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# On universal tensorial forms on a principal fibre bundle.

By Yoshihiro TASHIRO

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The concept of the connection of generalized spaces due to E. Cartan has been recently clarified by several authors in the light of the notion of fibre bundles. In particular, S. S. Chern [3], [4] and Ambrose-Singer [1] have generalized the covariant differentiation of tensors and tensorial forms in affinely connected manifolds to the case of principal fibre bundles with connection. S. S. Chern [3] has shown thereby in the case of affine connection that tensorial forms on the base space are in one-one correspondence with certain forms on the bundle of frames with some characteristic properties. In this paper, we shall generalize this result to the case of any principal fibre bundle with connection. After preliminaries (§ 1), we shall define namely (in § 2) the *universal tensorial forms* on the bundle space which are in one-one correspondence with tensorial forms on the base space (Theorem 1). The covariant differential in Ambrose-Singer's sense of a universal tensorial form will be given by an explicit formula (Theorems 2, 3). Finally we shall give a useful characterization (Theorem 4) of the universal tensorial form by means of the covariant differentiation, generalizing the results of S. S. Chern [4] and Boothby [2].

**§ 1. Preliminaries on connection.** Let $\mathcal{B}=\{B, X, G, G\}$ be a differentiable principal bundle. Thus we assume that the bundle space $B$ and the base space $X$ of $B$ are differentiable spaces, the fibre $G$ (indicated by the first $G$) and the structural group $G$ (indicated by the second $G$) are the same Lie group, and that the projection $p$, and the coordinate functions $\varphi_\alpha \in \Phi$, are differentiable maps.[1)]

1) Throughout this paper we shall always assume that spaces and maps are of differentiability of a suitable high class.

The tangent space $T(B)$[2] of the bundle space $B$ has a structure of principal bundle, $T(\mathcal{B})=\{T(B), T(X), T(G), T(G)\}$,[3] whose projection is identical with the induced tangential map $p_*$ and whose coordinate functions identical with the induced ones $\varphi_{\alpha *}, \varphi_\alpha \in \Phi$.

By means of the inclusion of the base space $X$ into the tangent bundle $T(X)$ as the trivial cross-section[4], we obtain a bundle $\mathcal{V}=\{V, X, T(G), G\}$[5], the portion of $T(\mathcal{B})$ over $X$, which we call the ***vertically tangent bundle*** of $\mathcal{B}$ and whose elements are called ***vertically tangent vectors***, or simply ***vertical vectors***, of $\mathcal{B}$. If $W$ is a vertical vector, $p_*(W)$ is clearly a null vector on $X$, $p_*(W)=0$. The linear space, spanned by vertical vectors at $b\in B$, is denoted by $V_b$, which can be identified with $T_b(G_x)$, $G_x$ being the fibre over $x=p(b)$.

The Lie algebra $L(G)$ of $G$ gives rise to an isomorphic Lie algebra $\mathcal{Q}$ of vertical vector fields $Q$[6]. This isomorphism of $L(G)$ onto $\mathcal{Q}$ is denoted by $q$. If $W\in V_b$, then it is clear that there exists a unique $Q\in\mathcal{Q}$ such that $Q(b)=W$; we then say the vector field $Q$ and its inverse image $q^{-1}(Q)\in L(G)$ are ***generated by W***.

The right translation[7] on $\mathcal{B}$ by $g\in G$, is also denoted by $\mathrm{r}(g)$. The inner automorphism of $G$ corresponding to $g\in G$ is denoted by $\mathrm{A}(g)$, i. e. $\mathrm{A}(g)h=ghg^{-1}$ for any $h\in G$; the induced tangential map $\mathrm{A}(g)_*$, or especially its contraction on $L(G)=T_e(G)$, is as usual denoted by $\mathrm{ad}(g)$. If, for $Q\in\mathcal{Q}$, we define $\mathrm{ad}(g)Q$ by $\mathrm{ad}(g)Q=q(\mathrm{ad}(g)q^{-1}(Q))$, then we have $\mathrm{r}(g)_*Q=\mathrm{ad}(g^{-1})Q$.

The following well-known definitions of connection on a principal bundle $\mathcal{B}=\{B, X, G, G\}$ are easily shown to be equivalent to each

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2) We denote the tangent and cotangent (differential) spaces at a point $x$ of a spaces $X$ by $T_x(X)$ and $T^*_x(X)$ respectively. By $T(X)$ and $T^*(X)$ we means the bundle space of the tangent or cotangent bundles respectively, i. e. $T(X)=\bigcup_{x\in X} T_x(X)$ and $T^*(X)=\bigcup_{x\in X} T^*_x(X)$. If $X, Y$ are two spaces and there is a map $f: X\to Y$, then the induced tangential map $T(X)\to T(Y)$ is denoted by $f_*$ and the induced differential map $T^*(Y)\to T^*(X)$ by $f^*$. $f^*$ will sometimes be used to represent the induced linear map between the bundles of exterior differential algebras. Cf. S. S. Chern [4].

3) Cf. Y. Tashiro [9].

4) N. Steenrod [8].

5) Y. Tashiro [9].

6) W. Ambrose and I. M. Singer [1].

7) N. Steenrod [8]; W. Ambrose and I. M. Singer [1].

other:

I.[8] There is an assignment $H$, called a *connection*, which

i) assigns to each point $b \in B$ a linear subspace $H_b$ complementary to $V_b$ in $T_b(B)$, and

ii) is invariant under the right translation by any $g \in G$, i. e. $H_{bg} = \mathrm{r}(g)_* H_b$ for any $b \in B$.

II.[9] There is an $L(G)$-valued 1-form $\pi$ on $B$, called a *connection form on* $B$, such that

i) $\pi(W) = q^{-1}(Q)$, for each vertical vector $W = Q(b)$, and

ii) $\mathrm{r}(g)^* \pi = \mathrm{ad}(g^{-1})\pi$.

III[10] There is a system $\theta = \{\theta_\alpha\}$ of $L(G)$-valued 1-forms in $X$, called *connection forms on* $X$, such that

i) each component $\theta_\alpha$ is defined in the corresponding neighborhood $U_\alpha$ of $X$, and

ii) if $U_\alpha \cap U_\beta \neq \phi$, then $\theta_\alpha$ and $\theta_\beta$ are in relation

$$\theta_\alpha = \mathrm{ad}(g_{\beta\alpha}^{-1})\theta_\beta + g_{\beta\alpha}^* \omega \tag{1.1}$$

where $g_{\beta\alpha}$ is the coordinate transformation $g_{\beta\alpha} : U_\alpha \cap U_\beta \to G$ and $\omega$ is the left invariant $L(G)$-valued form on $G$ giving the identity map $L(G) \to L(G)$.

A connection $H$ gives the unique decomposition of $T_b(B)$ of the form $T_b(B) = H_b + V_b$, and the unique projections of tangent vectors at $b$ into $H_b$ and $V_b$ are written by the same letters $H$ and $V$ respectively.

The relation between a connection $H$ and its form $\pi$ is given by[11]

$$H_b = \{ W \in T_b(B) \mid \pi_b(W) = 0 \} . \tag{1.2}$$

The relations between a connection form $\pi$ on $B$ and a component $\theta_\alpha$ in $U_\alpha$ is given by[12]

$$\pi = \mathrm{ad}(p_\alpha(b)^{-1}) p^* \theta_{\alpha, p(b)} + p_\alpha^* \omega_{p_\alpha(b)} , \tag{1.3}$$

where $p_\alpha : p^{-1}(U_\alpha) \to G$ defined by $p_\alpha | G_x = \varphi_{\alpha,x}^{-1}$.

8) C. Ehresmann [5], W. Ambrose and I. M. Singer [1], T. Ōtsuki [6].
9) See papers cited in 8) and S. S. Chern [3].
10) S. S. Chern [3], T. Ōtsuki [6].
11) W. Ambrose and I. M. Singer [1], T. Ōtsuki [6].
12) S. S. Chern [3] and T. Ōtsuki [6].

**§2. Tensorial form and universal tensorial form.** We consider a principal fible bundle $\mathcal{B}=\{B, X, G, G\}$ in which a connection is given by a system of connection forms $\{\theta_\alpha\}$. Let $E$ be an $N$-dimensional vector space, $L_N$ the group of all linear transformations of $E$. Let $M$ be a representation of $G$ on $E$: $M: G\rightarrow L_N$. A *tensorial form*[13] on the base space $X$ of degree $r$ and of type $M$, simply called an *M-tensorial r-form*, is a system $u=\{u_\alpha\}$ of $E$-valued $r$-forms, each component $u_\alpha$ of which is defined in a corresponding neighborhood $U_\alpha$, such that, if $U_\alpha \cap U_\beta \neq \phi$, $u_\alpha$ and $u_\beta$ are related by the equation

$$u_\alpha = M(g_{\beta\alpha}^{-1})u_\beta\,. \tag{2.1}$$

The representation $M$ induces the representation $\overline{M}$ of the Lie algebra $L(G)$ into $L(L_N)$. An element of $L(L_N)$ can be represented by an $(N, N)$-matrix, and we may identify it with a linear endomorphism on $E$. With this understanding, the equation (1.1) goes under $\overline{M}$ into the equation

$$dM(g_{\beta\alpha}) = M(g_{\beta\alpha})\overline{M}(\theta_\alpha) - \overline{M}(\theta_\beta)M(g_{\beta\alpha})\,. \tag{2.2}$$

Although the exterior differential $du=\{du_\alpha\}$ is in general not a tensorial form, the equation obtained by exterior differentiation of (2.1) shows, together with the above equation (2.2), that

$$Du_\alpha = du_\alpha + \overline{M}(\theta_\alpha)\wedge u_\alpha \tag{2.3}$$

is a component of a tensorial $(r+1)$-form on $X$ of the same type $M$, which is denoted by $Du$ and called the *covariant differential* of the original tensorial form $u$.

Now for an $M$-tensorial $r$-form $u=\{u_\alpha\}$ on the base space $X$, we define an $E$-valued form $\tilde{u}$ on the bundle space $B$ by

$$\tilde{u}_b = M(g^{-1})p^* u_{\alpha,x}\,, \qquad x=p(b)\in U_\alpha,\ g=p_\alpha(b)\,. \tag{2.4}$$

By making use of (2.1), it is easily seen that this definition is independent of the choice of coordinate neighborhood. The $E$-valued form $\tilde{u}$ on $B$ thus defined from $u$ is called the *universal M-tensorial form* of $u$. Then we shall call $u$ the covered form of $\tilde{u}$.

An (ordinary or $E$-valued) $r$-form $\tilde{\varphi}$ on $B$ is said to be *vertically*

13) S. S. Chern [3].

*null* if $\tilde{\varphi}(W_1 \wedge \cdots \wedge W_r) = 0$ where at least one of $W$'s is vertical. Then we have the following

THEOREM 1. *The necessary and sufficient condition that an $E$-valued $r$-form $\tilde{u}$ on $B$ is the universal tensorial form of a tensorial form on the base space $X$ is that it is vertically null and is transformed, under right translation* $\mathrm{r}(h), h \in G$, *according to the equation*

$$\mathrm{r}(h)^* \tilde{u} = M(h^{-1}) \tilde{u} \,. \tag{2.5}$$

The necessity of the first condition is clear and that of the latter condition is proved as follows: For $W_1, \cdots, W_r \in T_b(B)$, we have

$$\begin{aligned}
(\mathrm{r}(h)^* \tilde{u}_{bh})(W_1 \wedge \cdots \wedge W_r) &= \tilde{u}_{bh}(\mathrm{r}(h)_*(W_1 \wedge \cdots \wedge W_r)) \\
&= M((gh)^{-1}) p^* u_\alpha(\mathrm{r}(h)_*(W_1 \wedge \cdots \wedge W_r)) \\
&= M(h^{-1}) M(g^{-1}) u_\alpha((p \circ \mathrm{r}(h))_*(W_1 \wedge \cdots \wedge W_r)) \\
&= M(h^{-1}) M(g^{-1}) u_\alpha(p_*(W_1 \wedge \cdots \wedge W_r)) \\
&= M(h^{-1}) M(g^{-1}) p^* u_\alpha(W_1 \wedge \cdots \wedge W_r) \\
&= M(h^{-1}) \tilde{u}_b(W_1 \wedge \cdots \wedge W_r) \,,
\end{aligned}$$

that is, (2.5) holds. To prove the sufficiency, we consider a diagram

$$\begin{tikzcd}
 & G \arrow[r, "\varphi_{\alpha,x}"] & G_x \arrow[d, phantom, "\cap"] \\
U_\alpha \times G \arrow[rr, "\varphi_\alpha"'] & & p^{-1}(U_\alpha) \subset B \arrow[ul, "p_\alpha"'] \arrow[dl, "p"] \\
 & U_\alpha \arrow[ul, "\rho_\alpha"] &
\end{tikzcd}$$

where $\rho_\alpha$ is defined by $\rho_\alpha(x) = (x, e)$, $e$ being the neutral element of $G$. For $b = \varphi_\alpha(x, g) \in p^{-1}(U_\alpha)$, we have $b = \mathrm{r}(g)\varphi_\alpha \rho_\alpha p(b)$ and $b = \varphi_{\alpha,x} p_\alpha(b)$, and hence any vector $W \in T_b(B)$ is decomposed into

$$W = (\mathrm{r}(g) \circ \varphi_\alpha \circ \rho_\alpha \circ p)_* W + (\varphi_{\alpha,x} \circ p_\alpha)_* W \,, \tag{2.6}$$

where we have to note that the last term is a vertical vector. If we put

$$u_\alpha = \rho_\alpha^* \varphi_\alpha^* \tilde{u}\,, \tag{2.7}$$

then, by means of the vertical nullity of $\tilde{u}$ and the condition (2.5), we have

$$\begin{aligned}\tilde{u}(W_1 \wedge \cdots \wedge W_r) &= \tilde{u}((\mathrm{r}(g) \circ \varphi_\alpha \circ \rho_\alpha \circ p)_*(W_1 \wedge \cdots \wedge W_r)) \\ &= p^* \rho_\alpha^* \varphi_\alpha^* \mathrm{r}(g)^* \tilde{u}(W_1 \wedge \cdots \wedge W_r) \\ &= M(g^{-1}) p^* u_\alpha(W_1 \wedge \cdots \wedge W_r)\,,\end{aligned}$$

i. e.,

$$\tilde{u} = M(g^{-1}) p^* u_\alpha\,.$$

If $b \in p^{-1}(U_\alpha \cap U_\beta)$ and $g' = p_\beta(b)$, then $g'g^{-1} = g_{\beta\alpha}(x)$ and we have

$$p^* u_\alpha = M(g_{\beta\alpha}(x)^{-1}) p^* u_\beta\,.$$

Since the projection $p: B \rightarrow X$ is onto, we have finally

$$u_\alpha = M(g_{\beta\alpha}(x)^{-1}) u_\beta\,,$$

which shows that the set $\{u_\alpha\}$ constitutes an $M$-tensorial $r$-form $u$ on $X$.

**§ 3. Covariant differential.** According to W. Ambrose and I. W. Singer[14], we define the *covariant differential* $D\tilde{\varphi}$ of any (ordinary or $E$-valued) form $\tilde{\varphi}$ on $B$ with respect to a given connection $H$ by the $(r+1)$-form

$$D\tilde{\varphi} = H^* d\tilde{\varphi}\,. \tag{3.1}$$

Then we have

THEOREM 2. *For a universal $M$-tensorial $r$-form $\tilde{u}$, we have an explicit formula*

$$D\tilde{u} = d\tilde{u} + \overline{M}(\pi) \wedge \tilde{u}\,, \tag{3.2}$$

*where $\pi$ is the connection form of the connection $H$ on $B$.*

First of all we notice that the equation (1.3) goes under the representation $\overline{M}$ into

14) [1].

$$dM(p_\alpha)=M(p_\alpha)\overline{M}(\pi)-\overline{M}(p^*\theta_\alpha)M(p_\alpha)\,. \tag{3.3}$$

It is sufficient to prove the formula (3.2) for any set $W_1,\cdots,W_{r+1}$ of horizontal and vertical vector fields which span the tangent space at each point $b$ of $B$. From the definition (3.1), the left hand side of (3.2) is clearly vertically null. On the other hand, the right hand side becomes, by a well-known formula[15],

$$\begin{aligned}
&(d\tilde{u}+\overline{M}(\pi)\wedge\tilde{u})\,(W_1\wedge\cdots\wedge W_{r+1})\\
&=\sum_{i=1}^{r+1}(-1)^{i+1}W_i(\tilde{u}(W_1\wedge\cdots\wedge\hat{W}_i\wedge\cdots\wedge W_{r+1})\\
&+\sum_{i<j}(-1)^{i+j}\tilde{u}([W_i,W_j]\wedge W_1\wedge\cdots\wedge\hat{W}_i\wedge\cdots\wedge\hat{W}_j\wedge\cdots\wedge W_{r+1})\\
&+\sum_{i=1}^{r+1}(-1)^{i+1}\overline{M}(\pi(W_i))\tilde{u}(W_1\wedge\cdots\wedge\hat{W}_i\wedge\cdots\wedge W_{r+1})\,,
\end{aligned}$$

the symbol ^ denoting the omission of the factors. It vanishes clearly if at least three of $W_i$ are vertical, and so does it also if two of them, say $W_1=Q_1$ and $W_2=Q_2$, are vertical, because $[Q_1,Q_2]$ is vertical. If one of them, say $W_1=Q$, is vertical, then, by taking account of $p_*Q=0$, the equations (2.4), (3.3) and the vertical nullity of $\tilde{u}$, we obtain

$$\begin{aligned}
&d\tilde{u}(Q\wedge W_1\wedge\cdots\wedge W_r)\\
&\quad=(dM(p_\alpha^{-1})\wedge p^*u_\alpha+M(p_\alpha^{-1})p^*du_\alpha)\,(Q\wedge W_1\wedge\cdots\wedge W_r)\\
&\quad=((-\overline{M}(\pi)M(p_\alpha^{-1})+M(p_\alpha^{-1})\overline{M}(p^*\theta_\alpha))\wedge p^*u_\alpha)\,(Q\wedge W_1\wedge\cdots\wedge W_r)\\
&\quad=(-\overline{M}(\pi)\wedge\tilde{u})\,(Q\wedge W_1\wedge\cdots\wedge W_r)\,.
\end{aligned}$$

Hence we know that the right hand side of (3.2) vanishes also in this case. If all $W_i$ are horizontal vectors, i. e., $W_i=HW_i$, then we see, in consequence of $\pi(W_i)=0$, the both sides of (3.2) are identical with each other. Thus, in any case, the formula (3.2) holds.

Making use of the equation (3.3), we can easily verify

THEOREM 3. *The covariant differential $D\tilde{u}$ of a universal tensorial*

15) R. S. Palais [7].

*form $\tilde{u}$ is the universal tensorial form of the covariant differential Du of the covered form u on X, that is,*

$$D\tilde{u}=\widetilde{Du}\,.$$

It is well known that, *if the structural group G is connected, then a necessary and sufficient condition that a form $\tilde{\varphi}$ on B is the $p^*$ image of a form $\varphi$ on X, is that both $\tilde{\varphi}$ and $d\tilde{\varphi}$ are vertically null.* This fact will be used to prove the following

THEOREM 4. *Let the structural group G be connected. Then, a necessary and sufficient condition that an E-valued form $\tilde{u}$ on B is a universal tensorial form is that both $\tilde{u}$ and*

$$d\tilde{u}+\overline{M}(\pi)\wedge\tilde{u}$$

*are vertically null.*

This is a generalization of the theorems due to S. S. Chern and W. M. Boothby[16].

The necessity is clear. To prove the sufficiency, we define

$$\tilde{u}_\alpha=M(p_\alpha)\tilde{u}$$

in $p^{-1}(U_\alpha)$. Then, by the equation (3.3) and our conditions, it is easily seen that both $\tilde{u}_\alpha$ and

$$d\tilde{u}_\alpha=dM(p_\alpha)\wedge\tilde{u}+M(p_\alpha)d\tilde{u}$$

are vertically null. Hence $\tilde{u}_\alpha$ may be written as

$$\tilde{u}_\alpha=p^*u_\alpha\,,$$

$u_\alpha$ being an $E$-valued form in $U_\alpha\subset X$. In another neighborhood $U_\beta$ with $U_\alpha\cap U_\beta\neq\phi$, we have also an $E$-valued form $u_\beta$ such that

$$p^*u_\beta=M(p_\beta)\tilde{u}\,.$$

By $g_{\beta\alpha}=p_\beta p_\alpha^{-1}$, $u_\alpha$ and $u_\beta$ are in relation

$$p^*u_\alpha=M(g_{\beta\alpha}^{-1})p^*u_\beta$$

and, since $p: B\to X$ is onto, they are moreover in relation

16) S. S. Chern [4], W. M. Boothby [2].

$$u_\alpha = M(g^{-1}_{\beta\alpha})u_\beta\ ,$$

which shows that the set $u=\{u_\alpha\}$ is an $M$-tensorial form on the base space $X$.

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