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Factor System Approach to the Isomorphism and Reciprocity Theorems

Tadasi Nakayama

The Takagi-Artin class field theory establishes a canonical isomorphism between the Galois group of the full Abelian extension A_k of an algebraic number field k and a certain factor group of the idèle-class group $\mathfrak{C}_k$ of k, according to Chevalley's formulation ([2], [3]). Let K be a (finite) Galois extension of k and let A_K be the full Abelian extension of K. Then the Galois group of A_K over k is an extension of the Galois group of A_K/K by the Galois group $\mathfrak{G}$ of K/k. Thus it defines, by the cited canonical isomarphism, a factor system of $\mathfrak{G}$ in a factor group of the idèle-class group $\mathfrak{C}_K$ of K. Weil showed recently that this factor system can be represented by a factor system in the idèle-class group itself, so as some further requirements are met ([12]). Now, it is hoped to construct such factor systems directly and to use them conversely in establishing the class field theory. For the lodal class field theory such a factory system approach has been given in the note [8].[1] As to the global theory, i.e. the class field theory proper, Hasse has shown that his sum-relation of locol invariants of Brauer algebra-classes gives a central assertion in the reciprocity law ([5]). Noether has given a factor system formulation of the principal genus theorem ([10]). To proceed further (or, to start with, rather), it is desirable to define certain canonical factor systems in idèle-class groups and to derive from them the isomorphism and reciprocity theorems directly. This we propose to do in the present note.[2] Thus the work has little novelty in its true arithmetical bearing. But it provides, as the writer hopes, a rather elegant approach to those thorems.

1) Cf. also [9] and Akizuki [1]. Further, Hochschild [6] has constructed the whole theory without appealing to the theory of algebras but by dealing merely with factor systems (or cohomology classes).

2) Hochschild [7] has given recently a very direct and elegant proof to the reciprocity law (using also the idea of Hasse but without dealing with algebras explicitely). However, while he combines local canonical isomorphisms defined by local factor systems, in order to obtain global canonical isomorphism, so to speak, our present program is to carry out the transition to "global" already at the stage of factor systems, which also clarifies the combination of isomorphisms.

Besides his deepest veneration to Professor Takagi, the writer wishes to express his hearty thanks to Professors Weil and Hochschild, who have kindly given him the opportunity to read the manuscripts of their so far unappeared papers [12], [7].

1. Lemmas. Let k be an algebraic number field,[3] Denote the groups of idèles, principal idèles of k by J_k, P_k respectively, So $\mathfrak{C}_k = J_k/P_k$ is the idèle-class group of k. With a (finite) extension K of k, denote by $N = N_{K/k}$ the norm operation, for J_K, P_K or $\mathfrak{C}_K$. The following index relations are proved at a rather earlier stage of the standard treatment of the class field theory ([3]) :

Lemma 1. With an Abelian extension K/k, the degree $[K:k] \geq$ the index $[J_k : P_k N(J_K)]$.

If K/k is cyclic then $[K:k]$ ($\leq$ whence) $= [J_k : P_k N(J_K)]$. And, as a by-product we obtain usually

Lemma 2 (Norm theorem). If K/k is cyclic then $P_k \cap N(J_K) = N(P_K)$.

We shall also make use of the reciprocity law for cyclotomic fields ; as a matter of fact we shall use merely the cyclic case. We also need a very weak[4] existence theorem asserting that there exsits a cyclic cyclotomic field of a given degree over k. Naturally we assume the local class field theory for our disposal ; its formulation most suited for our purpose being given in [8], [6] or, [11].

We shall also use, though at the very end, the sum-relation :

Lemma 3. If $\mathfrak{A}$ is a central simple algebra over k and x_p denote, with primes p of k, the p-invariants of $\mathfrak{A}$, then $\sum x_p \equiv 0 \pmod 1$.[5]

This is a theorem of rather deep arithmetical nature, but is derived from Lemma 2 and the class field theory for cyclotomic fields (including certain existence theorem for them).

2. Idèle-class factor systems. Our starting point is

Lemma 4. With cyclic[6] K/k the 1-dimensional cohomology group

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3) Or, an algebraic function field of one variable over a finite field.

4) There is no significance in this weakness of the employed existence theorem, since a much stronger existence theorem, cyclotomic fields, is used in proving Lemma 3 below.

5) One could avoid the terminologies in the theory of algebras. Cf. [7].

6) As a matter of fact, the lemma holds for arbitrary Galois extension (or, even for arbitrary extension, if we employ Schur-Brauer factor systems), which amounts to formulate Noether's principal genus theorem in terms of idèles and idèle-classes. However, since the cyclic case suffices for our purpose of defining canonical factor systems and proving the isomorphism and reciprocity theorems, we restrict ourselves to the case here. The same for Lemmas 5, 6.

$H^1(\mathfrak{G}, \mathfrak{C}_K)$ of its Galois group $\mathfrak{G}$ in $\mathfrak{C}_K$ (i,e. the group of idèle-class "Transformationsgrössen" (modulo splitting ones)) is unity. In other words, if $N(\mathfrak{a})=1$ with $\mathfrak{a}\in\mathfrak{C}_K$, then $\mathfrak{a}=\mathfrak{b}^{1-\sigma}$ with a suitable $\mathfrak{b}\in\mathfrak{C}_K$, where σ denotes a generator of $\mathfrak{G}$.

Proof. Let $a\in J_K$ be a representative of $\mathfrak{a}$. We have, by assumption, $N(a)\in P_K$, whence $\in P_k$. It is norm-residue for every prime of k. By Lemma 2 there exists a $\xi\in P_K$ such that $N(a)=N(\xi)$, or $N(a\xi^{-1})=1$. It is easy to see[7] that there exists an idèle $b(\in J_K)$ such that $a\xi^{-1}=b^{1-\sigma}$.

From this follows then

Lemma 5. Let K/k be a Galois extension with Galois group $\mathfrak{G}$. Let $\mathfrak{H}$ be a cyclic no mal subgroup of $\mathfrak{G}$ and let L be the intermediate field belonging to $\mathfrak{H}$. Then the lifting operation λ maps the 2-dimensional cyhomology group $H^2(\mathfrak{G}/\mathfrak{H}, \mathfrak{C}_L)$ of $\mathfrak{G}/\mathfrak{H}$ in $\mathfrak{C}_L$ isomorphically into $H^2(\mathfrak{G}, \mathfrak{C}_K)$. In other words, if an (idèle-class) factor system of $\mathfrak{G}/\mathfrak{H}$ in $\mathfrak{C}_L$ splits in $\mathfrak{C}_K$, for $\mathfrak{G}$, then it is by itself a splitting factor system.

For the proof see [4] or [6], which applies to the present case too. We obtain also, as in [6],

Lemma 6. Let K/k, $\mathfrak{G}$, $\mathfrak{H}$, L be as in Lemma 5. The kernel of the restriction mapping from $\mathfrak{G}$ to $\mathfrak{H}$ in $H^2(\mathfrak{G}, \mathfrak{C}_K)$ coincides with $\lambda(H^2(\mathfrak{G}/\mathfrak{H}, \mathfrak{C}_L))$, In other words, if the restriction in $\mathfrak{H}$ of a factor system of $\mathfrak{G}$ in $\mathfrak{C}_K$ splits, then the factor system is equivalent to a liftıng of a factor system of $\mathfrak{G}/\mathfrak{H}$ in $\mathfrak{C}_L$.

Of algebraic nature is our next

Lemma 7. Let $\mathfrak{H}$ be a normal subgroup of the Galois group $\mathfrak{G}$ of K/k, and let L belong to $\mathfrak{H}$, Let h be the order of $\mathfrak{H}$. With any factor system $\mathfrak{a}$ of $\mathfrak{G}$ in $\mathfrak{C}_K$ the h-th power $\mathfrak{a}^h$ of $\mathfrak{a}$ is equivalent to the lifting of the factor system $\mathfrak{b}$ of $\mathfrak{G}/\mathfrak{H}$ in $\mathfrak{C}_L$ defined by $\mathfrak{b}(\bar\sigma, \bar\tau) = N_{K/L}\mathfrak{a}(\sigma_0\tau_0, \nu)$ $\Pi_{\nu\in\mathfrak{H}}\mathfrak{a}(\sigma_0\tau_0, \nu)\Pi_{\nu\in\mathfrak{H}}\mathfrak{a}((\sigma\tau)_0, \nu)^{-1}$, where $(\cdots, \sigma_0, \tau_0, \cdots)$ is a system of representatives of $\mathfrak{G}/\mathfrak{H}$ and σ denotes the coset of σ.

This formula, of Witt [13] and Akizuki [1], can be proved in exactly the same manner as in the case of usual factor systems in elements.

3. Canonical factor systems, the isomorphism and reciprocity theorems. Let K be a Galois extension of k and let $n=[K:k]$. We take a cyclic cyclotomic extension Z over k whose degree $m=[Z:k]$ is

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7) By component-wise application of the usual algebraic (Hilbert-Speiser) theorem of "Transformationsgrössen", (with a bit of arithmetical consideration).

divisible by n. Let $\mathfrak{Z}=\{1,\zeta,\zeta^2,\cdots,\zeta^{m-1}\}$ be the Galois group of Z/k. Each idèle $c\in J_k$ defines a normalized factor system (with respect to ζ) such as $c(\zeta^i,\zeta^j)=1$ or $c(i,j=0,1,\cdots,m-1)$ according as $i+j<$ or $\geq m$. We denote this normalized factor system simply by c. The (idèle-) class $\mathfrak{c}$ of c defines in similar fashion a normalized idèle-class factor system of $\mathfrak{Z}$, which we denote again by $\mathfrak{c}$. Now, by the reciprocity theorem for the cyclotomic field Z/k, there exists a c satisfying

$$(c,\ Z/k)=\zeta,$$

where $(c,\ Z/k)$ is the (essentially finite) product $\Pi_p\left(\frac{c,\ Z/k}{p}\right)^{-1}$ of norm-symbols. Then the factor system $\mathfrak{c}$ (with such choice of c) has the exact order m,

Consider the composite KZ and denote the Galois group of KZ/k by $\mathfrak{G}_1$. Let $\mathfrak{H}_1$ be the subgroup of $\mathfrak{G}_1$ belonging to K; $\mathfrak{H}_1$ is essentially the subgroup of $\mathfrak{Z}$ consisting of elements which leave K elementwise fixed. As is well known $(c,KZ/K)=\zeta^n$. It follows that the restriction in $\mathfrak{H}_1$ of the lifting $\mathfrak{c}^*$ of $\mathfrak{c}$ to $\mathfrak{G}_1$ has the order $m/n=[Z:k]/[K:k]$; the extension KZ/K is cyclotomic and we are allowed to apply the reciprocity law there. So the restriction in $\mathfrak{H}_1$ of $\mathfrak{c}^{*\frac{m}{n}}$ splits. By Lemma 6 there exists a factor system $\mathfrak{a}$ of $\mathfrak{G}$ in $\mathfrak{C}_K$ whose lifting $\mathfrak{a}^*=\lambda(\mathfrak{a})$ to $\mathfrak{G}_1$ is equivalent to $\mathfrak{c}^{*\frac{m}{n}}$;

$$\mathfrak{a}^*=\lambda(\mathfrak{a})\sim\mathfrak{c}^{*\frac{m}{n}}.$$

$\mathfrak{a}$ is determined, up to equivalence, in virtue of Lemma 5, and we want to call $\mathfrak{a}$ the *canonical factor sytem* of the extension K/k. (Our determination of $\mathfrak{a}$ depends on Z presently. Though it is easy to see from the reciprocity law for cyclotomic extensions that it is independent of the special choice the Z, we may think of it in terms of Z; that suffices for our purpose).

The order of $\mathfrak{c}^{*\frac{m}{n}}$, that is, the order of $\mathfrak{a}^*$, is equal to n; cf. Lemma 5. Then the order of $\mathfrak{a}$, our canonical factor system of K/k, is equal to n, much the more.

Let $k\subseteq L\subseteq K$ and $\mathfrak{H}$ be the subgroup of $\mathfrak{G}$ belonging to L. The restriction of $\mathfrak{a}$ to K/L, i.e. to $\mathfrak{H}$, is the canonical factor system for K/L, with respect to the cyclic cyclotomic field LZ over L, as we see without difficulty from that the restriction of $\mathfrak{a}^*$ to KZ/L is equivalent to the restriction of $\mathfrak{c}^{*\frac{m}{n}}$ to KZ/L and $(c,LZ/L)=\zeta^{[L:k]}$. Further, if L/k is normal then $\mathfrak{a}^h$, h being the order of $\mathfrak{H}$, is equivalent to the lifting of the canonical factor system $\mathfrak{b}$ for L/k (with respect to Z). For, the lifting of $\mathfrak{b}$ to LZ/k is equivalent to the lifting, to LZ/k, of $\mathfrak{c}^{\frac{m}{[L:k]}}$, whence the lifting of $\mathfrak{b}$ to KZ/k is equivalent to the lifting, to KZ/k, of $\mathfrak{c}^{\frac{m}{[L:k]}}$, i.e. to $\mathfrak{c}^{*\frac{m}{[L:k]}} \sim \mathfrak{b}^{*h}$. Thus, $\mathfrak{b}$ is given by the formula of of Lemma 7.

Now we put

$$\mathfrak{t}_{K/k}(\sigma)=\mathfrak{t}(\sigma)=\Pi_{\tau\in\mathfrak{G}}\mathfrak{a}(\sigma,\ \tau).$$

It is easy to see that a factor system equivalent to $\mathfrak{a}$ gives the same mapping modulo $N(\mathfrak{C}_K)$. We prove similarly as in [8]

Theorem 1. *Let K/k be Abelian.* $\sigma\rightarrow\mathfrak{t}(\sigma)$ mod $N(\mathfrak{C}_K)$ *defines an isomorphism of the Galois group $\mathfrak{G}$ and the idèle-class norm-class group* $\mathfrak{C}_k/N(\mathfrak{C})_K)\ (=J_k/P_kN(J_K))$,

Multiplying, namely, the equalities $\mathfrak{a}(\sigma,\tau)\mathfrak{a}(\rho\sigma,\tau)^{-1}\mathfrak{a}(\rho,\sigma\tau)\mathfrak{a}(\rho,\sigma)^{-\tau}=1$ over σ, τ respectively, we see that $\mathfrak{t}(\sigma)\in\mathfrak{C}_k$ and the mapping is homomorphic. With an intermediate field L, we see readily $\mathfrak{t}_{K/k}(\sigma)=N_{L/k}(\mathfrak{t}_{K/L}(\sigma))$ for $\sigma\in\mathfrak{H}$, where $\mathfrak{t}_{K/L}$ is defined by the restriction of $\mathfrak{a}$ to $\mathfrak{H}$, i.e. to K/L (which is the canonical factor system for K/L). That our homomorphism is an (into-) isomorphism is clear if $\mathfrak{G}$ is cyclic (since the order of the canonical factor systen is equal to the order n of $\mathfrak{G}$). Let $\mathfrak{G}$ be non-cyclic, and assume that the (into-) isomorphism assertion is true for every proper subfield of K/k. Let $\mathfrak{G}=\mathfrak{H}_1\times\mathfrak{H}_2$ ($\mathfrak{H}_i\neq 1$) and let K_1, K_2 be the subfields belonging to $\mathfrak{H}_1$, $\mathfrak{H}_2$. Considering the elements of $\mathfrak{H}_1$ as the representatives of $\mathfrak{G}/\mathfrak{H}_2$, we have, with $\sigma_1\in\mathfrak{H}_1$, $\mathfrak{t}_{K_2/k}(\sigma_1)=N_{K/K_2}(\mathfrak{t}_{K/K_1}(\sigma_1))$ $=N_{K_1/k}(\mathfrak{t}_{K/K_1}(\sigma_1))$, by our observation on canonical factor systems for subfields. The last is however $\mathfrak{t}_{K/k}(\sigma_1)$, as was observed just above. With $\sigma_i\in\mathfrak{H}_i$ we have thus $\mathfrak{t}(\sigma_1\sigma_2)=\mathfrak{t}_{K/k}(\sigma_1)\mathfrak{t}_{K/k}(\sigma_2)=\mathfrak{t}_{K_2/k}(\sigma_1)\mathfrak{t}_{K/k}(\sigma_2)$. Here $\mathfrak{t}_{K/k}(\sigma_2)$ $=N_{K_2/k}(\mathfrak{t}_{K/K_2}(\sigma))\in N_{K_2/k}(\mathfrak{C}_{K_2})$ by the same remark, with 1, 2 interchanged. Suppose now $\mathfrak{t}_{K/k}(\sigma_1\sigma_2)\in N_{K/k}(\mathfrak{C}_K)$. It follows that $\mathfrak{t}_{K_2/k}(\sigma_1)\in N_{K_2/k}(\mathfrak{C}_{K_2})$. This implies however $\sigma_1=1$ because of our induction assumption. Similarly $\sigma_2=1$. This shows that our isomorphism assertion holds for our case, and thus generally, by induction. But the order of $\mathfrak{C}_k/N(\mathfrak{C}_K)=J_k/P_kN(J_K)$ is, by Lemma 1, at most equal to n. So $\mathfrak{t}$ maps $\mathfrak{G}$ onto $\mathfrak{C}_k/N(\mathfrak{C}_K)$, isomorphically.[8]

8) On modifying our argument a little, we could restrict our use of Lemma 1 to cyclic case. Cf. Hochschild [6].

Theorem 2. *Our isomorphism* $\sigma \longleftrightarrow \mathfrak{t}(\sigma)$ mod $N(\mathfrak{C}_K)$ *in Theorem* 1 *is given by the law: an idèle a* $(\in J_k)$ *is in* $\mathfrak{t}(\sigma)$ mod $N(\mathfrak{C}_K)$ *if and only if*

$$(a,\ K/k) = \Pi_p(a_p,\ K/k) = \sigma,$$

where a_p *is, for each prime* p *of* k, *the* p*-component of* a *and* $(a_p,\ K/k)$ *is defined by the lacal reciprocity law.*[9]

Proof. Consider first the case where K/k is cyclic. Let σ be a generator of the Galois group $\mathfrak{G}$. Let $b \in J_k$ and consider the normalized factor system defined by the class $\mathfrak{b}$ and σ, which we denote by $\mathfrak{b}$ too. By normalizing the canonical factor system $\mathfrak{a}$ and by Theorem 1 we see that $\mathfrak{b}$ is a certain power, say i-th, of $\mathfrak{a}$. With our Z/k, $\mathfrak{Z} = \{\zeta\}$, $c \in J_k((c,Z/k)=\zeta)$, we have

$$b^* \sim \beta\, c^{*\frac{im}{n}}$$

with a factor system β of $\mathfrak{G}_1$ in P_{KZ}, where b^*, c^* denote the liftings to KZ/k of the normalized (idèle) factor systems defined by b, c with respect to σ, ζ, which we denote again by b, c; since b, c (not only represent idèle-class factor systems, but) are factor systems, β is (not only a system of elements, but) a factor system. The p-components of b^*, c^*, β foim factor systems of $\mathfrak{G}_1$ in $(KZ)_p/k_p$, p being a prime of k. Let μ_p, λ_p, $\varkappa_p$ be their (p-)invarirnts. We have $\mu_p \equiv \varkappa_p + \frac{im}{n}\lambda_p$. The sum $\sum \varkappa_p$, over all primes in k, is 0 (mod 1) by Lemma 3, while $\sum \lambda_p \equiv 1/m$ since $(c,\ Z/k) = \zeta$. Thus $\sum \mu_p \equiv i/n$, whence $(b,\ K/k) = \sigma^i$. This proves the assertion in cyclic case.

Let next K/k be a general Abelian extension and let L be an intermediate field, with $\mathfrak{H}$ belonging to it. Let $\mathfrak{b}$ be the canonical factor system for L/k, and $\mathfrak{t}_\mathfrak{b}$ be defined by $\mathfrak{b}$ similarly as $\mathfrak{t}$ is defined by the canonical system $\mathfrak{a}$ of K/k. $\mathfrak{t}_\mathfrak{b}(\sigma) \equiv \mathfrak{t}(\sigma)$ mod $N_{L/k}(\mathfrak{C}_L)$ as is seen similarly as in our proof of Theorem 1. Now, assume that the theorem is true for L/k, i.e. $(d,\ L/k) = \sigma$ if and only if $d \in \mathfrak{t}_\mathfrak{b}(\sigma) N_{L/k}(\mathfrak{C}_L)$. Then our isomorphism of $\mathfrak{G}$ and $\mathfrak{C}_k/N(\mathfrak{C}_K)$ given by $\mathfrak{t}$ induces that isomorphism of $\mathfrak{G}/\mathfrak{H}$ and $\mathfrak{C}_k/N_{L/k}(\mathfrak{C}_L)$, given by $\sigma \longleftrightarrow \mathfrak{d}$ (mod $N_{L/k}(\mathfrak{C}_L)$) $((d,\ L/k) = \sigma)$. So, expressing K as a direct composite $L_1 \times L_2 \times \cdots \times L_s$ of cyclic fields, we see the validity of the asseition of our theorem, for K/k, since $\cap \mathfrak{H}_i = 1$, where $\mathfrak{H}_i$ belong to L_i.

9) This direct formulation of canonical reciprocity isomorphism is also convenient in obtaining the decomposition and ramification theorems; cf. Hochschild [7].

By making full[10] use of Lemma 7 it is easy to extend Theorems 1, 2 to a non-Abelian K/k, where we deal with the factor group of $\mathfrak{G}$ with respect to its commutator subgroup; cf. [1]. Further, it turns out that our canonical factor system coincides the one given in Weil [12]. To these and to some related problems, the writer wants to come back elsewhere.

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Nagoya University and University of Illinois

(10) Only Chevalley's special case of Lemma 7, which deals with a direct component $\mathfrak{H}$ of $\mathfrak{G}$, was needed in our derivation of Theorems 1, 2.