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WEIGHTED NORM INEQUALITIES FOR MULTISUBLINEAR MAXIMAL OPERATOR ON MARTINGALE SPACES

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Abstract. Let $v,\ \omega_1,\ \omega_2$ be weights and let $1 < p_1,\ p_2 < \infty$. Suppose that $1/p = 1/p_1 + 1/p_2$ and the couple of weights (ω_1, ω_2) satisfies the reverse Hölder's condition. For the multisublinear maximal operator $\mathfrak{M}$ on martingale spaces, we characterize the weights for which $\mathfrak{M}$ is bounded from $L^{p_1}(\omega_1) \times L^{p_2}(\omega_2)$ to $L^{p,\infty}(v)$ or $L^p(v)$. If $v = \omega_2^{p/p_2}\omega_2^{p/p_2}$, we partially give the bilinear version of one-weight theory.

Introduction. Let R^n be the n-dimensional real Euclidean space and f a real valued measurable function, the classical Hardy-Littlewood maximal operator M, the maximal geometric mean operator G and the minimal operator $\mathbf{m}$ are defined by

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|dy\,,$$

$$G(f)(x) = \sup_{x\in Q} \exp \frac{1}{|Q|}\int_Q \log|f(y)|dy$$

and

$$\mathbf{m}f(x) = \inf_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|dy\,,$$

where Q is a non-degenerate cube with its sides parallel to the coordinate axes and $|Q|$ is the Lebesgue measure of Q.

Let $u,\ v$ be two weights, i.e., positive measurable functions. As is well known, for $p \geq 1$, Muckenhoupt [18] showed that the inequality

$$\lambda^p \int_{\{Mf>\lambda\}} u(x)dx \leq C\int_{R^n} |f(x)|^p v(x)dx\,, \quad \lambda > 0\,,\ \ f \in L^p(v)$$

holds if and only if $(u, v) \in A_p$, i.e., for any cube Q in R^n with sides parallel to the coordinates

$$\left(\frac{1}{|Q|}\int_Q u(x)dx\right)\left(\frac{1}{|Q|}\int_Q v(x)^{-\frac{1}{p-1}}dx\right)^{p-1} < C\,, \quad p > 1\,;$$

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$$\frac{1}{|Q|}\int_Q u(x)dx \le C \operatorname*{ess\,inf}_Q v(x)\,, \qquad p=1\,.$$

Suppose that $u=v$ and $p>1$, Muckenhoupt [18] also proved that

$$\int_{R^n}\big(Mf(x)\big)^p v(x)dx \le C\int_{R^n}|f(x)|^p v(x)dx\,, \quad \forall f\in L^p(v)$$

holds if and only if v satisfies

$$\left(\frac{1}{|Q|}\int_Q v(x)dx\right)\left(\frac{1}{|Q|}\int_Q v(x)^{-\frac{1}{p-1}}dx\right)^{p-1} < C\,, \quad \forall Q\,. \tag{1}$$

The crucial step is to show that if v satisfies A_p, then there is an $\varepsilon>0$ such that v also satisfies $A_{p-\varepsilon}$. However, the problem of finding all u and v such that

$$\int_{R^n}\big(Mf(x)\big)^p u(x)dx \le C\int_{R^n}|f(x)|^p v(x)dx\,, \quad \forall f\in L^p(v)$$

is much hard and complicated. In order to solve the problem, Sawyer [22] established the testing condition $S_{p,q}$, i.e., for any cube Q in R^n with sides parallel to the coordinates

$$\left(\int_Q\big(M(\chi_Q v^{1-p'})(x)\big)^q u(x)dx\right)^{\frac{1}{q}} \le C\left(\int_Q v(x)^{1-p'}dx\right)^{\frac{1}{p}}\,, \quad \forall Q$$

where $1<p\le q<\infty$. The condition $S_{p,q}$ is a necessary and sufficient condition such that the weighted inequality

$$\left(\int_{R^n}\big(Mf(x)\big)^q u(x)dx\right)^{\frac{1}{q}} \le C\left(\int_{R^n}|f(x)|^p v(x)dx\right)^{\frac{1}{p}}\,, \ \forall f\in L^p(v)$$

holds. In this case, the method of proof is very interesting. Motivated by [18, 22], the theory of weights developed so rapidly that it is difficult to give its history a full account here (see [6] and [5] for more information). However, it is possible to give a story of weighted inequalities for the different variants of Hardy-littlewood operator. Let $p\to\infty$ in (1), it follows that

$$\left(\frac{1}{|Q|}\int_Q v(x)dx\right)\exp\left(\frac{1}{|Q|}\int_Q \log\left(\frac{1}{v(x)}\right)dx\right) < C\,, \tag{2}$$

which is an alternative definition of A_∞ weight (see [10]). It is known that Sbordone and Wik [23] used (2) to characterize the boundedness of G from $L^1(v)$ to $L^1(v)$. In the case of two weights, Yin and Muckenhoupt [24] gave that

$$\left(\frac{1}{|Q|}\int_Q u(x)dx\right)\exp\left(\frac{1}{|Q|}\int_Q \log\left(\frac{1}{v(x)}\right)dx\right) < C\,, \ \ \forall Q \Leftrightarrow \sup_{\|f\|_{L^p(v)}=1}\|Gf\|_{L^{p,\infty}(u)}<\infty$$

and

$$\int_Q G(v^{-1}\chi_Q)(x)u(x)dx \le C|Q|\,, \quad \forall\, Q \Leftrightarrow \sup_{\|f\|_{L^p(v)}=1}\|\,Gf\,\|_{L^p(u)}<\infty\,,$$

which generalize the results of [11]. Recently, Cruz-Uribe [4] (see also the references therein) also studied the minimal operator and reverse Hölder's inequality. There are still other variants of Hardy-littlewood operator, for example, the generalized maximal operator and the strong maximal operator which were considered in [20, 21] and [14], respectively. Now, the multisublinear maximal function

$$\mathfrak{M}(f_1, \ldots, f_m)(x) = \sup_{x \in Q} \prod_{i=1}^{m} \frac{1}{|Q|} \int_Q |f_i(y_i)| dy_i$$

associated with cubes with sides parallel to the coordinate axes was studied in [15]. They introduced the multilinear $A_{\vec{p}}$ condition which is an analogue of the A_p weight for multiple weights. The more general case was extensively discussed in [9, 8].

The above operators can be defined in martingale space, and the weighted inequalities also have their martingale versions. In fact, all of them have been discussed in [26, 17, 1, 12, 3, 16] (see also the references therein), except the one for multisublinear maximal function. In this paper, with stopping times and a kind of reverse Hölder's condition, we discuss weighted inequalities for multisublinear maximal operator on martingale spaces. One of our main results is the martingale-variant of $A_{\vec{p}}$, and the other is the equivalence of $S_{\vec{p}}$ and strong weighted inequality in martingale space. We also discuss the convergence of martingale, which is partly a bilinear version of the results in [13].

The rest of this section consists of the preliminaries for our paper.

Let $(\Omega, \mathcal{F}, \mu)$ be a complete probability space and $(\mathcal{F}_n)_{n \geq 0}$ an increasing sequence of sub-σ-fields of $\mathcal{F}$ with $\mathcal{F} = \bigvee_{n \geq 0} \mathcal{F}_n$. A weight ω is a random variable with $\omega > 0$ and $E(\omega) < \infty$. For any $n \geq 0$ and $f \in L^1$, we denote the conditional expectation with respect to $\mathcal{F}_n$ by $E_n(f)$, $E(f|\mathcal{F}_n)$ or f_n, then $(f_n)_{n \geq 0}$ is an uniformly integrable martingale. Suppose that functions f, g are integrable, the maximal operator and multisublinear maximal operator are defined by

$$Mf = \sup_{n \geq 0} |E_n(f)| \text{ and } \mathfrak{M}(f, g) = \sup_{n \geq 0} |E_n(f)||E_n(g)|,$$

respectively. For $B \in \mathcal{F}$, we always denote $\int_\Omega \chi_B d\mu$ and $\int_\Omega \chi_B \omega d\mu$ by $|B|$ and $|B|_\omega$, respectively. For $(\Omega, \mathcal{F}, \mu)$ and $(\mathcal{F}_n)_{n \geq 0}$, the family of all stopping times is denoted by $\mathcal{T}$. Throughout this paper, C will denote a constant not necessarily the same at each occurrence.

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1. Results and their proofs.

DEFINITION 1.1. Let ω_1, ω_2 be weights and $1 < p_1$, $p_2 < \infty$. Suppose that $1/p = 1/p_1 + 1/p_2$ and $\sigma_i = \omega_i^{-\frac{1}{p_i - 1}} \in L^1$, $i = 1, 2$. We say that the couple of weights (ω_1, ω_2) satisfies the reverse Hölder's condition $RH(p_1, p_2)$, if there exists a positive constant C such that

$$\left(\int_{\{\tau < \infty\}} \sigma_1 d\mu \right)^{\frac{p}{p_1}} \left(\int_{\{\tau < \infty\}} \sigma_2 d\mu \right)^{\frac{p}{p_2}} \leq C \int_{\{\tau < \infty\}} \sigma_1^{\frac{p}{p_1}} \sigma_2^{\frac{p}{p_2}} d\mu, \quad \forall \tau \in \mathcal{T}.$$

REMARK 1.2. In literature there exist many inverse Hölder's inequalities of the type

$$\|f\|_p\|g\|_q \le C\|fg\|_1\,,$$

where $1/p + 1/q = 1$, C is a constant and the functions f and g are subjected to suitable restrictions. The suitable restrictions can be found in [19, 25]. In our paper, we find that the reverse Hölder's condition is useful for bilinear weighted theory in martingale context.

DEFINITION 1.3. Let $v,\ \omega_1,\ \omega_2$ be weights and $1 < p_1,\ p_2 < \infty$. Suppose that $1/p = 1/p_1 + 1/p_2$. Denote that $\overrightarrow{p} = (p_1, p_2)$ and $\sigma_i = \omega_i^{-\frac{1}{p_i-1}} \in L^1,\ i = 1,\ 2$. We say that the triple of weights $(v,\ \omega_1,\ \omega_2)$ satisfies the condition $A_{\overrightarrow{p}}$, if there exists a positive constant C such that

$$\sup_{n\ge 0} E_n(v)^{\frac{1}{p}} E_n(\omega_1^{1-p_1'})^{\frac{1}{p_1'}} E_n(\omega_2^{1-p_2'})^{\frac{1}{p_2'}} \le C\,,$$

where $1/p_i + 1/p_i' = 1,\ i = 1,\ 2$.

DEFINITION 1.4. Let $v,\ \omega_1,\ \omega_2$ be weights and $1 < p_1,\ p_2 < \infty$. Suppose that $1/p = 1/p_1 + 1/p_2$. Denote that $\overrightarrow{p} = (p_1, p_2)$ and $\sigma_i = \omega_i^{-\frac{1}{p_i-1}} \in L^1,\ i = 1,\ 2$. We say that the triple of weights $(v,\ \omega_1,\ \omega_2)$ satisfies the condition $S_{\overrightarrow{p}}$, if there exists a positive constant C such that

$$\left(\int_{\{\tau<\infty\}} \mathfrak{M}(\sigma_1\chi_{\{\tau<\infty\}}, \sigma_2\chi_{\{\tau<\infty\}})^p v\,d\mu\right)^{\frac{1}{p}} \le C|\{\tau<\infty\}|_{\sigma_1}^{\frac{1}{p_1}}|\{\tau<\infty\}|_{\sigma_2}^{\frac{1}{p_2}}\,, \quad \forall \tau \in \mathcal{T}.$$

REMARK 1.5. If we substitute $p_1 = p_2$ and $\omega_1 = \omega_2$ into Definition 1.3 and Definition 1.4, they reduce to the A_{p_1} condition and the S_{p_1} condition in martingale spaces, respectively (see, e.g., [17]).

1.1. Bilinear version of two-weight inequalities.

THEOREM 1.6. *Let $v,\ \omega_1,\ \omega_2$ be weights and $1 < p_1,\ p_2 < \infty$. Suppose that $1/p = 1/p_1 + 1/p_2$ and $(\omega_1, \omega_2) \in RH(p_1, p_2)$, then the following statements are equivalent*:

(a) *There exists a positive constant C such that for any $\tau \in \mathcal{T}$, any $f \in L^{p_1}(\omega_1)$ and any $g \in L^{p_2}(\omega_2)$,*

$$\left(\int_{\{\tau<\infty\}} (|f_\tau||g_\tau|)^p v\,d\mu\right)^{\frac{1}{p}} \le C\|f\|_{L^{p_1}(\omega_1)}\|g\|_{L^{p_2}(\omega_2)}\,; \tag{3}$$

(b) *There exists a positive constant C such that*

$$\|\mathfrak{M}(f, g)\|_{L^{p,\infty}(v)} \le C\|f\|_{L^{p_1}(\omega_1)}\|g\|_{L^{p_2}(\omega_2)},\ \forall f \in L^{p_1}(\omega_1),\ g \in L^{p_2}(\omega_2); \tag{4}$$

(c) *The triple of weights $(v,\ \omega_1,\ \omega_2)$ satisfies the condition $A_{\overrightarrow{p}}$.*

PROOF. We shall follow the scheme: (a)⇔(b)⇐(c)⇐(a).

(a)⇒(b). It is trivial and we omit it.

(b)$\Rightarrow$(a). Fix $n \in N$ and $B \in \mathcal{F}_n$. For $f \in L^{p_1}(\omega_1)$ and $g \in L^{p_2}(\omega_2)$, let

$$F = f\chi_B \text{ and } G = g\chi_B\,,$$

respectively. Then $E_n(F) = f_n\chi_B$ and $E_n(G) = g_n\chi_B$. Moreover

$$|f_n g_n|\chi_B \le \mathfrak{M}(F,G)\,.$$

Combining with (4), we have

$$\begin{aligned}
\lambda^p \int_{B\cap\{|f_n g_n|>\lambda\}} v\,d\mu &\le \lambda^p \int_{\{\mathfrak{M}(F,G)>\lambda\}} v\,d\mu \\
&\le C\Big(\int_\Omega |F|^{p_1}\omega_1 d\mu\Big)^{\frac{p}{p_1}}\Big(\int_\Omega |G|^{p_2}\omega_2 d\mu\Big)^{\frac{p}{p_2}} \\
&= C\Big(\int_B |f|^{p_1}\omega_1 d\mu\Big)^{\frac{p}{p_1}}\Big(\int_B |g|^{p_2}\omega_2 d\mu\Big)^{\frac{p}{p_2}}.
\end{aligned}$$

For $k \in Z$, let

$$B_k = \{2^k < |f_n||g_n| \le 2^{k+1}\}\,.$$

Note that

$$\{2^k < |f_n||g_n| \le 2^{k+1}\} \subseteq \{2^k < |f_n||g_n|\}\,.$$

Then

$$\begin{aligned}
\int_\Omega (|f_n||g_n|)^p v\,d\mu &= \sum_{k\in Z}\int_{B_k}(|f_n||g_n|)^p v\,d\mu \\
&\le C\sum_{k\in Z}\int_{B_k\cap\{|f_n||g_n|>2^k\}} 2^{kp} v\,d\mu \\
&\le C\sum_{k\in Z}\Big(\int_{B_k}|f|^{p_1}\omega_1 d\mu\Big)^{\frac{p}{p_1}}\Big(\int_{B_k}|g|^{p_2}\omega_2 d\mu\Big)^{\frac{p}{p_2}} \\
&\le C\Big(\sum_{k\in Z}\int_{B_k}|f|^{p_1}\omega_1 d\mu\Big)^{\frac{p}{p_1}}\Big(\sum_{k\in Z}\int_{B_k}|g|^{p_2}\omega_2 d\mu\Big)^{\frac{p}{p_2}} \\
&= C\Big(\int_\Omega |f|^{p_1}\omega_1 d\mu\Big)^{\frac{p}{p_1}}\Big(\int_\Omega |g|^{p_2}\omega_2 d\mu\Big)^{\frac{p}{p_2}},
\end{aligned}$$

where we have used Hölder's inequality. As for $\tau \in \mathcal{T}$, it is easy to see that

$$\begin{aligned}
\int_{\{\tau<\infty\}}(|f_\tau||g_\tau|)^p v\,d\mu &= \sum_{n\ge 0}\int_{\{\tau=n\}}(|f_n||g_n|)^p v\,d\mu \\
&\le C\sum_{n\ge 0}\Big(\int_\Omega |f\chi_{\{\tau=n\}}|^{p_1}\omega_1 d\mu\Big)^{\frac{p}{p_1}}\Big(\int_\Omega |g\chi_{\{\tau=n\}}|^{p_2}\omega_2 d\mu\Big)^{\frac{p}{p_2}}
\end{aligned}$$

$$\le C\Big(\sum_{n\ge0}\int_\Omega |f\chi_{\{\tau=n\}}|^{p_1}\omega_1 d\mu\Big)^{\frac{p}{p_1}}\Big(\sum_{n\ge0}\int_\Omega |g\chi_{\{\tau=n\}}|^{p_2}\omega_2 d\mu\Big)^{\frac{p}{p_2}}$$
$$\le C\Big(\int_\Omega |f|^{p_1}\omega_1 d\mu\Big)^{\frac{p}{p_1}}\Big(\int_\Omega |g|^{p_2}\omega_2 d\mu\Big)^{\frac{p}{p_2}}.$$

Therefore,

$$\Big(\int_{\{\tau<\infty\}}(|f_\tau||g_\tau|)^p v d\mu\Big)^{\frac{1}{p}} \le C\|f\|_{L^{p_1}(\omega_1)}\|g\|_{L^{p_2}(\omega_2)}.$$

(c)⇒(b). For $f\in L^{p_1}(\omega_1)$, $g\in L^{p_1}(\omega_2)$ and $n\in N$, we get

$$|E_n(f)|\le E_n(|f^{p_1}\omega_1|)^{\frac{1}{p_1}}E_n(\omega_1^{-\frac{1}{p_1-1}})^{\frac{1}{p_1'}} \quad\text{and}\quad |E_n(g)|\le E_n(|g^{p_2}\omega_2|)^{\frac{1}{p_2}}E_n(\omega_2^{-\frac{1}{p_2-1}})^{\frac{1}{p_2'}}.$$

Furthermore,

$$\begin{aligned}|E_n(f)E_n(g)|^p &\le E_n(|f^{p_1}\omega_1|)^{\frac{p}{p_1}}E_n(|g^{p_2}\omega_2|)^{\frac{p}{p_2}}E_n(\omega_1^{-\frac{1}{p_1-1}})^{\frac{p}{p_1'}}E_n(\omega_2^{-\frac{1}{p_2-1}})^{\frac{p}{p_2'}}\\ &= E_n^v(|f^{p_1}\omega_1 v^{-1}|)^{\frac{p}{p_1}}E_n^v(|g^{p_2}\omega_2 v^{-1}|)^{\frac{p}{p_2}}E_n(v)E_n(\omega_1^{-\frac{1}{p_1-1}})^{\frac{p}{p_1'}}E_n(\omega_2^{-\frac{1}{p_2-1}})^{\frac{p}{p_2'}},\end{aligned}$$

where $E_n^v(\cdot)$ is the conditional expectation relative to the probability measure $\frac{v}{|\Omega|_v}d\mu$. Because of $(v,\ \omega_1,\ \omega_2)\in A_{\vec{p}}$, we get

$$|E_n(f)E_n(g)|\le CE_n^v(|f^{p_1}\omega_1 v^{-1}|)^{\frac{1}{p_1}}E_n^v(|g^{p_2}\omega_2 v^{-1}|)^{\frac{1}{p_2}}.$$

Thus

$$\mathfrak{M}(f,g)\le CM^v(f^{p_1}\omega_1 v^{-1})^{\frac{1}{p_1}}M^v(g^{p_2}\omega_2 v^{-1})^{\frac{1}{p_2}}.$$

From this, using Hölder's inequality for weak spaces (see, e.g., [7, p. 15]), we obtain

$$\begin{aligned}\|\mathfrak{M}(f,g)\|_{L^{p,\infty}(v)} &\le C\|M^v(f^{p_1}\omega_1 v^{-1})^{\frac{1}{p_1}}\|_{L^{p_1,\infty}(v)}\|M^v(g^{p_2}\omega_2 v^{-1})^{\frac{1}{p_2}}\|_{L^{p_2,\infty}(v)}\\ &= C\|M^v(f^{p_1}\omega_1 v^{-1})\|_{L^{1,\infty}(v)}^{\frac{1}{p_1}}\|M^v(g^{p_2}\omega_2 v^{-1})\|_{L^{1,\infty}(v)}^{\frac{1}{p_2}}\\ &\le C\|f^{p_1}\omega_1 v^{-1}\|_{L^1(v)}^{\frac{1}{p_1}}\|g^{p_2}\omega_2 v^{-1}\|_{L^1(v)}^{\frac{1}{p_2}}\\ &= C\|f^{p_1}\omega_1\|_{L^1}^{\frac{1}{p_1}}\|g^{p_2}\omega_2\|_{L^1}^{\frac{1}{p_2}}\\ &= C\|f\|_{L^{p_1}(\omega_1)}\|g\|_{L^{p_2}(\omega_2)}.\end{aligned}$$

(a)⇒(c). For any $n\in N$ and $B\in\mathcal{F}_n$, set $f=\omega_1^{-\frac{1}{p_1-1}}\chi_B$ and $g=\omega_2^{-\frac{1}{p_2-1}}\chi_B$. Then

$$\Big(\int_B E_n(\omega_1^{-\frac{1}{p_1-1}})^p E_n(\omega_2^{-\frac{1}{p_2-1}})^p v d\mu\Big)^{\frac{1}{p}}$$
$$\le C\Big(\int_\Omega \omega_1^{-\frac{1}{p_1-1}}\chi_B d\mu\Big)^{\frac{1}{p_1}}\Big(\int_\Omega \omega_2^{-\frac{1}{p_2-1}}\chi_B d\mu\Big)^{\frac{1}{p_2}}.$$

Furthermore,

$$\Big(\int_B E_n(\omega_1^{-\frac{1}{p_1-1}})^p E_n(\omega_2^{-\frac{1}{p_2-1}})^p E_n(v)d\mu\Big)^{\frac{1}{p}} \tag{5}$$
$$\le C\Big(\int_B E_n(\omega_1^{-\frac{1}{p_1-1}})d\mu\Big)^{\frac{1}{p_1}}\Big(\int_B E_n(\omega_2^{-\frac{1}{p_2-1}})d\mu\Big)^{\frac{1}{p_2}}.$$

We claim that there exists a constant C such that

$$\Big(E_n(\omega_1^{-\frac{1}{p_1-1}})^p E_n(\omega_2^{-\frac{1}{p_2-1}})^p E_n(v)\Big)^{\frac{1}{p}} \le C E_n(\omega_1^{-\frac{1}{p_1-1}})^{\frac{1}{p_1}} E_n(\omega_2^{-\frac{1}{p_2-1}})^{\frac{1}{p_2}}.$$

Otherwise, for any $C > 0$, let

$$B = \{E_n(\omega_1^{-\frac{1}{p_1-1}})^p E_n(\omega_2^{-\frac{1}{p_2-1}})^p E_n(v) > C E_n(\omega_1^{-\frac{1}{p_1-1}})^{\frac{p}{p_1}} E_n(\omega_2^{-\frac{1}{p_2-1}})^{\frac{p}{p_2}}\},$$

then $\mu(B) > 0$. Consequently,

$$\int_B E_n(\omega_1^{-\frac{1}{p_1-1}})^p E_n(\omega_2^{-\frac{1}{p_2-1}})^p E_n(v)d\mu > C\int_B E_n(\omega_1^{-\frac{1}{p_1-1}})^{\frac{p}{p_1}} E_n(\omega_2^{-\frac{1}{p_2-1}})^{\frac{p}{p_2}}d\mu$$
$$\ge C\int_B E_n(\omega_1^{-\frac{1}{p_1-1}\frac{p}{p_1}}\omega_2^{-\frac{1}{p_2-1}\frac{p}{p_2}})d\mu \tag{6}$$
$$= C\int_B \omega_1^{-\frac{1}{p_1-1}\frac{p}{p_1}}\omega_2^{-\frac{1}{p_2-1}\frac{p}{p_2}}d\mu$$
$$\ge C\Big(\int_B \omega_1^{-\frac{1}{p_1-1}}d\mu\Big)^{\frac{p}{p_1}}\Big(\int_B \omega_2^{-\frac{1}{p_2-1}}d\mu\Big)^{\frac{p}{p_2}}, \tag{7}$$

where (6) and (7) follow from Hölder's inequality for $E_n(\cdot)$ and the $RH(p_1, p_2)$ condition, respectively. It follows that

$$\int_B E_n(\omega_1^{-\frac{1}{p_1-1}})^p E_n(\omega_2^{-\frac{1}{p_2-1}})^p E_n(v)d\mu > C\Big(\int_B \omega_1^{-\frac{1}{p_1-1}}d\mu\Big)^{\frac{p}{p_1}}\Big(\int_B \omega_2^{-\frac{1}{p_2-1}}d\mu\Big)^{\frac{p}{p_2}},$$

which contradicts (5). By contradiction, we have

$$\Big(E_n(\omega_1^{-\frac{1}{p_1-1}})^p E_n(\omega_2^{-\frac{1}{p_2-1}})^p E_n(v)\Big)^{\frac{1}{p}} \le C E_n(\omega_1^{-\frac{1}{p_1-1}})^{\frac{1}{p_1}} E_n(\omega_2^{-\frac{1}{p_2-1}})^{\frac{1}{p_2}}.$$

Then

$$E_n(v)^{\frac{1}{p}} E_n(\omega_1^{1-p_1'})^{\frac{1}{p_1'}} E_n(\omega_2^{1-p_2'})^{\frac{1}{p_2'}} \le C.$$

This completes the proof. □

THEOREM 1.7. *Let v, ω_1, ω_2 be weights and $1 < p_1, p_2 < \infty$. Suppose that $1/p = 1/p_1 + 1/p_2$ and $(\omega_1, \omega_2) \in RH(p_1, p_2)$, then the following statements are equivalent*:

(a) *There exists a positive constant C such that*

$$\|\mathfrak{M}(f, g)\|_{L^p(v)} \le C\|f\|_{L^{p_1}(\omega_1)}\|g\|_{L^{p_2}(\omega_2)}, \quad \forall f \in L^{p_1}(\omega_1), \quad g \in L^{p_2}(\omega_2);$$

(b) *There exists a positive constant C such that*

(8) $$\|\mathfrak{M}(f\sigma_1, g\sigma_2)\|_{L^p(v)} \le C\|f\|_{L^{p_1}(\sigma_1)}\|g\|_{L^{p_2}(\sigma_2)}\,, \quad \forall f \in L^{p_1}(\sigma_1)\,, \quad g \in L^{p_2}(\sigma_2)\,,$$

where $\sigma_i = \omega_i^{-\frac{1}{p_i-1}}$, $i = 1,\ 2$;

(c) *The triple of weights* $(v,\ \omega_1,\ \omega_2)$ *satisfies the condition* $S_{\vec{p}}$.

REMARK 1.8. We mention that the first author has also obtained a similar characterization for the multisublinear maximal function in function space. The multilinear testing condition was further discussed by [2] in function space, which generalized the result in [22].

PROOF. It is clear that (a)⇔(b)⇒(c), so we omit them. To prove (c)⇒(b), we proceed in the following way. Let $f \in L^{p_1}(\sigma_1)$, $g \in L^{p_2}(\sigma_2)$. For all $k \in Z$, define stopping times

$$\tau_k = \inf\{n : |E(f\sigma_1|\mathcal{F}_n)E(g\sigma_2|\mathcal{F}_n)| > 2^k\}\,.$$

Set

$$A_{k,j} = \{\tau_k < \infty\} \cap \{2^j < E(\sigma_1|\mathcal{F}_{\tau_k})E(\sigma_2|\mathcal{F}_{\tau_k}) \le 2^{j+1}\}\,;$$

$$B_{k,j} = \{\tau_k < \infty, \tau_{k+1} = \infty\} \cap \{2^j < E(\sigma_1|\mathcal{F}_{\tau_k})E(\sigma_2|\mathcal{F}_{\tau_k}) \le 2^{j+1}\}\,, \quad j \in Z\,.$$

Then $A_{k,j} \in \mathcal{F}_{\tau_k}$, $B_{k,j} \subseteq A_{k,j}$. Moreover, $\{B_{k,j}\}_{k,j}$ is a family of disjoint sets and

$$\{2^k < \mathfrak{M}(f\sigma_1, g\sigma_2) \le 2^{k+1}\} = \{\tau_k < \infty, \tau_{k+1} = \infty\} = \bigcup_{j\in Z} B_{k,j}\,, k \in Z\,.$$

Trivially,

$$E(f\sigma_1|\mathcal{F}_{\tau_k}) = E^{\sigma_1}(f|\mathcal{F}_{\tau_k})E(\sigma_1|\mathcal{F}_{\tau_k}) \quad \text{and} \quad E(g\sigma_2|\mathcal{F}_{\tau_k}) = E^{\sigma_2}(g|\mathcal{F}_{\tau_k})E(\sigma_2|\mathcal{F}_{\tau_k})\,.$$

On each $A_{k,j}$, we have

$$\begin{aligned} 2^{kp} &\le \operatorname*{ess\,inf}_{A_{k,j}} |E(f\sigma_1|\mathcal{F}_{\tau_k})^p E(g\sigma_2|\mathcal{F}_{\tau_k})^p| \\ &\le \operatorname*{ess\,inf}_{A_{k,j}} |E^{\sigma_1}(f|\mathcal{F}_{\tau_k})E^{\sigma_2}(g|\mathcal{F}_{\tau_k})|^p \operatorname*{ess\,sup}_{A_{k,j}} \big(E(\sigma_1|\mathcal{F}_{\tau_k})E(\sigma_2|\mathcal{F}_{\tau_k})\big)^p \\ &\le 2^p \operatorname*{ess\,inf}_{A_{k,j}} |E^{\sigma_1}(f|\mathcal{F}_{\tau_k})E^{\sigma_2}(g|\mathcal{F}_{\tau_k})|^p |B_{k,j}|_v^{-1} \int_{B_{k,j}} \big(E(\sigma_1|\mathcal{F}_{\tau_k})E(\sigma_2|\mathcal{F}_{\tau_k})\big)^p v\,d\mu\,. \end{aligned}$$

To estimate $\int_\Omega \mathfrak{M}(f\sigma_1, g\sigma_2)^p v\,d\mu$, firstly we have

$$\begin{aligned} &\int_\Omega \mathfrak{M}(f\sigma_1, g\sigma_2)^p v\,d\mu \\ &\quad = \sum_{k\in Z} \int_{\{2^k < \mathfrak{M}(f\sigma_1, g\sigma_2) \le 2^{k+1}\}} \mathfrak{M}(f\sigma_1, g\sigma_2)^p v\,d\mu \\ &\quad \le 2^p \sum_{k\in Z} \int_{\{2^k < \mathfrak{M}(f\sigma_1, g\sigma_2) \le 2^{k+1}\}} 2^{kp} v\,d\mu \\ &\quad = 2^p \sum_{k\in Z, j\in Z} 2^{kp} \int_{B_{k,j}} v\,d\mu \end{aligned}$$

$$\leq 4^p \sum_{k\in Z, j\in Z} \operatorname*{ess\,inf}_{A_{k,j}} |E^{\sigma_1}(f|\mathcal{F}_{\tau_k})E^{\sigma_2}(g|\mathcal{F}_{\tau_k})|^p \int_{B_{k,j}} \big(E(\sigma_1|\mathcal{F}_{\tau_k})E(\sigma_2|\mathcal{F}_{\tau_k})\big)^p v d\mu\,.$$

It is clear that ϑ is a measure on $X = Z^2$ with

$$\vartheta(k, j) = \int_{B_{k,j}} \big(E(\sigma_1|\mathcal{F}_{\tau_k})E(\sigma_2|\mathcal{F}_{\tau_k})\big)^p v d\mu\,.$$

For the above $f \in L^{p_1}(\sigma_1)$, $g \in L^{p_2}(\sigma_2)$, define

$$T_{f,g}(k, j) = \operatorname*{ess\,inf}_{A_{k,j}} |E^{\sigma_1}(f|\mathcal{F}_{\tau_k})E^{\sigma_2}(g|\mathcal{F}_{\tau_k})|^p$$

and denote

$$E_\lambda = \Big\{(k, j);\ \operatorname*{ess\,inf}_{A_{k,j}} |E^{\sigma_1}(f|\mathcal{F}_{\tau_k})E^{\sigma_2}(g|\mathcal{F}_{\tau_k})|^p > \lambda\Big\} \quad \text{and} \quad G_\lambda = \bigcup_{(k,j)\in E_\lambda} A_{k,j}$$

for each $\lambda > 0$. Then we have

$$\begin{aligned}
|\{T_{f,g} > \lambda\}|_\vartheta &= \sum_{(k,j)\in E_\lambda} \int_{B_{k,j}} \big(E(\sigma_1|\mathcal{F}_{\tau_k})E(\sigma_2|\mathcal{F}_{\tau_k})\big)^p v d\mu \\
&= \sum_{(k,j)\in E_\lambda} \int_{B_{k,j}} \big(E(\sigma_1\chi_{G_\lambda}|\mathcal{F}_{\tau_k})E(\sigma_2\chi_{G_\lambda}|\mathcal{F}_{\tau_k})\big)^p v d\mu \\
&\leq \int_{G_\lambda} \mathfrak{M}(\sigma_1\chi_{G_\lambda}, \sigma_2\chi_{G_\lambda})^p v d\mu\,.
\end{aligned}$$

Let $\tau = \inf\big\{n\colon |E^{\sigma_1}(f|\mathcal{F}_n)E^{\sigma_2}(g|\mathcal{F}_n)|^p > \lambda\big\}$. We have $G_\lambda \subseteq \big\{\mathfrak{M}^{\sigma_1,\sigma_2}(f, g)^p > \lambda\big\} = \{\tau < \infty\}$, where $\mathfrak{M}^{\sigma_1,\sigma_2}(f, g) = \sup_{n\geq 0} |E^{\sigma_1}(f|\mathcal{F}_n)||E^{\sigma_2}(g|\mathcal{F}_n)|$. It follows from $S_{\vec{p}}$ and $RH(p_1, p_2)$ that

$$\begin{aligned}
|\{T_{f,g} > \lambda\}|_\vartheta &\leq \int_{\{\tau<\infty\}} \mathfrak{M}(\sigma_1\chi_{\{\tau<\infty\}}, \sigma_2\chi_{\{\tau<\infty\}})^p v d\mu \\
&\leq C|\{\tau < \infty\}|_{\sigma_1}^{\frac{p}{p_1}} |\{\tau < \infty\}|_{\sigma_2}^{\frac{p}{p_2}} \\
&\leq C \int_{\{\tau<\infty\}} \sigma_1^{\frac{p}{p_1}} \sigma_2^{\frac{p}{p_2}} d\mu\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_\Omega \mathfrak{M}(f\sigma_1, g\sigma_2)^p v d\mu &\leq 4^p \int_X T_{f,g} d\vartheta = 4^p \int_0^\infty |\{T_{f,g} > \lambda\}|_\vartheta d\lambda \\
&\leq C \int_0^\infty \int_{\{\tau<\infty\}} \sigma_1^{\frac{p}{p_1}} \sigma_2^{\frac{p}{p_2}} d\mu d\lambda \\
&= C \int_0^\infty \int_{\{\mathfrak{M}^{\sigma_1,\sigma_2}(f,g)^p>\lambda\}} \sigma_1^{\frac{p}{p_1}} \sigma_2^{\frac{p}{p_2}} d\mu d\lambda \\
&= C \int_\Omega \mathfrak{M}^{\sigma_1,\sigma_2}(f, g)^p \sigma_1^{\frac{p}{p_1}} \sigma_2^{\frac{p}{p_2}} d\mu
\end{aligned}$$

$$
\begin{aligned}
&\le C \int_\Omega M^{\sigma_1}(f)^p M^{\sigma_2}(g)^p \sigma_1^{\frac{p}{p_1}} \sigma_2^{\frac{p}{p_2}} d\mu \\
&\le C\Big(\int_\Omega M^{\sigma_1}(f)^{p_1} \sigma_1 d\mu \Big)^{\frac{p}{p_1}} \Big(\int_\Omega M^{\sigma_1}(f)^{p_2} \sigma_2 d\mu \Big)^{\frac{p}{p_2}} \\
&\le C \|f\|^p_{L^{p_1}(\sigma_1)} \|g\|^p_{L^{p_2}(\sigma_2)} ,
\end{aligned}
$$

where we have used Hölder's inequality. Hence (8) is valid. □

COROLLARY 1.9. *Let v, ω be weights and $1 < p < \infty$. Suppose that $\omega^{-\frac{1}{p-1}} \in L^1$. Then the following statements are equivalent*:

(a) *There exists a positive constant C such that*

$$
\Big(\int_{\{\tau<\infty\}} |f_\tau|^p v d\mu \Big)^{\frac{1}{p}} \le C\|f\|_{L^p(\omega)} , \quad \forall \tau \in \mathcal{T} , \quad f \in L^p(\omega) ;
$$

(b) *There exists a positive constant C such that*

$$
\|Mf\|_{L^{p,\infty}(v)} \le C\|f\|_{L^p(\omega)} , \quad \forall f \in L^p(\omega) ;
$$

(c) *The couple of weights (v, ω) satisfies the condition A_p.*

COROLLARY 1.10. *Let v, ω be weights and $1 < p < \infty$. Suppose that $\omega^{-\frac{1}{p-1}} \in L^1$. Then the following statements are equivalent*:

(a) *There exists a positive constant C such that*

$$
\|Mf\|_{L^p(v)} \le C\|f\|_{L^p(\omega)} , \quad \forall f \in L^p(\omega) ;
$$

(b) *There exists a positive constant C such that*

$$
\|M(f\sigma)\|_{L^p(v)} \le C\|f\|_{L^p(\sigma)} , \quad \forall f \in L^p(\sigma) ,
$$

where $\sigma = \omega^{-\frac{1}{p-1}}$;

(c) *The couple of weights (v, ω) satisfies the condition S_p.*

PROOF. If we substitute $p_1 = p_2$ and $\omega_1 = \omega_2$ into Theorem 1.6 and Theorem 1.7, then the reverse Hölder's condition is trivial and we get Corollary 1.9 and Corollary 1.10. □

1.2. Bilinear version of one-weight theory. We recall the following Proposition 1.11 which characterizes an A_p weight in martingale context (see, e.g., [13, 16]). Then, we partially give its bilinear analogue.

PROPOSITION 1.11. *Let ω be a weight and let $1 < p < \infty$. Suppose that $\omega^{-\frac{1}{p-1}} \in L^1$. Then the following statements are equivalent:*

(a) *The weight ω satisfies the condition A_p, i.e.,*

$$
\sup_{n\ge 0} E_n(\omega) E_n(\omega^{-\frac{1}{p-1}})^{p-1} \le C ;
$$

(b) *There exists a positive constant C such that*

$$\|E_n(f)\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}, \quad \forall n \in N,\ f \in L^p(\omega);$$

(c) *If $f \in L^p(\omega)$, then $E_n(f) \in L^p(\omega)$, for any $n \in N$, and*

$$\lim_{n\to\infty}\left(\int_\Omega |E_n(f) - f|^p \omega d\mu\right)^{\frac{1}{p}} = 0;$$

(d) *There exists a positive constant C such that*

$$\|Mf\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}, \quad \forall f \in L^p(\omega).$$

REMARK 1.12. In the proof of Theorem 1.6, the condition $(\omega_1, \omega_2) \in RH(p_1, p_2)$ has been used only to show that (3) implies $(v,\ \omega_1,\ \omega_2) \in A_{\vec{p}}$. Moreover, under the same assumptions as in Theorem 1.6, the following statements are equivalent:

(a) There exists a positive constant C such that for any $n \in N$, any $f \in L^{p_1}(\omega_1)$ and any $g \in L^{p_2}(\omega_2)$,

$$\left(\int_\Omega |E_n(f)E_n(g)|^p v d\mu\right)^{\frac{1}{p}} \le C\|f\|_{L^{p_1}(\omega_1)}\|g\|_{L^{p_2}(\omega_2)}; \tag{9}$$

(b) The triple of weights $(v,\ \omega_1,\ \omega_2)$ satisfies the condition $A_{\vec{p}}$.

LEMMA 1.13. *Let $\omega_1,\ \omega_2$ be weights and $1 < p_1,\ p_2 < \infty$. Suppose that $1/p = 1/p_1 + 1/p_2$, $\omega_i^{-\frac{1}{p_i-1}} \in L^1$, $i = 1,\ 2$ and $v = \omega_1^{p/p_1}\omega_2^{p/p_2}$. If $f \in L^{p_1}(\omega_1),\ g \in L^{p_2}(\omega_2)$ and $E_n(f)E_n(g) \in L^p(v)$, for any $n \in N$, then*

$$\lim_{n\to\infty}\left(\int_\Omega |E_n(f)E_n(g) - fg|^p v d\mu\right)^{\frac{1}{p}} = 0, \tag{10}$$

if and only if, for any $\varepsilon > 0$, there is a nonnegative function $y \in L^p(v)$ such that

$$\sup_{n\ge 0}\left(\int_\Omega |E_n(f)E_n(g)\chi_{\{|E_n(f)E_n(g)|\ge y\}}|^p v d\mu\right)^{\frac{1}{p}} \le \varepsilon. \tag{11}$$

PROOF. Suppose that (11) is valid. We will prove (10). For any $\varepsilon > 0$, there is a nonnegative function $y \in L^p(v)$ such that

$$\sup_{n\ge 0}\left(\int_\Omega |E_n(f)E_n(g)\chi_{\{|E_n(f)E_n(g)|\ge y\}}|^p v d\mu\right)^{\frac{1}{p}} \le \varepsilon.$$

Since $\|fg\|_{L^p(v)} \le \|f\|_{L^{p_1}(\omega_1)}\|g\|_{L^{p_2}(\omega_2)} < \infty$, we can assume that $y > |fg|$. We also have $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} g_n = g$, because the martingales $(f_n)_{n\ge 0}$ and $(g_n)_{n\ge 0}$ are uniformly integrable. Thus

$$(2y)^p \ge |f_n g_n \chi_{\{|f_n g_n| < y\}} - fg|^p \to 0, \quad \text{as } n \to \infty.$$

It follows from the dominated convergence theorem

$$\lim_{n\to\infty}\|f_n g_n \chi_{\{|f_n g_n|<y\}} - fg\|_{L^p(v)} = 0.$$

For the above ε, there is an $n_0 \in N$, such that

$$\|f_n g_n \chi_{\{|f_n g_n|<y\}} - fg\|_{L^p(v)} < \varepsilon\,, \quad \forall n > n_0\,.$$

Moreover,

$$\begin{aligned}\|f_n g_n - fg\|_{L^p(v)} &= \|f_n g_n(\chi_{\{|f_n g_n|<y\}} + \chi_{\{|f_n g_n|\geq y\}}) - fg\|_{L^p(v)} \\ &\leq (2^{\frac{1-p}{p}} \vee 1)\big(\|f_n g_n \chi_{\{|f_n g_n|<y\}} - fg\|_{L^p(v)} + \|f_n g_n \chi_{\{|f_n g_n|\geq y\}}\|_{L^p(v)}\big) \\ &< 2(2^{\frac{1-p}{p}} \vee 1)\varepsilon,\ \forall n > n_0\,,\end{aligned}$$

which implies (10).

Conversely, we assume that (10) is valid. Since $fg \in L^p(v)$, we obtain that for any $0 < \varepsilon < 1$, there exists $\delta > 0$ such that whenever $E \in \mathcal{F}$ satisfies $|E|_v < \delta$, then $\big(\int_E |fg|^p v d\mu\big)^{1/p} < \frac{1}{2(2^{(1-p)/p}\vee 1)}\varepsilon$. For the above $\varepsilon > 0$, there exists an n_0, such that

$$\left(\int_\Omega |E_n(f)E_n(g) - fg|^p v d\mu\right)^{\frac{1}{p}} < \left(\frac{1}{2(2^{\frac{1-p}{p}} \vee 1)} \wedge \delta^{\frac{1}{p}}\right)\varepsilon\,, \quad \forall n \geq n_0\,.$$

Moreover, for the above $\varepsilon > 0$ and $n \geq n_0$, we obtain that

$$\begin{aligned}|\{|E_n(f)E_n(g)| - |fg| > \varepsilon\}|_v &= \frac{1}{\varepsilon^p}\int_{\{|E_n(f)E_n(g)|-|fg|>\varepsilon\}} \varepsilon^p v d\mu \\ &\leq \frac{1}{\varepsilon^p}\int_\Omega |E_n(f)E_n(g) - fg|^p v d\mu < \delta\,.\end{aligned}$$

Let $y = \max\{2|f_1 g_1|,\ 2|f_2 g_2|, \ldots, 2|f_{n_0} g_{n_0}|,\ |fg| + 2\varepsilon\}$. It follows that $y \in L^p(v)$ and

$$\begin{aligned}&\sup_{n\geq 0}\left(\int_\Omega |E_n(f)E_n(g)\chi_{\{|E_n(f)E_n(g)|\geq y\}}|^p v d\mu\right)^{\frac{1}{p}} \\ &\quad = \sup_{n>n_0}\left(\int_{\{|E_n(f)E_n(g)|\geq y\}} |E_n(f)E_n(g)|^p v d\mu\right)^{\frac{1}{p}} \\ &\quad = \sup_{n>n_0}\left(\int_{\{|E_n(f)E_n(g)|\geq y\}} |E_n(f)E_n(g) - fg + fg|^p v d\mu\right)^{\frac{1}{p}} \\ &\quad \leq (2^{\frac{1-p}{p}} \vee 1)\sup_{n>n_0}\left(\int_\Omega |E_n(f)E_n(g) - fg|^p v d\mu\right)^{\frac{1}{p}} \\ &\qquad + (2^{\frac{1-p}{p}} \vee 1)\sup_{n>n_0}\left(\int_{\{|E_n(f)E_n(g)|-|fg|>\varepsilon\}} |fg|^p v d\mu\right)^{\frac{1}{p}} \\ &\quad < \varepsilon\,.\end{aligned}$$

This completes the proof. □

PROPOSITION 1.14. *Let $\omega_1,\ \omega_2$ be weights and $1 < p_1,\ p_2 < \infty$. Suppose that $1/p = 1/p_1 + 1/p_2$ and $v = \omega_1^{p/p_1}\omega_2^{p/p_2}$. If the triple of weights $(v,\ \omega_1,\ \omega_2)$ satisfies the* *condition* $A_{\vec{p}}$, *then*

$$\lim_{n\to\infty}\left(\int_\Omega |E_n(f)E_n(g)-fg|^p v\,d\mu\right)^{\frac{1}{p}}=0,\ \forall f\in L^{p_1}(\omega_1)\,,\quad g\in L^{p_2}(\omega_2)\,. \tag{12}$$

PROOF. Let $f\in L^{p_1}(\omega_1)$ and $g\in L^{p_2}(\omega_2)$. It follows from the condition $A_{\vec{p}}$ and Remark 1.12 that

$$\left(\int_\Omega |E_n(f)E_n(g)|^p v\,d\mu\right)^{\frac{1}{p}}\le C\|f\|_{L^{p_1}(\omega_1)}\|g\|_{L^{p_2}(\omega_2)},\ \forall n\in N\,,$$

which is the assumption of the Lemma 1.13. If (11) is valid , we have (12) by the Lemma 1.13. We will prove (11) in the following way. Since f and g are integrable, the martingales $(f_n)_{n\ge 0}$ and $(g_n)_{n\ge 0}$ are uniformly integrable. It follows from Doob's inequality that

$$\sup_{\lambda>0}\lambda|\{Mf>\lambda\}|\le\int_\Omega |f|d\mu \text{ and } \sup_{\lambda>0}\lambda|\{Mg>\lambda\}|\le\int_\Omega |g|d\mu\,. \tag{13}$$

For $n\in N$, fix $\lambda>0$, which will be determined later. Then,

$$\begin{aligned}
&\left(\int_\Omega |E_n(f)E_n(g)\chi_{\{|E_n(f)E_n(g)|\ge\lambda\}}|^p v\,d\mu\right)^{\frac{1}{p}}\\
&\quad=\left(\int_\Omega |E_n(f\chi_{\{|E_n(f)E_n(g)|\ge\lambda\}})E_n(g\chi_{\{|E_n(f)E_n(g)|\ge\lambda\}})|^p v\,d\mu\right)^{\frac{1}{p}}\\
&\quad\le\left(\int_\Omega E_n(|f\chi_{\{MfMg\ge\lambda\}}|)^p E_n(|g\chi_{\{MfMg\ge\lambda\}}|)^p v\,d\mu\right)^{\frac{1}{p}}\\
&\quad\le C\|f\chi_{\{MfMg\ge\lambda\}}\|_{L^{p_1}(\omega_1)}\|g\chi_{\{MfMg\ge\lambda\}}\|_{L^{p_2}(\omega_2)}\,, \qquad (14)
\end{aligned}$$

where (14) is a result of Remark 1.12. It is clear that

$$\{MfMg\ge\lambda\}\subseteq\{Mf\ge\lambda^{\frac{p}{p_1}}\}\cup\{Mg\ge\lambda^{\frac{p}{p_2}}\}\,.$$

Thus $|\{MfMg\ge\lambda\}|\le|\{Mf\ge\lambda^{p/p_1}\}|+|\{Mg\ge\lambda^{p/p_2}\}|$. Combing with (13), we get $\lim_{\lambda\to\infty}|\{MfMg\ge\lambda\}|=0$. Then, (11) follows from (14), because of the absolute continuity of the integral. □

PROPOSITION 1.15. *Let* $\omega_1,\ \omega_2$ *be weights and* $1<p_1,\ p_2<\infty$. *Suppose that* $1/p=1/p_1+1/p_2$ *and* $v=\omega_1^{p/p_1}\omega_2^{p/p_2}$. *If there exists a positive constant C such that*

$$\|\mathfrak{M}(f,g)\|_{L^p(v)}\le C\|f\|_{L^{p_1}(\omega_1)}\|g\|_{L^{p_2}(\omega_2)}\,,\quad\forall f\in L^{p_1}(\omega_1)\,,\quad g\in L^{p_2}(\omega_2)\,,$$

we have $(v,\ \omega_1,\ \omega_2)\in A_{\vec{p}}$, (9) *and* (12).

REMARK 1.16. The proof of Proposition 1.15 is clear and we omit it. But we can not give the converse of the Proposition 1.15 in martingale spaces.

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