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CORRIGENDUM TO "ON A CLASS OF FOLIATED NON-KÄHLERIAN COMPACT COMPLEX SURFACES"

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Abstract. In this note we correct a mistake in author's paper Tohoku Mathematical Journal Vol. 63, No. 3 (2011), 441–460.

In the paper [Br, p. 444], we stated a certain property of the pair $(\tilde{S}, \tilde{C})$, namely that $\tilde{S} \setminus \tilde{C}$ is simply connected. The proof given there, however, is not convincing; the problem is that, when we take a Nakamura deformation (S', C') of (S, C), the complement $S \setminus C$ is certainly *not* diffeomorphic to $S' \setminus C'$; it is instead diffeomorphic to $S' \setminus (C' \cup \Delta)$, where $\Delta \subset S'$ is a smooth disc with boundary on C' (vanishing cycle). Hence starting with a rational curve R' in S' which intersects C' at a single point, it is not clear how to find a smooth sphere Σ in S intersecting C again at a single point, because R' could intersect Δ.

In this note we correct that mistake. The main point is the following weaker statement.

PROPOSITION 1. *In above situation, we have* $\operatorname{rank} H_1(\tilde{S} \setminus \tilde{C}, \boldsymbol{Z}) = 0$.

In [Br], the property "$\pi_1(\tilde{S} \setminus \tilde{C}) = 0$" is used exclusively in the proof of [Br, Lemma 2.2], and it is immediate to check that the weaker property provided by Proposition 1 is largely sufficient to prove [Br, Lemma 2.2].

Let us now prove Proposition 1.

We use Mayer-Vietoris sequence, with integer coefficients, applied to the covering $\{U, V\}$ of $\tilde{S}$, with U a tubular neighborhood of $\tilde{C}$ and $V = \tilde{S} \setminus \tilde{C}$:

$$H_2(U) \oplus H_2(V) \xrightarrow{i_*} H_2(\tilde{S}) \xrightarrow{\partial_*} H_1(U \cap V) \longrightarrow H_1(U) \oplus H_1(V) \longrightarrow H_1(\tilde{S})\,.$$

We have $H_1(\tilde{S}) = 0$, $H_1(U) = H_1(\tilde{C}) = 0$, and $H_1(U \cap V) = H_1(\partial U) = \boldsymbol{Z}^2$, since the boundary ∂U is diffeomorphic to $\boldsymbol{T}^2 \times \boldsymbol{R}$. Hence, the statement of the proposition is equivalent to say that there exist two classes in $H_2(\tilde{S})$ whose images by ∂_* in $H_1(U \cap V) = \boldsymbol{Z}^2$ are linearly independent.

The group $H_2(U)$ is generated by the rational curves $\{C_j\}_{j \in \boldsymbol{Z}}$ composing $\tilde{C}$, each one of selfintersection -3. It follows that for every nontrivial class $A \in H_2(U)$ we have $A \cdot A \leq 3$ (to see this, observe that the intersection matrix Q of a chain of (-3)-curves can be written as $Q_0 - 1$, where Q_0 is the intersection matrix of a chain of (-2)-curves, which is still negative

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definite). On the other side, for every nontrivial class $B \in H_2(V)$ we have $B \cdot B \leq -1$, because the intersection form is negative definite.

Return now to the deformation argument of [Br, p. 444]. It shows, at least, that we can find a smooth oriented sphere Σ in S homologous to the exceptional rational curve $R' \subset S'$. In particular, $\Sigma \cdot \Sigma = -1$ or -2 and $\Sigma \cdot C = R' \cdot C' = 1$ (as already observed, in spite of this Σ could intersect C many times). Let $\tilde{\Sigma} \subset \tilde{S}$ be a diffeomorphic lifting of Σ to $\tilde{S}$. We claim that $[\tilde{\Sigma}] \in H_2(\tilde{S})$ is not in the image of $H_2(U) \oplus H_2(V)$ by i_*, and consequently not in the kernel of ∂_*. Indeed, suppose by contradiction that $[\tilde{\Sigma}] = A + B$, with $A \in H_2(U)$ and $B \in H_2(V)$, so that $\tilde{\Sigma} \cdot \tilde{\Sigma} = A \cdot A + B \cdot B$. Since this selfintersection is -1 or -2, the only possibility is that $A = 0$. Thus $\tilde{\Sigma}$ is homologous to a simplicial complex with support disjoint from $\tilde{C}$, and, by projection, Σ is homologous to a simplicial complex with support disjoint from C. But this is in contradiction with $\Sigma \cdot C = 1$.

Take now a second lifting of Σ to $\tilde{S}$, say $\tilde{\Sigma}_1 = \varphi(\tilde{\Sigma})$ where φ is the generator of the deck transformations. By the previous argument, the two classes $\sigma = \partial_*([\tilde{\Sigma}])$ and $\sigma_1 = \partial_*([\tilde{\Sigma}_1])$ are both nonzero in $H_1(U \cap V) = \boldsymbol{Z}^2$, and we claim that they are also linearly independent. Indeed, these two classes are related by $\sigma_1 = M(\sigma)$, where $M \in \mathrm{SL}(2, \boldsymbol{Z})$ is the monodromy of the $\boldsymbol{T}^2$-bundle over $\boldsymbol{S}^1$ corresponding to the boundary of a tubular neighborhood $U_0 = U/\varphi$ of C in S. This monodromy is of hyperbolic type ($|\mathrm{Tr}(M)| > 2$), and hence for every nonzero $(n, m) \in \boldsymbol{Z}^2$ we have that (n, m) and $M(n, m)$ are linearly independent. In particular, this applies to $(n, m) = \sigma$.

EXAMPLE 2. Let us conclude with an example showing that the argument of [Br, p. 444], used in a different situation, leads to a wrong conclusion. We take Kato surface S of intermediate type, $b_2(S) = 2$. There is a cycle $C \subset S$ and a smooth rational curve $D \subset S$, with $D \cdot D = -2$ and $D \cdot C = 1$. Again by Nakamura's deformation theorem, we can deform S to a blown up Hopf surface S', in such a way that C is deformed to an elliptic curve C' and, moreover, D is preserved, i.e., deformed to a rational curve D' with $D' \cdot D' = -2$ and $D' \cdot C' = 1$. Necessarily, there is on S' another rational curve E', with $E' \cdot E' = -1$, $E' \cdot D' = 1$ and $E' \cdot C' = 0$. If it would be possible to deform E' to a smooth sphere $\Sigma \subset S$ with $\Sigma \cap D = \{1 \text{ point}\}$ and $\Sigma \cap C = \emptyset$, then we would obtain that a loop linked around D would be homotopic to zero in $S \setminus (C \cup D)$. But this is not true.

REFERENCES

[Br] M. BRUNELLA, On a class of foliated non-Kählerian compact complex surfaces, Tohoku Math. J. 63 (2011), 441–460.

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