Journal of the Mathematical Society of Japan Vol. 6, No. 1, March, 1954

Note on Betti numbers of Riemannian manifolds III.

By Yasuro TOMONAGA.

(Received July 27, 1953)

In continuation of our former papers,[1] we consider a Riemannian manifold which is compact orientable and whose fundamental differential form is positive definite. Consider an infinitesimal motion

(1) $$\bar{x}^i = x^i + \varepsilon X^i ,$$

where X^i satisfies Killing's equations

(2) $$X_{i;j} + X_{j;i} = 0 .$$

It follows from (2) that

(3) $$X_{i;j;k} + R_{ijkl} X^l = 0 .$$

If X^i satisfies only (3), the transformation (1) defines an infinitesimal affine collineation.

From (3) we have

(4) $$\varDelta X^i = -R^i{}_j X^j .$$

Consider a symmetric tensor A_{ij} whose quadratic form

$$A_{ij} f^i f^i$$

is positive definite. Then we get by the same way as before

(5) $$-\int\left(\frac{1}{2} \varDelta A_{ij} + A_{ik} R^k_j\right) X^i X^j \, dv + \int A_{ij} X^{i;k} X^j{}_{;k} \, dv = 0 .$$

Hence, if the equadratic form

(6) $$Q = \left(\frac{1}{2} \varDelta A_{ij} + A_{ik} R^k_j\right) f^i f^j$$

is everywhere negative definite, it follows from (5) that

$$X^i = 0 ,$$

1) Note on Betti numbers of Riemannian manifolds I, II, Jour. Math. Soc. Japan, 5 (1953), 59-69.

i. e. our manifold admits no infinitesimal motion (affine collineation). If Q is everywhere negative semi-definite, we have

$$X_{i;j}=0,$$

hence the vector X^i contains at most n arbitrary constants.

Especially in the case in which

$$A_{ij}=\rho^2 g_{ij} \qquad (\rho\neq 0)$$

we have

$$Q=\frac{1}{2}(\Delta\rho^2)f_i f^i+\rho^2 R_{ij}f^i f^j \tag{7}$$

$$\leqq\left\{\frac{1}{2}(\Delta\rho^2)+\frac{R}{n}\rho^2+\rho^2\sqrt{R_{ab}R^{ab}-\frac{R^2}{n}}\right\}f_i f^i.$$

Hence, if there exists a scalar such that the inequality

$$\frac{1}{2}\left(\frac{\Delta\rho^2}{\rho^2}\right)+\frac{R}{n}+\sqrt{R_{ab}R^{ab}-\frac{R^2}{n}}<0 \tag{8}$$

holds everywhere, then our manifold admits no infinitesimal motion (affine collineation). If it holds that

$$\frac{1}{2}\left(\frac{\Delta\rho^2}{\rho^2}\right)+\frac{R}{n}+\sqrt{R_{ab}R^{ab}-\frac{R^2}{n}}\leqq 0, \tag{9}$$

then X^i satisfies

$$X_{i;j}=0.$$

Especially in the case

$$\rho^2=1+c^2R^2 \tag{10}$$

we have the

THEOREM 1. *If the inequality*

$$\frac{1}{2}\left(\frac{c^2\Delta R^2}{1+c^2R^2}\right)+\frac{R}{n}+\sqrt{R_{ab}R^{ab}-\frac{R^2}{n}}<0 \tag{11}$$

holds everywhere for a certain constant c, our manifold admits no infinitesimal motion (affine collineation). If

$$\frac{1}{2}\left(\frac{c^2\Delta R^2}{1+c^2R^2}\right)+\frac{R}{n}+\sqrt{R_{ab}R^{ab}-\frac{R^2}{n}}\leqq 0 \tag{12}$$

holds, then X^i satisfies

$$X_{i;j}=0\,.$$

By the same way, we have, for Betti numbers, the

THEOREM 2. *If the inequality*

$$\frac{1}{2}\left(\frac{c^2 \varDelta R^2}{1+c^2R^2}\right)\leqq\frac{R}{n}-\sqrt{R_{ab}R^{ab}-\frac{R^2}{n}} \tag{13}$$

holds everywhere for a certain constant c, then we have

$$B_1\leqq n\,.$$

If, in (13), *the equality sign can be omitted, then we have*

$$B_1=0\,.$$

THEOREM 3. *If the inequality*

$$\frac{1}{2}\left(\frac{c^2 \varDelta R^2}{1+c^2R^2}\right)\leqq p\left\{\frac{n-p}{n(n-1)}R\right.$$
$$\left.-\sqrt{\left(\frac{p-1}{2}\right)^2 R_{ijkl}R^{ijkl}+\frac{n-4p+2}{4}R_{ij}R^{ij}+\left\{\frac{1}{4}-\frac{(n-p)^2}{2n(n-1)}\right\}R^2}\right\} \tag{14}$$

holds everywhere for a certain constant c, it holds that

$$B_p\leqq\binom{n}{p},$$

and the covariant derivative of any harmonic tensor of degree p vanishes. If, in (14), *the equality sign can be omitted, then we have*

$$B_p=0\,.$$

Utunomiya University.