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# CONTACT 3-MANIFOLDS WITH THE REEB FLOW SYMMETRY

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**Abstract.** We prove that the Ricci operator on a contact Riemannian 3-manifold $M$ is invariant along the Reeb flow if and only if $M$ is Sasakian or locally isometric to SU(2) (or SO(3)), SL(2, $\boldsymbol{R}$) (or $O(1,2)$), the group $E(2)$ of rigid motions of Euclidean 2-plane with a contact left invariant Riemannian metric.

**1. Introduction.** In a contact manifold $(M, \eta)$, we find a fundamental fact that the Reeb vector field $\xi$ generates a contact transformation, that is, $\pounds_\xi \eta = 0$. For an associated Riemannian metric $g$, if $\xi$ generates an isometric flow, that is, $M$ satisfies $\pounds_\xi g = 0$, then $M$ is said to be $K$-contact. We note that a $K$-contact manifold is already Sasakian in dimension three. In this paper, we study a 3-dimensional contact Riemannian manifold whose Ricci operator $S$ is Reeb flow invariant, that is, $\pounds_\xi S = 0$. Then, we have

MAIN THEOREM. *Let $M$ be a 3-dimensional contact Riemannian manifold. Then $\pounds_\xi S = 0$ if and only if $M$ is Sasakian or locally isometric to* SU(2) (*or* SO(3)), SL(2, $\boldsymbol{R}$) (*or* $O(1,2)$), $E(2)$ (*the group of rigid motions of Euclidean 2-plane*) *with a left invariant contact Riemannian metric.*

All manifolds in the present paper are assumed to be connected and of class $C^\infty$.

**2. Preliminaries.** A 3-dimensional manifold $M$ is said to be a contact manifold if it admits a global 1-form $\eta$ such that $\eta \wedge (d\eta) \neq 0$ everywhere. Given a contact form $\eta$, we have a unique vector field $\xi$, which is called the characteristic vector field, satisfying $\eta(\xi) = 1$ and $\pounds_\xi \eta = 0$ (or $i_\xi d\eta = 0$), where $\pounds_\xi$ denotes Lie differentiation for $\xi$ and $i_\xi$ denotes the interior product operator by $\xi$. It is well-known that there exists a Riemannian metric $g$ and a $(1,1)$-tensor field $\varphi$ such that

$$\eta(X) = g(X, \xi)\,, \quad d\eta(X, Y) = g(X, \varphi Y)\,, \quad \varphi^2 X = -X + \eta(X)\xi\,, \tag{1}$$

where $X$ and $Y$ are vector fields on $M$. From (1) it follows that

$$\varphi\xi = 0\,, \quad \eta \circ \varphi = 0\,, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)\,. \tag{2}$$

A manifold $M$ equipped with structure tensors $(\varphi, \xi, \eta, g)$ satisfying (1) and (2) is said to be a contact Riemannian manifold and is denoted by $M = (M; \eta, g)$. Given a contact Riemannian manifold $M$, we define a $(1,1)$-tensor field $h$ by $h = \frac{1}{2}\pounds_\xi \varphi$. Then $h$ is self-adjoint and satisfies

$$h\xi = 0 \quad \text{and} \quad h\varphi = -\varphi h\,, \tag{3}$$

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$$\nabla_X \xi = -\varphi X - \varphi h X\,, \tag{4}$$

where $\nabla$ is Levi-Civita connection. From (3) and (4) we see that each trajectory of $\xi$ is a geodesic and $\mathrm{div}(\xi) = 0$. We denote by $R$ the Riemannian curvature tensor defined by

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z$$

for all vector fields $X, Y, Z$. Along a trajectory of $\xi$, the Jacobi operator $\ell = R(\cdot, \xi)\xi$ is a symmetric $(1, 1)$-tensor field, that is, $g(\ell X, Y) = g(X, \ell Y)$. We have

$$\mathrm{trace}\ \ell = \rho(\xi, \xi) = 2n - \mathrm{trace}\ (h^2)\,, \tag{5}$$

$$\nabla_\xi h = \varphi - \varphi\ell - \varphi h^2\,, \tag{6}$$

$$g(R(X, Y)\xi, Z) = g((\nabla_Z\varphi)X, Y) + g((\nabla_Y\varphi h)X - (\nabla_X\varphi h)Y, Z) \tag{7}$$

for all vector fields $X, Y, Z$ on $M$, where $\rho(X, Y) = g(SX, Y)$. A contact Riemannian manifold for which $\xi$ is Killing is called a $K$-contact Riemannian manifold. It is easy to see that a contact Riemannian manifold is $K$-contact if and only if $h = 0$. For a contact Riemannian manifold $M$ one may define naturally an almost complex structure $J$ on $M \times \boldsymbol{R}$;

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

where $X$ is a vector field tangent to $M$, $t$ the coordinate of $\boldsymbol{R}$ and $f$ a function on $M \times \boldsymbol{R}$. If the almost complex structure $J$ is integrable, $M$ is said to be normal or Sasakian. It is known that $M$ is normal if and only if $M$ satisfies

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0\,,$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of $\varphi$. A Sasakian manifold is characterized by a condition

$$(\nabla_X\varphi)Y = g(X, Y)\xi - \eta(Y)X \tag{8}$$

for all vector fields $X$ and $Y$ on the manifold. For more details about contact Riemannian manifolds we refer to [1].

**3. Contact 3-manifolds with the Reeb flow symmetry.** In this section, we prove the Main Theorem. First we recall that a contact Riemannian 3-manifold $M$ satisfies

$$(\nabla_X\varphi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX) \tag{9}$$

(cf. [5]). From (7) and (9) we have

$$R(X, Y)\xi = \eta(Y)(X + hX) - \eta(X)(Y + hY) + \varphi((\nabla_Y h)X - (\nabla_X h)Y) \tag{10}$$

for all vector fields $X$ and $Y$. From (8) and (9), we have at once

LEMMA 1. *A 3-dimensional contact Riemannian manifold is Sasakian if and only if* $h = 0$.

Moreover, we have

PROPOSITION 2. *A Sasakian 3-manifold is $\eta$-Einstein, that is, $S = \alpha I + \beta\eta\otimes\xi$, where $\alpha$ and $\beta$ are functions with $d\alpha(\xi) = d\beta(\xi) = 0$.*

Since $£_\xi \xi = £_\xi \eta = 0$, we have

COROLLARY 3. *For a Sasakian 3-manifold*, $£_\xi S = 0$.

Now, we prove the Main Theorem.

PROOF OF MAIN THEOREM. Let $M = (M^3; \eta, g)$ be a 3-dimensional contact Riemannian manifold. Then it is well-known that the curvature tensor $R$ of a 3-dimensional Riemannian manifold is expressed by

$$
\begin{aligned}
R(Y, X)Z =& \rho(X, Z)Y - \rho(Y, Z)X + g(X, Z)SY - g(Y, Z)SX \\
& - \frac{\tau}{2}\{g(X, Z)Y - g(Y, Z)X\}
\end{aligned}
\tag{11}
$$

for all vector fields $X, Y, Z$, where $\tau$ denotes the scalar curvature. If $h = 0$ on $M$, then from Lemma 1 we see that $M$ is Sasakian. Moreover, $M$ satisfies $£_\xi S = 0$ (Corollary 3). So, we consider on $M$ the maximal open subset $U_1$ on which $h \neq 0$ and the maximal open subset $U_2$ on which $h$ is identically zero. ($U_2$ is the union of all points $p$ in $M$ such that $h = 0$ in a neighborhood of $p$). $U_1 \cup U_2$ is open and dense in $M$. Suppose that $M$ is non-Sasakian. Then $U_1$ is non-empty and there is a local orthonormal frame field $\{e_1 = e, e_2 = \varphi e, e_3 = \xi\}$ on $U_1$ such that $h(e_1) = \lambda e_1$, $h(e_2) = -\lambda e_2$ for some positive function $\lambda$. We denote $\Gamma_{ijk} = g(\nabla_{e_i} e_j, e_k)$, $\rho_{ij} = \rho(e_i, e_j)$ for $i, j, k = 1, 2, 3$. Then from (4) we get

$$
\Gamma_{132} = -\Gamma_{123} = -(1 + \lambda)\,, \quad \Gamma_{231} = -\Gamma_{213} = 1 - \lambda \tag{12}
$$

and

$$
\Gamma_{131} = \Gamma_{113} = \Gamma_{232} = \Gamma_{223} = 0\,. \tag{13}
$$

Also, from (6) and taking account of (5) and (11), we have

$$
\xi\lambda = \rho_{12} \tag{14}
$$

and

$$
4\lambda\Gamma_{312} = \rho_{22} - \rho_{11}\,. \tag{15}
$$

LEMMA 4. *In* $U_1$, $£_\xi S = 0 \Leftrightarrow \nabla_\xi S = 0$ *and* $S\xi = \sigma\xi$, *where* $\sigma$ *is a function.*

PROOF. Suppose that $M$ satisfies $£_\xi S = 0$. Then, we compute

$$
\begin{aligned}
0 &= £_\xi(SX) - S(£_\xi X) \\
&= [\xi, SX] - S[\xi, X]\,.
\end{aligned}
$$

From this using (4) we get an equivalent equation to $£_\xi S = 0$:

$$
(\nabla_\xi S)X = (S\varphi - \varphi S)X + (S\varphi h - \varphi h S)X\,. \tag{16}
$$

Since $\nabla_\xi S$ and $S\varphi - \varphi S$ are self-adjoint operators, we get

$$
S\varphi h - \varphi h S = S h\varphi - h\varphi S\,.
$$

Since $h\varphi = -\varphi h$, it follows that

$$
S\varphi h = \varphi h S\,. \tag{17}
$$

Since $h\xi = 0$, from (17) we see that $hS\xi = 0$. From this and (5), we obtain $S\xi = \sigma\xi$, $\sigma = 2 - 2\lambda^2$ on $U_1$. And from (16) and (17) we get

$$\nabla_\xi S = S\varphi - \varphi S\,. \tag{18}$$

So, we get $(\nabla_\xi \rho)(\xi, \xi) = 0$, and then $\xi\lambda = 0$, where we have used $\nabla_\xi \xi = 0$. Then from (14) we have

$$\rho_{12} = \rho_{21} = 0\,. \tag{19}$$

Applying $e_1$ to (17) and taking an inner product with $e_2$ (with respect to $g$), we get

$$\rho_{22} = \rho_{11} \tag{20}$$

on $U_1$. Since $S\xi = \sigma\xi$, together with (19), we have $S\varphi = \varphi S$ on $U_1$. Thus, from (18) we obtain $\nabla_\xi S = 0$ on $U_1$. Conversely, we assume that $\nabla_\xi S = 0$ and $S\xi = \sigma\xi$ on $U_1$. Then, it follows from (5) that $\sigma = 2 - 2\lambda^2$, and

$$\rho_{13} = \rho_{31} = 0\,, \quad \rho_{23} = \rho_{32} = 0\,. \tag{21}$$

And from $(\nabla_\xi \rho)(\xi, \xi) = 0$ and $\nabla_\xi \xi = 0$ we have

$$\xi\lambda = 0\,, \tag{22}$$

which together with (14) yields

$$\rho_{12} = \rho_{21} = 0\,. \tag{23}$$

Using $\nabla_\xi S = 0$ again, we obtain from (23)

$$\Gamma_{312}(\rho_{11} - \rho_{22}) = 0\,. \tag{24}$$

By (15) and (24) we find that

$$\rho_{11} = \rho_{22}\,. \tag{25}$$

Since $S\xi = \sigma\xi$, equations (23) and (25) give $S\varphi = \varphi S$. Moreover, we see that $S\varphi h = \varphi h S$ on $U_1$. Therefore, by (16) we find that $\pounds_\xi S = 0$. This completes the proof of Lemma 4. □

Now we prove

LEMMA 5. *$\lambda$ is constant.*

PROOF. Among the proof of Lemma 4, from (15) and (25) we get in addition

$$\Gamma_{312} = \Gamma_{321} = 0\,. \tag{26}$$

From (11) with the help of Lemma 4 we have in $U_1$:

$$
\begin{aligned}
&R(e_1, e_2)e_2 = Se_1 - (1-\lambda^2)e_1\,,\\
&R(e_1, e_2)e_1 = -Se_2 + (1-\lambda^2)e_2\,,\\
&R(e_2, e_3)e_2 = R(e_1, e_3)e_1 = -(1-\lambda^2)e_3\,,\\
&R(e_1, e_3)e_3 = (1-\lambda^2)e_1\,,\\
&R(e_2, e_3)e_3 = (1-\lambda^2)e_2\,,\\
&R(e_i, e_j)e_k = 0 \text{ for } i \neq j \neq k \neq i\,.
\end{aligned}
\tag{27}
$$

Using (12), (13), (26), and (27), we have

$$
\begin{aligned}
&(\nabla_{e_1} R)(e_2, e_3)e_2 = e_1(\lambda^2 - 1)e_3\,,\\
&(\nabla_{e_2} R)(e_3, e_1)e_2 = (\nabla_{e_3} R)(e_1, e_2)e_2 = 0
\end{aligned}
\tag{28}
$$

and

$$
\begin{aligned}
&(\nabla_{e_2} R)(e_1, e_3)e_1 = e_2(\lambda^2 - 1)e_3\,,\\
&(\nabla_{e_1} R)(e_3, e_2)e_1 = (\nabla_{e_3} R)(e_2, e_1)e_1 = 0\,.
\end{aligned}
\tag{29}
$$

By the second Bianchi identity, (28) and (29) yield that $e_1(\lambda) = 0$ and $e_2(\lambda) = 0$ respectively. Hence, together with (22), we see that $\lambda$ is constant on $M$, where we have used the continuity argument of $\lambda$. □

On account of (27) we find that $R(e_1, e_2)\xi = 0$ in $M$. Here we use (10). Since $\lambda$ is constant, we have

$$
\Gamma_{212}e_1 - \Gamma_{121}e_2 = 0\,. \tag{30}
$$

From (30) we get

$$
\Gamma_{212} = \Gamma_{221} = \Gamma_{121} = \Gamma_{112} = 0\,. \tag{31}
$$

Thus, together with (12), (13), (26), and (31), we have

$$
[e_1, e_2] = 2e_3\,, \quad [e_2, e_3] = (1-\lambda)e_1\,, \quad [e_3, e_1] = (1+\lambda)e_2\,. \tag{32}
$$

Actually, from (32) we compute the Ricci operator $S$:

$$
\begin{aligned}
&Se_1 = 0\,,\\
&Se_2 = 0\,,\\
&Se_3 = \left(2 - 2\lambda^2\right)e_3\,.
\end{aligned}
\tag{33}
$$

Moreover, we can check $(£_\xi S)e_1 = (£_\xi S)e_2 = 0$.

After all, owing to J. Milnor's result (Section 4 or [2]), we see from (32) that $M$ is locally isometric to one of the following Lie groups:

(i) SU(2) (or SO(3)) with a left invariant metric when $0 < \lambda < 1$;
(ii) SL(2, $\boldsymbol{R}$) (or O(1, 2)) with a left invariant metric when $\lambda > 1$;
(iii) $E(2)$ when $\lambda = 1$.

Thus we have proved our Main Theorem. □

We see from Proposition 2 that a Sasakian 3-manifold satisfies $S\xi = \sigma\xi$ and $\nabla_\xi S = 0$, where $\sigma$ is a function.

COROLLARY 6. *Let $M$ be a 3-dimensional contact Riemannian manifold. Then $S\xi = \sigma\xi$ and $\nabla_\xi S = 0$ if and only if $M$ is Sasakian or locally isometric to* SU(2) (*or* SO(3)), SL(2, R) (*or* $O(1,2)$), *the group* $E(2)$ *of rigid motions of Euclidean 2-plane with a left invariant contact Riemannian metric.*

We can not remove the condition $S\xi = \sigma\xi$ in Corollary 6. Indeed, we have a counter example. See Remark 2 in the next section.

**4. 3-dimensional Lie groups.** By a theorem due to K. Sekigawa [4] and the classification due to J. Milnor [2] of 3-dimensional Lie groups with a left invariant metric, Perrone [3] classified all simply connected homogeneous contact Riemannian 3-manifolds. Recall that $M$ is called unimodular if its left invariant Haar measure is also right invariant. In terms of the Lie algebra $\mathfrak{m}$, $M$ is unimodular if and only if the adjoint transformation $ad_X$ has trace zero for every $X \in \mathfrak{m}$. Then we have

PROPOSITION 7 ([5]). *Let $M$ be a* 3*-dimensional unimodular Lie group with a left invariant contact Riemannian structure, then there exists an orthonormal basis* $\{e_1, e_2 = \varphi e_1, e_3 = \xi\} \in \mathfrak{m}$ *such that*

$$[e_1, e_2] = 2e_3,\ [e_2, e_3] = c_2 e_1,\ [e_3, e_1] = c_3 e_2\,. \tag{34}$$

REMARK 1 ([3]). In fact, every three-dimensional unimodular Lie group, with only exception of the commutative Lie group $\boldsymbol{R}^3$, admits a left-invariant contact metric structure.

We put

$$\Gamma_{ijk} = g\,(\nabla_{e_i} e_j,\ e_k) \quad \text{for} \quad i,\ j,\ k = 1,\ 2,\ 3\,.$$

Then by using the Koszul formula we have

$$\begin{cases} \Gamma_{123} = \dfrac{1}{2}(c_3 - c_2 + 2)\,, \\ \Gamma_{213} = \dfrac{1}{2}(c_3 - c_2 - 2)\,, \\ \Gamma_{312} = \dfrac{1}{2}(c_3 + c_2 - 2)\,, \\ \text{all others are zero.} \end{cases} \tag{35}$$

From (35) we easily see that $M$ is $K$-contact (or Sasakian) if and only if $c_2 = c_3$. Then, using (35), we find by a direct calculation

$$
\begin{aligned}
R(e_1, e_2)e_2 &= \left(\frac{1}{4}(c_3 - c_2)^2 - 3 + c_3 + c_2\right) e_1 \\
R(e_1, e_3)e_3 &= \left(-\frac{1}{4}(c_3 - c_2)^2 - \frac{1}{2}({c_3}^2 - {c_2}^2) + 1 - c_2 + c_3\right) e_1 \\
R(e_2, e_1)e_1 &= \left(\frac{1}{4}(c_3 - c_2)^2 - 3 + c_3 + c_2\right) e_2 \\
R(e_2, e_3)e_3 &= \left(\frac{1}{4}(c_3 + c_2)^2 - {c_2}^2 + 1 + c_2 - c_3\right) e_2 \\
R(e_3, e_1)e_1 &= \left(-\frac{1}{4}(c_3 - c_2)^2 - \frac{1}{2}({c_3}^2 - {c_2}^2) + 1 - c_2 + c_3\right) e_3 \\
R(e_3, e_2)e_2 &= \left(\frac{1}{4}(c_3 + c_2)^2 - {c_2}^2 + 1 + c_2 - c_3\right) e_3 .
\end{aligned} \tag{36}
$$

By using (36) we get

$$
\begin{aligned}
Se_1 &= \left(-\frac{1}{2}({c_3}^2 - {c_2}^2) - 2 + 2c_3\right) e_1 \\
Se_2 &= \left(\frac{1}{2}({c_3}^2 - {c_2}^2) - 2 + 2c_2\right) e_2 \\
Se_3 &= \left(-\frac{1}{2}(c_3 - c_2)^2 + 2\right) e_3 .
\end{aligned} \tag{37}
$$

Since $c_1 = 2 > 0$, the possible combinations of the signs of $c_1, c_2$ and $c_3$ and the associated Lie groups are indicated in the following table (see [2]):

| Signature of $(c_1, c_2, c_3)$ | Associated Lie group |
|---|---|
| (+, +, +) | SU(2) or SO(3) |
| (+, +, −) | SL(2, $\boldsymbol{R}$) or $O(1, 2)$ |
| (+, +, 0) | $E(2)$ |
| (+, −, −) | SL(2, $\boldsymbol{R}$) or $O(1, 2)$ |
| (+, −, 0) | $E(1, 1)$ |
| (+, 0, 0) | Heisenberg group $\mathrm{Nil}_3$ |

SU(2): group of $2 \times 2$ unitary matrices of determinant 1; homeomorphic to the unit 3-sphere.

SO(3): rotation group of Euclidean 3-space, isomorphic to $SU(2)/\{\pm I\}$.

SL(2, $\boldsymbol{R}$): group of $2 \times 2$ real matrices of determinant 1.

$O(1, 2)$: Lorentz group consisting of linear transformations preserving the quadratic form $t^2 - x^2 - y^2$. Its identity component is isomorphic to $SL(2, \boldsymbol{R})/\{\pm I\}$, or to the group of rigid motions of hyperbolic 2-space.

$E(2)$: group of rigid motions of Euclidean 2-space.

$E(1, 1)$: group of rigid motions of Minkowski 2-space.

Finally, the Heisenberg group can be described as the group of all $3 \times 3$ real matrices of the form

$$\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}.$$

PROPOSITION 8 ([3]). *Let $M$ be a 3-dimensional non-unimodular Lie group with left invariant contact Riemannian structure. Then there exists an orthonormal basis $\{e_1, e_2 = \varphi e_1, e_3 = \xi\} \in \mathfrak{m}$ such that*

$$[e_1, e_2] = \alpha e_2 + 2e_3, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = \gamma e_2, \tag{38}$$

*where $\alpha \neq 0$. Moreover, $M$ is Sasakian if and only if $\gamma = 0$.*

By using the Koszul formula we see from (38)

$$\begin{cases} \Gamma_{123} = \dfrac{\gamma + 2}{2} \\ \Gamma_{212} = -\alpha \\ \Gamma_{213} = \dfrac{\gamma - 2}{2} \\ \Gamma_{312} = \dfrac{\gamma - 2}{2} \\ \text{all others are zero.} \end{cases} \tag{39}$$

Then, using (39), we obtain by a direct calculation

$$\begin{aligned} R(e_1, e_2)e_2 &= \left(\frac{\gamma^2 + 4\gamma - 12}{4} - \alpha^2\right) e_1 \\ R(e_1, e_3)e_3 &= \left(\frac{-3\gamma^2 + 4\gamma + 4}{4}\right) e_1 \\ R(e_2, e_1)e_1 &= \left(\frac{\gamma^2 + 4\gamma - 12}{4} - \alpha^2\right) e_2 + \alpha\gamma e_3 \\ R(e_2, e_3)e_3 &= \frac{(\gamma - 2)^2}{4}\, e_2 \\ R(e_3, e_1)e_1 &= \alpha\gamma e_2 + \left(\frac{-3\gamma^2 + 4\gamma + 4}{4}\right) e_3 \\ R(e_3, e_2)e_2 &= \frac{(\gamma - 2)^2}{4}\, e_3, \end{aligned} \tag{40}$$

and thus

$$\begin{aligned} Se_1 &= \left(-\alpha^2 - 2 + 2\gamma - \frac{\gamma^2}{2}\right) e_1 \\ Se_2 &= \left(-\alpha^2 - 2 + \frac{\gamma^2}{2}\right) e_2 + \alpha\gamma e_3 \\ Se_3 &= \alpha\gamma e_2 + \left(2 - \frac{\gamma^2}{2}\right) e_3. \end{aligned} \tag{41}$$

THEOREM 9. *Let $M$ be a 3-dimensional Lie group with left invariant contact Riemannian structure. Suppose that $M$ satisfies $\nabla_\xi S = 0$.*
(a) *If $M$ is unimodular, then $M$ is isometric to one of the following Lie groups:*
   (i) $\mathrm{SU}(2)$ (*or* $\mathrm{SO}(3)$) *with Sasakian metric or* $(non-Sasakian)$ *contact Riemannian metric,*
   (ii) $\mathrm{SL}(2, \boldsymbol{R})$ (*or* $O(1,2)$) *with Sasakian metric or* $(non-Sasakian)$ *contact Riemannian metric,*
   (iii) *Heisenberg group with Sasakian metric,*
   (iv) $E(2)$ *with contact Riemannian metric.*
(b) *If $M$ is non-unimodular, then the Lie algebra structure is given by* (38) *with* $\gamma = 0$ $(Sasakian)$ *or* $\gamma = 2$.

PROOF. (a) By using (35) and (37), we obtain

$$(\nabla_{e_3} S)e_1 = \frac{1}{2}(c_2 - c_3)(c_3 + c_2 - 2)^2 e_2$$

and

$$\begin{aligned}(\nabla_{e_3} S)e_2 &= \frac{1}{2}(c_2 - c_3)(c_3 + c_2 - 2)^2 e_1,\\ (\nabla_{e_3} S)e_3 &= 0\,.\end{aligned}$$

Thus we see that $\nabla_\xi S = 0$ if and only if $c_3 = c_2$ or $c_3 + c_2 = 2$. Then, referring the Table we obtain (a).

(b) By using (39) and (41), we obtain

$$\begin{aligned}(\nabla_{e_3} S)e_1 &= -\frac{1}{2}\gamma(\gamma - 2)^2 e_2 - \frac{1}{2}\alpha\gamma(\gamma - 2)e_3\,,\\ (\nabla_{e_3} S)e_2 &= -\frac{1}{2}\gamma(\gamma - 2)^2 e_1\,,\\ (\nabla_{e_3} S)e_3 &= -\frac{1}{2}\alpha\gamma(\gamma - 2)e_1\,.\end{aligned}$$

Since $\alpha \neq 0$ from the above equations, we see that $\nabla_\xi S = 0$ if and only if $\gamma = 0$ or $\gamma = 2$. □

REMARK 2. From Theorem 9, we find that the non-unimodular Lie group whose Lie algebra structure is given by (38) with $\gamma = 2$ satisfies $\nabla_\xi S = 0$, but $S\xi \neq \sigma\xi$. In fact, we see from (41) that $S\xi = 2\alpha e_2$ $(\alpha \neq 0)$.

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