Tohoku Math. J.
67 (2015), 507–511

ON THE QUATERNIONIC MANIFOLDS WHOSE TWISTOR SPACES ARE FANO MANIFOLDS

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(Received March 19, 2014, revised May 29, 2014)

Abstract. Let M be a quaternionic manifold, $\dim M = 4k$, whose twistor space is a Fano manifold. We prove the following:
(a) M admits a reduction to $\mathrm{Sp}(1) \times \mathrm{GL}(k, \mathbb{H})$ if and only if $M = \mathbb{H}P^k$,
(b) either $b_2(M) = 0$ or $M = \mathrm{Gr}_2(k+2, \mathbb{C})$.
This generalizes results of S. Salamon and C. R. LeBrun, respectively, who obtained the same conclusions under the assumption that M is a complete quaternionic-Kähler manifold with positive scalar curvature.

1. Introduction. An *almost quaternionic structure* on a manifold M is a reduction of its frame bundle to $\mathrm{Sp}(1) \cdot \mathrm{GL}(k, \mathbb{H})$. Then the obstruction for M to admit a 'reduction' to $\mathrm{Sp}(1) \times \mathrm{GL}(k, \mathbb{H})$ is an element of $H^2(M, \mathbb{Z}_2)$ [8]. Equivalently, this is the second Stiefel-Whitney class of the oriented Riemannian vector bundle Q induced by the Lie groups morphism $\mathrm{Sp}(1) \cdot \mathrm{GL}(k, \mathbb{H}) \to \mathrm{SO}(3)$, $\pm(a, A) \mapsto \pm a$.

If $\dim M \geq 8$ then the almost quaternionic structure is *integrable* if there exists a torsion free connection on M which is compatible (with the structural group) [12]. Equivalently (see [3]), there exists a compatible connection ∇ on M such that the almost complex structure induced by ∇ on the sphere bundle Z of Q is integrable. Then the complex manifold Z is the *twistor space* of M and the fibres of $\pi : Z \to M$ are the 'real' *twistor lines*; furthermore, Z is endowed with a conjugation (given by the antipodal map on the fibres of π). Conversely, Z together with its conjugation and a real twistor line determines M (see [9]). Furthermore, by [12] and [10], there exists a holomorphic line bundle $\mathcal{L}$ over Z whose restriction to any twistor line has Chern number 2. It follows quickly that M admits a reduction to $\mathrm{Sp}(1) \times \mathrm{GL}(k, \mathbb{H})$ if and only if $\mathcal{L}$ admits a square root.

Further natural restrictions can be obtained by assuming that there exists a Riemannian metric on M for which the holonomy group of its Levi-Civita connection is contained by $\mathrm{Sp}(1) \cdot \mathrm{Sp}(k)$; then M is called *quaternionic-Kähler*. It follows [11] that any quaternionic-Kähler manifold is an Einstein manifold, and, assuming, further, completeness and the scalar curvature positive, the corresponding twistor space is a Fano manifold. Also, by [11, Theorem 6.3], $\mathbb{H}P^k$ is the only such quaternionic-Kähler manifold which admits a reduction to $\mathrm{Sp}(1) \times \mathrm{GL}(k, \mathbb{H})$.

2010 *Mathematics Subject Classification.* Primary 53C28, Secondary 53C26.
Key words and phrases. Quaternionic manifolds.

Another result, in the same vein, is [7] that for any complete quaternionic-Kähler manifold M with positive scalar curvature we have that either its second Betti number $b_2(M)$ is zero, or M is the Grassmannian $\mathrm{Gr}_2(k+2,\mathbb{C})$, where, as above, $\dim M = 4k$.

In this paper, we generalize these two results of [11] and [7], respectively, to the class of quaternionic manifolds whose twistor spaces are Fano manifolds.

I am very grateful to Paltin Ionescu for an illuminating exchange of e-mail messages, during which he generously provided, for example, the proof of Proposition 2.3. I am, also, grateful to Liviu Ornea and to Victor Vuletescu for useful comments on a preliminary version of the paper.

2. The results. As the four-dimensional case was elucidated in [2], we consider only quaternionic manifolds of dimension at least 8.

The following result generalizes [11, Theorem 6.3].

THEOREM 2.1. *Let M be a quaternionic manifold*, $\dim M = 4k \geq 8$, *which admits a reduction to* $\mathrm{Sp}(1)\times\mathrm{GL}(k,\mathbb{H})$; *denote by Z the twistor space of M.*

Then the following assertions are equivalent:

(i) $M = \mathbb{H}P^k$;

(ii) *M is simply-connected*, $b_2(M)=0$ *and Z is projective* (*that is, Z can be embedded as a compact complex submanifold of a complex projective space*);

(iii) *Z is a Fano manifold* (*that is, Z is compact and its anticanonical line bundle is ample*).

PROOF. It is obvious that if (i) holds then both (ii) and (iii) are statisfied, as $Z=\mathbb{C}P^{2k+1}$ and $M=\mathbb{H}P^k$.

Further, as the restriction of the holomorphic cotangent bundle to each twistor line is $\mathcal{O}(-2)\oplus 2k\mathcal{O}(-1)$, where $\mathcal{O}(-1)$ is the tautological line bundle, essentially the same proof as for [2, Proposition 2.2(ii)] implies that any holomorphic form of positive degree on Z is zero. Consequently, if Z is projective, from the exact sequence of cohomology groups associated to the exact sequence of complex Lie groups $0\to\mathbb{Z}\to\mathbb{C}\to\mathbb{C}\setminus\{0\}\to 0$ (determined by the exponential) we deduce that the Picard group $\mathrm{Pic}(Z)$ is isomorphic to $H^2(Z,\mathbb{Z})$. Furthermore, if (ii) holds then, also, Z is simply-connected (by the homotopy exact sequence determined by the smooth bundle $Z\to M$), and, hence, $\mathrm{Pic}(Z)$ has no torsion. Also, as $b_2(Z)=b_2(M)+1$ (see [7]), $\mathrm{Pic}(Z)$ has rank 1. We have, thus, proved that $\mathrm{Pic}(Z)$ is isomorphic to $\mathbb{Z}$.

Let $\mathcal{L}$ be the restriction to Z of the dual of the tautological line bundle over the complex projective space in which Z is embedded. As both the restriction of $\mathcal{L}$ and of the anticanonical line bundle K_Z^* of Z, to a twistor line, are positive we deduce that $\left(K_Z^*\right)^p=\mathcal{L}^q$, for some positive integers p and q. Thus, also $\left(K_Z^*\right)^p$ is very ample, and (ii)$\Longrightarrow$(iii) is proved.

To complete the proof it is sufficient to show that (iii)$\Longrightarrow$(i). We claim that, if (iii) holds, there exists a holomorphic line bundle $\mathcal{L}$ over Z such that:

(a) $\mathcal{L}$ is ample;

(b) $\mathcal{L}$ restricted to each twistor line is (isomorphic to) $\mathcal{O}(1)$.

Indeed, from the assumption that M admits a reduction to $\mathrm{Sp}(1)\times\mathrm{GL}(k,\mathbb{H})$, by [12] and [10] there exists a holomorphic line bundle $\mathcal{L}_1$ over Z which satisfies condition (b), above; moreover, $\mathcal{L}_1$ is endowed with a morphism of (real) vector bundles whose square is -1 and which is an anti-holomorphic diffeomorphism covering the conjugation of Z (given, on each fibre of $Z \to M$, by the antipodal map). We shall show that after tensorising, if necessary, $\mathcal{L}_1$ with a holomorphic line bundle, whose restriction to each twitor line is trivial, we obtain a line bundle satisfying (a).

For this, firstly, note that $K_Z^* \,(= \Lambda_{\mathbb{C}}^{2k+1} TZ)$ restricted to each twistor line is $\mathcal{O}(2k+2)$. Hence, $K_Z \otimes \mathcal{L}_1^{2k+2}$ restricted to each twistor line is trivial; moreover, this holomorphic line bundle is endowed with a conjugation (that is, an involutive morphism of vector bundles which is an anti-holomorphic diffeomorphism) covering the conjugation of Z. Therefore $K_Z \otimes \mathcal{L}_1^{2k+2}$ corresponds, through the Ward transform, to a (real) line bundle L over M endowed with an anti-self-dual connection (that is, a connection whose curvature form is such that its $(0,2)$-part, with respect to any admissible linear complex structure on M, is zero).

As M is simply-connected (because Z is Fano and therefore simply-connected, and the fibres of the projection $Z \to M$ are connected), L is orientable and, hence, there exists a line bundle L_1 such that $L = L_1^{2k+2}$; furthermore, this isomorphism is connection preserving with respect to a unique anti-self-dual connection on L_1. Hence, L_1 corresponds to a holomorphic line bundle $\mathcal{L}_2$ over Z whose restriction to each twistor line is trivial, and such that $K_Z \otimes \mathcal{L}_1^{2k+2} = \mathcal{L}_2^{2k+2}$.

Thus, since K_Z^* is ample, $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2^*$ satisfies (a) and (b), above. Moreover, $\mathcal{L}$ is endowed with a morphism of vector bundles τ whose square is -1 and which is an anti-holomorphic diffeomorphism covering the conjugation of Z. Hence, τ induces a linear complex structure J on $H^0(Z,\mathcal{L})$ which anti-commutes with its canonical complex structure.

By [4, Corollary 2.4], Z is a complex projective space and the twistor lines are just the complex projective lines; moreover, Z is the projectivisation of the dual of $H^0(Z,\mathcal{L})$. Furthermore, J induces on the dual E of $H^0(Z,\mathcal{L})$ a linear quaternionic structure with respect to which the fibres of $Z \to M$ are those complex projective lines obtained through the complex projectivisation of the quaternionic vector subspaces of E of real dimension 4. Thus, $Z = PE$, M is the quaternionic projective space $P_{\mathbb{H}}E$, and $Z \to M$ is the canonical projection $PE \to P_{\mathbb{H}}E$. The proof is complete. □

The following result generalizes [7, Theorem 1].

THEOREM 2.2. *Let M be a quaternionic manifold*, $\dim M = 4k \geq 8$, *whose twistor space is a Fano manifold.*

Then either $b_2(M) = 0$ or $M = \mathrm{Gr}_2(k+2,\mathbb{C})$.

PROOF. Let Z be the twistor space of M. Similarly to the proof of Theorem 2.1, we obtain a holomorphic line bundle $\mathcal{L}$ over Z such that $\mathcal{L}^{k+1} = K_Z^*$. Furthermore, $\mathcal{L}$ admits a square root if and only if M admits a reduction to $\mathrm{Sp}(1)\times\mathrm{GL}(k,\mathbb{H})$. Therefore, by Theorem

2.1, either $M = \mathbb{H}P^k$ or $k+1$ is the greatest natural number n for which K_Z^* admits an n-th root. From now on, in this proof, we shall assume that the latter holds.

Now, just like in the proof of [7, Theorem 1], by using [13], we obtain that if $b_2(M) \neq 0$ then one of the following three statements holds:

(i) $Z = \mathbb{C}P^k \times Q_{k+1}$, where Q_{k+1} is the nondegenerate hyperquadric in $\mathbb{C}P^{k+2}$,

(ii) Z is the projectivisation of the holomorphic cotangent bundle of $\mathbb{C}P^{k+1}$,

(iii) Z is $\mathbb{C}P^{2k+1}$ blown up along $\mathbb{C}P^{k-1}$.

The fact that (i) cannot occur is a consequence of Proposition 2.3, below.

In the remaining two cases, it follows that M can be locally identified (through quaternionic diffeomorphisms) with $\mathrm{Gr}_2(k+2, \mathbb{C})$ or with $\mathbb{H}P^k$, respectively. By using that M is compact and simply-connected, a standard argument shows that either $M = \mathrm{Gr}_2(k+2, \mathbb{C})$ or $M = \mathbb{H}P^k$. As the latter leads to a contradiction, the proof is complete. $\square$

The following result, also interesting in itself, was used in the proof of Theorem 2.2.

PROPOSITION 2.3 ([6]). *Let Q_{k+1} be the nondegenerate hyperquadric in $\mathbb{C}P^{k+2}$. Then no open subset of $\mathbb{C}P^k \times Q_{k+1}$ can be the twistor space of a quaternionic manifold.*

PROOF. We shall prove that $Y = \mathbb{C}P^k \times Q_{k+1}$ does not admit an embedded Riemann sphere whose normal bundle is $2k\mathcal{O}(1)$. Indeed, let L_1 and L_2 be the restrictions to $\mathbb{C}P^k$ and Q_{k+1} of the duals of the tautological line bundles on $\mathbb{C}P^k$ and $\mathbb{C}P^{k+2}$, respectively. We have that both L_1 and L_2 are very ample and, also, $K^*_{\mathbb{C}P^k} = (L_1)^{k+1}$, $K^*_{Q_{k+1}} = (L_2)^{k+1}$ (for the latter, use the adjunction formula mentioned in [1, p. 147]). Thus, on denoting by π_1 and π_2 the projections from Y onto its factors, respectively, we obtain that, also, $L = \pi_1^* L_1 \otimes \pi_2^* L_2$ is very ample, and $K_Y^* = L^{k+1}$. Therefore if Y would admit an embedded Riemann sphere t whose normal bundle is $2k\mathcal{O}(1)$ then $L|_t = \mathcal{O}(2)$. On embedding Y into the projectivisation of the dual of $H^0(Y, L)$, we obtain that t has degree two and therefore it is a conic. It follows that any two points of Y are joined by a conic. But, according to [5], Y cannot have this property, thus completing the proof. $\square$

REFERENCES

[1] P. A. GRIFFITHS AND J. HARRIS, Principles of algebraic geometry, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1978.

[2] N. J. HITCHIN, Kählerian twistor spaces, Proc. London Math. Soc. 43 (1981), 133–150.

[3] S. IANUŞ, S. MARCHIAFAVA, L. ORNEA AND R. PANTILIE, Twistorial maps between quaternionic manifolds, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 9 (2010), 47–67.

[4] P. IONESCU, Birational geometry of rationally connected manifolds via quasi-lines, Projective varieties with unexpected properties, 317–335, Walter de Gruyter GmbH & Co. KG, Berlin, 2005.

[5] P. IONESCU AND F. RUSSO, Conic-connected manifolds, J. Reine Angew. Math. 644 (2010), 145–157.

[6] P. IONESCU, Private correspondence.

[7] C. R. LEBRUN, On the topology of quaternionic manifolds, Twistor Newsletter 32 (1991), 6–7.

[8] S. MARCHIAFAVA AND G. ROMANI, Sui fibrati con struttura quaternionale generalizzata, Ann. Mat. Pura Appl. (4) 107 (1976), 131–157.

[9] R. PANTILIE, On the twistor space of a (co-)CR quaternionic manifold, New York J. Math. 20 (2014), 959–971.

[10] H. PEDERSEN, Y. S. POON AND A. F. SWANN, Hypercomplex structures associated to quaternionic manifolds, Differential Geom. Appl. 9 (1998), 273–292.

[11] S. SALAMON, Quaternionic Kähler manifolds, Invent. Math. 67 (1982), 143–171.

[12] S. SALAMON, Differential geometry of quaternionic manifolds, Ann. Sci. École Norm. Sup. (4) 19 (1986), 31–55.

[13] J. A. WIEŚNIEWSKI, On Fano manifolds of large index, Manuscripta Math. 70 (1991), 145–152.

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