,$$

whence, taking account of Killing's equations satisfied by u and w [**11**], we find

$$0 = g^{\lambda\beta}\nabla_\beta(u_\alpha w^\alpha) = -(\nabla_\alpha u^\lambda)w^\alpha - u^\alpha \nabla_\alpha w^\lambda,$$

where $g_{\lambda\mu}$ is the metric tensor and ∇ is the covariant differentiation. On the other hand, u belonging to the center of G^L, we have [**11**]:

$$0 = \pounds_w u^\lambda = w^\alpha \nabla_\alpha u^\lambda - u^\alpha \nabla_\alpha w^\lambda \qquad \text{for any } w \text{ in } U^L.$$

From these two equations, we deduce

$$w^\alpha \nabla_\alpha u^\lambda = 0 \qquad \text{for any } w \text{ in } U^L.$$

Further the length of u, an element of the center of G^L, is constant on M, and so, from Killing's equation, follows

$$u^\alpha \nabla_\alpha u = 0.$$

The last two equations allow us to conclude that u is parallel, which completes the proof of (2.1).

(2.2) *Any harmonic vector field h belongs to T^L.*

A vector field h on M is said harmonic, if h satisfies two equations:

$$\nabla_\lambda h_\mu = \nabla_\mu h_\lambda \qquad \text{and} \qquad g^{\alpha\beta}\nabla_\alpha h_\beta = 0.$$

If (2.2) is proved under the assumption of orientability of M, (2.2) is valid also for the general case, as one finds by inducing the geometric objects in question to the double covering of M (which will be orientable). Hence we suppose that M is orientable. By (2.1), there is a harmonic vector field $\hat{h}$ which coincides with h mod T^L and is orthogonal to $T(p)$ at a point p. Assume $\hat{h}$ not equal to zero. The vector fields in S^L which are orthogonal to $\hat{h}$ at p form a vector subspace V^L of S^L; $\dim V^L = \dim S^L - 1$. Since the inner product of a harmonic vector field and a Killing one is constant on M [**2**], h is orthogonal to any element in V^L at every point of M;

$$h_\alpha v^\alpha = 0 \qquad \text{for any } v \text{ in } V^L.$$

From the fact that a harmonic form is invariant by any Killing vector field [**12**], follows

$$0=\pounds_w(h_\alpha v^\alpha)=h_\alpha \pounds_w v^\alpha$$

for any w in S^L, which means that V^L is an ideal in S^L. But $\dim V^L=\dim S^L-1$, and S^L must contain a one-dimensional ideal, contrary to the semi-simpleness of S^L. Thus we have $h(p)=0$, hence $h=0$ on M and (2.2) is proved.

We have just proved Theorem 2.

3. The proof of Theorem 1.

(3.1) *A G-invariant exact* 1*-form is zero.*

Let df be the 1-form where f is a differentiable function on M. Since M is compact, M admits a critical point of f, at which df vanishes. Being invariant by G, df is therefore a zero-valued form.

By (2.1) and (2.2) and the famous theorem of Hodge ([8, Corollaire 4, p. 159]), we see $\dim T^L=B_1$ and so $\dim S(p)+B_1=n$, provided that M is orientable. If M is not orientable, B_1 does not exceed the first Betti number of the double covering space of M ([**9**, Proposition 2 in the appendix]). Therefore we find $\dim S(p)+B_1\leqq n$. On the other hand, for any $u\in T^L$, the dual 1-form of u is closed due to (2.1). By (3.1) it is not exact unless it equals zero. Thus we deduce $\dim T^L\leqq B_1$ from de Rham's theorem ([8, Théorème 17′, p. 114]). Combining this with the other inequality above, we conclude Theorem 1.

The proof of Corollary 3. If $n\leqq B_1$, we have $\dim S(p)=0$, i.e. the effective group G_e homomorphic to G is abelian, because of Theorem 1. G_e is simply transitive, for an effective and transitive transformation group does not contain a non-trivial normal subgroup in its isotropy subgroup. Hence M is homeomorphic to G_e, which is an n-dimensional toral group.

The proof of Corollary 5. If u is parallel, it is Killing and harmonic. Hence u is invariant by G. u being parallel, the dual 1-form u' is closed. If u is tangent to $S(p)$, u' naturally induces a closed S-invariant 1-form u'' on $S(p)$, which must vanish by Corollary 1 and (3.1). Conversely assume that u belongs to the centralizer of G^L in the Killing algebra with respect to a G-invariant Riemannian metric and $u(p)$ is not tangent to $S(p)$. Then there exists a compact connected transitive group whose maximal semi-simple subgroup is S and whose Lie algebra contains u and G^L. We shall denote it by G'. We can define T in **2** so that its Lie algebra contains u and obtain a G'-invariant metric as in **2**. It follows from (2.1) that u is parallel.

The proof of Corollary 6. Assume that h belongs to K^L. Since h is invariant by G, the closure W of the one-parameter group generated by h is a toral group whose each element commutes with each element of G. Thus $G\cdot W$ is a compact Lie transformation group transitive on M. M admits a $G\cdot W$-invariant Riemannian metric. On the other hand $h(\neq 0)$ is not tangent to $S(p)$, for otherwise h induces on $S(p)$ a closed 1-form invariant by S, which is not exact by (3.1), contrary to Corollary 1. Applying Corollary 5, we conclude that h is parallel with respect to some G-invariant metric. The other parts of Corollary 6 are now obvious.

4. The symmetric space. Assume that $M=G/H$ is symmetric. We remove the metric considered in the preceding sections.

(4.1) *If a vector field u on M is invariant by G, $u(p)$ is orthogonal to $S(p)$ at each point p in M.*

Let $v(p)$ denote the orthogonal projection of $u(p)$ to the tangent space of $S(p)$. Then the vector field v which assigns $v(p)$ to each point p is invariant by G. On $S(p)$, v is an S-invariant vector field. Since the involutive automorphism of G leaves S invariant, we find that $S(p)=S/S\cap H$ is also a symmetric space. By Cartan's theorem [**2**] the dual 1-form of v is closed. It vanishes at any point on $S(p)$ by Corollary 1 and (3.1). Hence v is zero on M, which proves (4.1).

THE PROOF OF THEOREM 3. In the notation of the paragraph 2, $T(p)$ is orthogonal to $S(p)$, owing to (4.1). Hence the conclusions in Theorem 2 hold for our space M. Further by (4.1), the condition 3) in Theorem 2 is equivalent to 4) in Theorem 3.

THE PROOF OF THEOREM 4. We have only to consider the case where M is orientable, as one sees from the remarks in **2** and **3**. Any harmonic form of degree $(2k+1)$ is invariant under G [**12**], and so by the symmetry with respect to any point p in M. It induces the linear transformation $\lambda: X\to -X$ on the tangent space of M at p. Any $(2k+1)$-form invariant by λ is obviously zero. Theorem 4 follows now from Hodge's theorem.

Remark on the proof of Theorem 1.

Y. Matsushima informed the author an algebraic proof of Theorem 1, whose outline we shall give here. By a well known theorem [**2**], he needs no orientable covering. Let L be the totality of linear forms α on G^L satisfying the conditions; 1) $\alpha([G^L, G^L])=\alpha(S^L)=0$, 2) $\alpha(ad\ h\cdot X)=\alpha(X)$ for any $h\in H$ and $X\in G^L$, and 3) $\alpha(H^L)=0$. By Cartan's theorem we have $\dim L=B_1$. Let M^L be the orthocomplement of H^L in G^L with respect to a positive definite bilinear form ϕ on G^L invariant under $ad(G)$, the adjoint group of G. Let ρ denote the mapping of L into G^L having the properties: $\alpha(X)=\phi(\rho(\alpha), X)$ for each $X\in G^L$. We have $\rho(L)=C^L\cap M^L$ where C is the center of G. It follows that $\rho(L)$ is the orthocomplement of H^L+S^L. Denoting by N^L the orthocomplement of H^L in S^L+H^L, we obtain $G^L=N^L+H^L+\rho(L)$ (direct sum), which implies $\dim M=\dim G^L-\dim H^L=\dim N^L+\dim \rho(L)=\dim N^L+B_1$. On the other hand $\dim N^L=\dim(S^L+H^L/H^L)=\dim(S/S\cap H)=\dim S(p)$ $(p\in M)$. Thus the proof of Theorem 1 is completed.

University of Tokyo.

Bibliography

[1] A. Borel, Some remarks about Lie groups transitive on spheres and tori, Bull. Amer. Math. Soc., **55** (1949), 580-587.

[2] E. Cartan, Sur les invariants intégraux de certains espaces. Ann. Soc. Polon. Math., **8** (1929), 181-225.

[3] T. Frankel, Homology and flows on manifolds, Ann. of Math., **65** (1957), 331-339.

[4] B. Kostant, Holonomy and the Lie algebra of infinitesimal motions of a Riemannian manifold, Trans. Amer. Math. Soc., **80** (1955), 528-542.

[5] M. Matsumoto, Some applications of Bochner's method to Riemannian spaces, Mem. Coll. Sci. Univ. Kyôto, **27** (1952), 168-188.

[6] D. Montgomery and H. Samelson, Groups transitive on the n-dimensional torus, Bull. Amer. Math. Soc., **49** (1943), 455-456.

[7] T. Nagano, Sur des hypersurfaces et quelques groupes d'isométries d'un espace riemannian, to appear in Tôhoku Math. J.

[8] G. de Rham, Variétés différentiables, Act. Sci. et Ind., **122** (1955).

[9] J. P. Serre, Homologie singulière des espaces fibrés, Applications. Ann. of Math., **54** (1951), 425-505.

[10] H. C. Wang, Closed manifolds with homogeneous complex structure, Amer. J. Math., **76** (1954), 1-32.

[11] K. Yano, The theory of Lie derivatives and its applications, North-Holland Publ. Co., 1957.

[12] K. Yano and S. Bochner, Curvature and Betti numbers, Ann. of Math. Studies, **32** (1953).

[13] K. Yano and T. Nagano, The de Rham decomposition, isometries and affine transformations in Riemannian spaces, to appear.