Tohoku Math. J.
61 (2009), 417–440

ON THE SOLUTIONS OF SET-VALUED STOCHASTIC DIFFERENTIAL EQUATIONS IN M-TYPE 2 BANACH SPACES

JINPING ZHANG, SHOUMEI LI*, ITARU MITOMA† AND YOSHIAKI OKAZAKI

(Received July 24, 2008, revised April 28, 2009)

Abstract. In a certain Banach space called an M-type 2 Banach space (including Hilbert spaces), we consider a set-valued stochastic differential equation with a set-valued drift term and a single valued diffusion term, under the Lipschitz continuity conditions, and we prove the existence and uniqueness of strong solutions which are continuous in the Hausdorff distance.

1. Introduction. Theory of stochastic differential inclusions, as an important generalization of that of stochastic differential equations, has been received much attention with general applications to the mathematical economics, the control field, etc. In this area, we would like to refer to the nice survey [10, 11] written by Kisielewicz et al. In the n-dimensional Euclidean space $\boldsymbol{R}^n$, much work has been done on stochastic differential or integral inclusions. Aubin and Da Prato [2] studied the viability theorem for the following stochastic differential inclusions

$$dx_t \in F_t(x_t)dt + g_t(x_t)dB_t\,, \quad x_0 = \xi\,,$$

where F is set-valued, g is single valued and $\{B_t\}$ is an $\boldsymbol{R}^n$-valued Brownian motion. Kisielewicz (e.g., [12, 13, 14, 15, 20]) considered the following integral inclusions

$$x_t - x_0 \in \mathrm{cl}\bigg(\int_0^t F_\tau(x_\tau)d\tau + \int_0^t G_\tau(x_\tau)dB_\tau\bigg)\,, \quad t \in [0, T]\,,$$

where both F and G are set-valued.

However there are only a few literatures considering the set-valued stochastic differential equations or integral equations because of the complexity of derivative of set-valued functions and the difficulties in defining set-valued stochastic integrals. For instance, in the 1-dimensional Euclidean space $\boldsymbol{R}$, even if the integrand is $G_t(\omega) = [-1, 1]$ a.s., Ogura [21] pointed out that the integral $\int_0^t G_\tau dB_\tau$ is unbounded a.s. Thus it is difficult to consider the strong solution of the following stochastic differential equation,

$$X_t = X_0 + \int_0^t F_\tau(X_\tau)d\tau + \int_0^t G_\tau(X_\tau)dB_\tau\,, \quad t \in [0, T]\,,$$

2000 *Mathematics Subject Classification.* Primary 65C30; Secondary 26E25, 54C65.

Key words and phrases. M-type 2 Banach space, integrals of set-valued stochastic processes, set-valued stochastic differential equation.

*Partly supported by NSFC (No. 10771010), Research Fund of Beijing ducational Committee, PHR(IHLB) and 111 Talent Project Fund of BJUT, P.R. China.

†Partly supported by the Grand-in-Aid for Scientific Research (C), Japan Society for the Promotion of Science, No. 20540120.

where F, G and X_t are set-valued. In a separable Banach space, Michta [19] studied compact convex set-valued random differential equation without the diffusion term, where the derivative of a set-valued function is the Hukuhara derivative. Michta also discussed in [19] the relationship between set-valued differential equation and differential inclusion.

In this paper, inspired by Aubin and Da Prato [2], in a separable M-type 2 Banach space $\mathfrak{X}$ defined precisely in Section 3, we will study the set-valued stochastic differential equation with a set-valued drift term and a single valued diffusion term, which is presented as follows:

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dB_s\,, \quad t \in [0, T]\,, \tag{1.1}$$

where both X_s and $a(s, X_s)$ are set-valued, $b(s, X_s)$ is single valued, and $\{B_t\}$ is a real valued Brownian motion. The sum of a set X and a single point y is defined as $X + y = \{x + y\,; x \in X\}$.

There exist quite a few literatures treating stochastic differential or integral inclusions, and even if there exist, the most of them deal with these subjects in finite dimensional settings. However the set-valued stochastic differential equation of the type (1.1) is a rather new subject if we compare it with those stochastic inclusions, and further the space, where (1.1) is considered, belongs to a certain class of Banach spaces including Hilbert spaces.

Under these circumstances, we obtain

THEOREM 1. *Suppose $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are jointly measurable, H-bounded and satisfy Lipschitz conditions in the following sense*:

$$H(\{0\}, a(t, X)) + \|b(t, X)\| \le C(1 + H(\{0\}, X))\,, \quad X \subset \mathfrak{X}\,, \ \ t \in [0, T]$$

for some constant C, and

$$H(a(t, X), a(t, Y)) + \|b(t, X) - b(t, Y)\| \le DH(X, Y)\,, \quad X, Y \subset \mathfrak{X}\,, \ \ t \in [0, T]$$

for some constant D, where $H(A, B)$ is the Hausdorff distance between sets A and B. Then for any given L^2-integrably bounded, weakly compact, set-valued random variable X_0, the equation (1.1) *has a unique H-continuous solution.*

The paper is organized as follows. Section 2 is for preliminaries of set-valued random variables and set-valued stochastic processes. Section 3 is devoted to integrals of set-valued stochastic processes with respect to a Brownian motion and with respect to time t. In Section 4, first we state our set-valued stochastic differential equation, and then prove the existence and uniqueness of solutions.

2. Preliminaries. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and $\{\mathcal{F}_t\}_{t\ge 0}$ a filtration satisfying the usual conditions such that $\mathcal{F}_0$ includes all P-null sets in $\mathcal{F}$. The filtration is non-decreasing and right continuous. Let $\mathcal{B}(E)$ be the Borel field of a topological space E, $(\mathfrak{X}, \|\cdot\|)$ a separable Banach space $\mathfrak{X}$ equipped with the norm $\|\cdot\|$, $\mathfrak{X}^*$ the dual Banach space of $\mathfrak{X}$ and $\mathbf{K}(\mathfrak{X})$ (resp. $\mathbf{K}_b(\mathfrak{X})$, $\mathbf{K}_c(\mathfrak{X})$) the family of all nonempty closed (resp. closed bounded, closed convex) subsets of $\mathfrak{X}$. Let p be $1 \le p < +\infty$ and $L^p(\Omega, \mathcal{F}, P\,; \mathfrak{X})$, denoted briefly by $L^p(\Omega\,;\mathfrak{X})$, the Banach space of equivalence classes of $\mathfrak{X}$-valued $\mathcal{F}$-measurable functions $f:\Omega\to\mathfrak{X}$ such that the norm

$$\|f\|_p=\left\{\int_\Omega\|f(\omega)\|^p dP\right\}^{1/p}$$

is finite. f is called *L^p-integrable* if $f\in L^p(\Omega;\mathfrak{X})$.

A set-valued function $F:\Omega\to\mathbf{K}(\mathfrak{X})$ is said to be *measurable* if for any open set $O\subset\mathfrak{X}$, the inverse $F^{-1}(O):=\{\omega\in\Omega\,;F(\omega)\cap O\neq\emptyset\}$ is in $\mathcal{F}$. Such a function F is called a *set-valued random variable*. Let $\mathcal{M}(\Omega,\mathcal{F},P\,;\mathbf{K}(\mathfrak{X}))$ be the family of all set-valued random variables, briefly denoted by $\mathcal{M}(\Omega\,;\mathbf{K}(\mathfrak{X}))$.

A mapping g from a measurable space $(E_1,\mathcal{A}_1)$ into another measurable space $(E_2,\mathcal{A}_2)$ is called *$\mathcal{A}_1/\mathcal{A}_2$-measurable* if $g^{-1}(B)=\{x\in E\,;g(x)\in B\}$ is in $\mathcal{A}_1$ for all $B\in\mathcal{A}_2$.

For any open subset $O\subset\mathfrak{X}$, set

$$Z_O:=\{E\in\mathbf{K}(\mathfrak{X})\,;E\cap O\neq\emptyset\}\,,$$

$$\mathcal{C}:=\{Z_O\,;O\subset\mathfrak{X},\ O\text{ is open}\}\,,$$

and let $\sigma(\mathcal{C})$ be the σ-algebra generated by $\mathcal{C}$.

PROPOSITION 2.1. *A set-valued function $F:\Omega\to\mathbf{K}(\mathfrak{X})$ is measurable if and only if F is $\mathcal{F}/\sigma(\mathcal{C})$-measurable.*

PROOF. If $F:(\Omega,\mathcal{F})\to(\mathbf{K}(\mathfrak{X}),\sigma(\mathcal{C}))$ is $\mathcal{F}/\sigma(\mathcal{C})$-measurable, then for every open subset $O\subset\mathfrak{X}$, we have

$$F^{-1}(O)=\{\omega\in\Omega\,;F(\omega)\cap O\neq\emptyset\}=\{\omega\in\Omega\,;F(\omega)\in Z_O\}\in\mathcal{F}\,,$$

so that F is measurable.

Conversely, if F is measurable, then for each open subset $O\subset\mathfrak{X}$, it holds that $\{\omega\in\Omega\,;F(\omega)\in Z_O\}\in\mathcal{F}$, so that for every $Z\in\sigma(\mathcal{C})$, $\{\omega\in\Omega\,;F(\omega)\in Z\}$ is in $\mathcal{F}$. □

For $A,B\in\mathbf{K}(\mathfrak{X})$, $H(A,B)\geq 0$ is defined by

$$H(A,B):=\max\left\{\sup_{x\in A}\inf_{y\in B}\|x-y\|,\ \sup_{y\in B}\inf_{x\in A}\|x-y\|\right\}.$$

If $A,B\in\mathbf{K}_b(\mathfrak{X})$, then $H(A,B)$ is called the *Hausdorff distance* of A and B. It is well known that $\mathbf{K}_b(\mathfrak{X})$ equipped with the H-metric, denoted by $(\mathbf{K}_b(\mathfrak{X}),H)$, is a complete metric space.

The following results are also well known (see for example [6], [17]).

PROPOSITION 2.2. (i) *For $A,B,C,D\in\mathbf{K}(\mathfrak{X})$, we have*

$$H(A+B,C+D)\leq H(A,C)+H(B,D)\,.$$

(ii) *For $A,B\in\mathbf{K}(\mathfrak{X})$, $\mu\in\boldsymbol{R}$, we have*

$$H(\mu A,\mu B)=|\mu|H(A,B)\,.$$

For $F\in\mathcal{M}(\Omega,\mathbf{K}(\mathfrak{X}))$, the family of all L^p-integrable selections is defined by

$$S_F^p(\mathcal{F}):=\{f\in L^p(\Omega,\mathcal{F},P\,;\mathfrak{X})\,;f(\omega)\in F(\omega)\text{ a.s.}\}\,.$$

In the following, $S_F^p(\mathcal{F})$ is denoted briefly by S_F^p. If S_F^p is nonempty, F is said to be *L^p-integrable*. F is called *L^p-integrably bounded* if there exits a function $h \in L^p(\Omega, \mathcal{F}, P; \boldsymbol{R})$ such that $\|x\| \leq h(\omega)$ for any x and ω with $x \in F(\omega)$. It is equivalent to that $\|F\|_{\mathbf{K}} \in L^p(\Omega; \boldsymbol{R})$, where $\|F(\omega)\|_{\mathbf{K}} := \sup_{a \in F(\omega)} \|a\|$. The family of all measurable $\mathbf{K}(\mathfrak{X})$-valued (resp. $\mathbf{K}_c(\mathfrak{X})$-valued) L^p-integrably bounded functions is denoted by $L^p(\Omega, \mathcal{F}, P; \mathbf{K}(\mathfrak{X}))$ (resp. $L^p(\Omega, \mathcal{F}, P; \mathbf{K}_c(\mathfrak{X}))$). Write them for brevity as $L^p(\Omega; \mathbf{K}(\mathfrak{X}))$ (resp. $L^p(\Omega; \mathbf{K}_c(\mathfrak{X}))$).

Let Γ be a set of measurable functions $f : \Omega \to \mathfrak{X}$. Γ is called *decomposable* with respect to the σ-algebra $\mathcal{F}$ if, for any finite $\mathcal{F}$-measurable partition $A_1, \ldots, A_n$ and for any $f_1, \ldots, f_n \in \Gamma$, $\chi_{A_1} f_1 + \cdots + \chi_{A_n} f_n$ is in Γ, where χ_A is the indicator function of set A, i.e.,

$$\chi_A(\omega) := \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A. \end{cases}$$

PROPOSITION 2.3 (Hiai-Umegaki [6]). *Let Γ be a nonempty closed subset of $L^p(\Omega, \mathcal{F}, P; \mathfrak{X})$. Then there exists an $F \in \mathcal{M}(\Omega; \mathbf{K}(\mathfrak{X}))$ such that $\Gamma = S_F^p$ if and only if Γ is decomposable with respect to $\mathcal{F}$.*

LEMMA 2.4. *Let F be in $\mathcal{M}(\Omega; \mathbf{K}(\mathfrak{X}))$. Then F is L^p-integrably bounded if and only if S_F^p is nonempty and bounded in $L^p(\Omega; \mathfrak{X})$.*

PROOF. The case of $p = 1$ is due to Hiai-Umegaki [6]. By a manner similar to that of $p = 1$, we can also prove the statement for $1 < p < +\infty$. □

Let $\boldsymbol{R}_+$ be the set of all nonnegative real numbers and $\mathcal{B}_+ = \mathcal{B}(\boldsymbol{R}_+)$. An $\mathfrak{X}$-valued stochastic process $f = \{f_t; t \geq 0\}$ (or denoted by $f = \{f(t); t \geq 0\}$) is defined as a function $f : \boldsymbol{R}_+ \times \Omega \to \mathfrak{X}$ with $\mathcal{F}$-measurable section f_t for each $t \geq 0$. We say f is *measurable* if f is $\mathcal{B}_+ \otimes \mathcal{F}$-measurable. The process $f = \{f_t; t \geq 0\}$ is called *$\mathcal{F}_t$-adapted* if f_t is $\mathcal{F}_t$-measurable for every $t \geq 0$.

In a fashion similar to the $\mathfrak{X}$-valued stochastic process, a *set-valued stochastic process* $F = \{F_t; t \geq 0\}$ is defined as a set-valued function $F : \boldsymbol{R}_+ \times \Omega \to \mathbf{K}(\mathfrak{X})$ with $\mathcal{F}$-measurable section F_t for each $t \geq 0$. It is called *measurable* if it is $\mathcal{B}_+ \otimes \mathcal{F}$-measurable, and *$\mathcal{F}_t$-adapted* if for any fixed t, F_t is $\mathcal{F}_t$-measurable.

PROPOSITION 2.5. *Let $F = \{F_t; t \geq 0\}$ be an $\mathcal{F}_t$-adapted and measurable set-valued stochastic process. Then there exists an $\mathcal{F}_t$-adapted and measurable selection $f = \{f_t; t \geq 0\}$ such that*

$$f_t(\omega) \in F_t(\omega) \quad \textit{for all } \ (t, \omega) \in \boldsymbol{R}_+ \times \Omega .$$

PROOF. Let $\Sigma := \bigcap_{t \geq 0} \{Z \in \mathcal{B}_+ \otimes \mathcal{F}; Z_t \in \mathcal{F}_t\}$, where $Z_t = \{\omega; (t, \omega) \in Z\}$. We know that Σ is a σ-algebra on $\boldsymbol{R}_+ \times \Omega$. A function $f : \boldsymbol{R}_+ \times \Omega \to \mathfrak{X}$ (or a set-valued function $F : \boldsymbol{R}_+ \times \Omega \to \mathbf{K}(\mathfrak{X})$) is measurable and $\mathcal{F}_t$-adapted if and only if it is Σ-measurable. Therefore according to Kuratowski-Ryll-Nardzewski Measurable Selection Theorem (see e.g., [5]), for every $\mathcal{F}_t$-adapted and measurable $\mathbf{K}(\mathfrak{X})$-valued stochastic process $F = \{F_t; t \geq 0\}$, there exists an $\mathcal{F}_t$-adapted and measurable $\mathfrak{X}$-valued selection $f = \{f_t; t \geq$

0} such that

$$f_t(\omega) \in F_t(\omega) \quad \text{for all } (t, \omega) \in \boldsymbol{R}_+ \times \Omega .$$

$\square$

3. Integrals of set-valued stochastic processes. In this section we describe the definitions and properties of integrals of set-valued stochastic processes in a Banach space.

Let p be $1 \le p < \infty$, $\mathfrak{X}$ a separable Banach space, T a positive real number, $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ a complete probability space with filtration $\{\mathcal{F}_t ; t \in [0, T]\}$ and λ the Lebesgue measure on the interval $[0, T]$. In the following, the Lebesgue integral $\int_{[s,t]} f d\lambda$ will be denoted by $\int_s^t f_\tau d\tau$ for $[s, t] \subset [0, T]$, where f is a Lebesgue integrable function. Let $L^p(([0, T] \times \Omega), \mathcal{B}([0, T]) \otimes \mathcal{F}, \lambda \times P ; \mathfrak{X})$, denoted briefly by $L^p([0, T] \times \Omega ; \mathfrak{X})$, be the Banach space of equivalence classes of $\mathfrak{X}$-valued, $\mathcal{B}([0, T]) \otimes \mathcal{F}$-measurable functions $f : [0, T] \times \Omega \to \mathfrak{X}$ such that

$$\int_{[0,T]\times\Omega} \|f(t, \omega)\|^p d\lambda dP < +\infty . \tag{3.1}$$

Let $\mathcal{L}^p(\mathfrak{X})$ be the family of all $\mathcal{B}([0, T]) \otimes \mathcal{F}$-measurable, $\mathcal{F}_t$-adapted, $\mathfrak{X}$-valued stochastic processes $f = \{f_t, \mathcal{F}_t ; t \in [0, T]\}$ such that $E[\int_0^T \|f_s\|^p ds] := \int_{[0,T]\times\Omega} \|f(t, \omega)\|^p d\lambda dP < +\infty$, and $\mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$ the family of all $\mathcal{B}([0, T]) \otimes \mathcal{F}$-measurable, $\mathcal{F}_t$-adapted, set-valued stochastic processes $F = \{F_t, \mathcal{F}_t ; t \in [0, T]\}$ such that $\{\|F_t\|_{\mathbf{K}}\}_{t\in[0,T]}$ is in $\mathcal{L}^p(\boldsymbol{R})$.

For a $\mathcal{B}([0, T]) \otimes \mathcal{F}$-measurable set-valued stochastic process $\{F_t, \mathcal{F}_t ; t \in [0, T]\}$, a $\mathcal{B}([0, T]) \otimes \mathcal{F}$-measurable selection $f = \{f_t, \mathcal{F}_t ; t \in [0, T]\}$ is called *$\mathcal{L}^p$-selection* if $f = \{f_t, \mathcal{F}_t ; t \in [0, T]\}$ is in $\mathcal{L}^p(\mathfrak{X})$. The family of all $\mathcal{L}^p$-selections is denoted by $S^p(F(\cdot))$.

In fact, letting F be in $\mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$ and setting

$$\Sigma_T := \bigcap_{t\in[0,T]} \{Z \in \mathcal{B}([0, T]) \otimes \mathcal{F} ; Z_t \in \mathcal{F}_t\} ,$$

we have, by a manner similar to the proof of Proposition 2.5, that $S^p(F(\cdot))$ is nonempty and

$$\begin{aligned} S^p(F(\cdot)) = \{f \in L^p([0, T] \times \Omega, \Sigma_T, \lambda \times P ; \mathfrak{X}) ; f_t(\omega) \in F_t(\omega) \\ \text{for a.e. } (t, \omega) \in [0, T] \times \Omega\} , \end{aligned} \tag{3.2}$$

where $L^p([0, T] \times \Omega, \Sigma_T, \lambda \times P ; \mathfrak{X})$ is the Banach space of equivalence classes of $\mathfrak{X}$-valued, Σ_T-measurable functions $f : [0, T] \times \Omega \to \mathfrak{X}$ satisfying (3.1).

3.1. Stochastic integral with respect to Brownian motion in an M-type 2 Banach space. Let $\{B_t, \mathcal{F}_t ; t \in [0, T]\}$ be a real valued $\mathcal{F}_t$-Brownian motion, i.e., an $\mathcal{F}_t$-adapted continuous martingale and for any $0 \le t \le u \le T$, $E[(B_u - B_t)^2] = u - t$ (see [16]), with $B_0(\omega) = 0$ a.s.

DEFINITION 3.1 ([3]). A Banach space $(\mathfrak{X}, \|\cdot\|)$ is called M-type 2 if and only if there exists a constant $C_{\mathfrak{X}} > 0$ such that, for any $\mathfrak{X}$-valued martingale $\{M_k\}$, the inequality

$$\sup_k E[\|M_k\|^2] \le C_{\mathfrak{X}} \sum_k E[\|M_k - M_{k-1}\|^2] \tag{3.3}$$

holds.

The class of M-type 2 Banach spaces is a wider class than that of Hilbert spaces. The Lebesgue function spaces $L^p (p \geq 2)$ are examples of M-type 2 Banach spaces (see e.g. [23]).

Now, we review briefly about the stochastic integral studied in [25].

Let $\mathcal{L}^2_{\text{step}}(\mathfrak{X})$ be the subspace of those $f \in \mathcal{L}^2(\mathfrak{X})$ for which there exists a partition $0 = t_0 < t_1 < \cdots < t_n = T$ for some $n \in \boldsymbol{N}$, such that $f_t = f_{t_k}$ for each $t \in [t_k, t_{k+1}), 0 \leq k \leq n-1$.

For $f \in \mathcal{L}^2_{\text{step}}(\mathfrak{X})$, define an $\mathfrak{X}$-valued martingale

$$I_T(f) := \sum_{k=0}^{n-1} f_{t_k}(B_{t_{k+1}} - B_{t_k}) .$$

In an M-type 2 Banach space, due to the crucial inequality (3.3), for $f = \{f_t, \mathcal{F}_t\,; t \in [0, T]\} \in \mathcal{L}^2_{\text{step}}(\mathfrak{X})$, we have

LEMMA 3.2. $E[\|I_T(f)\|^2] \leq C_{\mathfrak{X}} \int_0^T E[\|f_t\|^2]dt.$

Further we proved that every element of $L^2([0, T]\,; \mathfrak{X})$ is approximated by a sequence of bounded continuous functions, which, together with the separability of $L^2([0, T]\,; \mathfrak{X})$, yields the following lemma.

LEMMA 3.3. *$\mathcal{L}^2_{\text{step}}(\mathfrak{X})$ is dense in $\mathcal{L}^2(\mathfrak{X})$.*

Lemmata 3.2 and 3.3 enable us to extend the above $I_T(f)$ from $\mathcal{L}^2_{\text{step}}(\mathfrak{X})$ to $\mathcal{L}^2(\mathfrak{X})$. The extension is the definition of the stochastic integral and has the following properties.

PROPOSITION 3.4. *For $f \in \mathcal{L}^2(\mathfrak{X})$, we have*

(i) *$E[I_t(f)] = 0$, $I_t(f) \in L^2(\Omega, \mathcal{F}, P\,; \mathfrak{X})$ and $\{I_t(f) : t \in [0, T]\}$ is a measurable $\mathcal{F}_t$-martingale,*

(ii)

$$E[\|I_t(f)\|^2] \leq C_{\mathfrak{X}} E\left[\int_0^t \|f_s\|^2 ds\right] \quad \textit{for all } t \in [0, T]\,, \quad \textit{and}$$

(iii) *there exists a t-continuous (in the norm of $\mathfrak{X}$) version of*

$$\int_0^t f_s(\omega)dB_s(\omega) \quad \textit{for } t \in [0, T]\,,$$

that is, there exists a t-continuous $\mathfrak{X}$-valued stochastic process J_t on $(\Omega, \mathcal{F}, P)$ such that

$$P\left(J_t = \int_0^t f_s dB_s\right) = 1 \quad \textit{for all } t\,,\ 0 \leq t \leq T\,.$$

From now on, we always assume that $\int_0^t f_s(\omega)dB_s(\omega)$ means a t-continuous version of the integral.

3.2. Set-valued integrals with respect to Lebesgue measure on time interval $[s, t]$. For a set-valued stochastic process $\{F_t, \mathcal{F}_t\,; t \in [0, T]\} \in \mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$, and for $0 \leq s \leq t \leq T$, define

$$\Lambda_{s,t} := \left\{ \int_s^t f_u du \,;\, (f_u)_{u\in[0,T]} \in S^p(F(\cdot)) \right\}, \tag{3.4}$$

where $\int_s^t f_u(\omega)du$ is the Bochner integral with respect to the Lebesgue measure λ on the interval $[s,t]$. By [5, Theorem 9.41], f is Bochner integrable on the interval $[s,t]$ if and only if its norm function $\|f\|$ is Lebesgue integrable, that is, $\int_s^t \|f_u\|du < +\infty$.

For each $f \in S^p(F(\cdot))$, we have $\int_0^T \|f_u\|^p du < +\infty$ a.s., which means there is a P-null set N_f, such that for all $\omega \in \Omega \setminus N_f$ and for $0 \le s < t \le T$, $\int_s^t \|f(u)\|^p du < +\infty$. For $\omega \in N_f$, we define $\int_s^t f_u du = 0$. Then for each $f \in S^p(F(\cdot))$, $\int_s^t f_u du$ is well defined for all $\omega \in \Omega$. Moreover, the process $\{\int_0^t f_u du : t \in [0,T]\}$ is continuous, measurable and $\mathcal{F}_t$-adapted. Hence $\Lambda_{s,t}$ is a subset of $L^p(\Omega, \mathcal{F}_t, P\,;\mathfrak{X}) \subset L^p(\Omega\,;\mathfrak{X})$.

We define the *decomposable closed hull* of $\Lambda_{s,t}$ with respect to $\mathcal{F}_t$ by

$$\begin{aligned}\overline{\mathrm{de}}\Lambda_{s,t} := \Big\{ & g \in L^p(\Omega, \mathcal{F}_t, P\,;\mathfrak{X})\,;\ \text{for any } \varepsilon > 0,\ \text{there exist a finite } \mathcal{F}_t\text{-measurable} \\ & \text{partition } \{A_1,\dots,A_n\} \text{ of } \Omega \text{ and } f^1,\dots,f^n \in S^p(F(\cdot)) \text{ such that} \\ & \Big\| g - \sum_{i=1}^n \chi_{A_i} \int_s^t f_u^i du \Big\|_{L^p(\Omega,\mathcal{F}_t,P\,;\mathfrak{X})} < \varepsilon \Big\}.\end{aligned}$$

By Proposition 2.3, $\overline{\mathrm{de}}\Lambda_{s,t}$ determines an $\mathcal{F}_t$-measurable set-valued function $I_{s,t}(F) : \Omega \to \mathbf{K}(\mathfrak{X})$ such that the family of all L^p-integrable selections of $I_{s,t}(F)$ is

$$S^p_{I_{s,t}(F)}(\mathcal{F}_t) = \overline{\mathrm{de}}\Lambda_{s,t}\,.$$

Particularly, $I_{0,t}(F)$ will be denoted by $I_t(F)$ for brevity. Therefore $\{I_t(F)\,;\,t \in [0,T]\}$ is an $\mathcal{F}_t$-adapted set-valued stochastic process. The joint measurability of $\{I_t(F)\,;\,t \in [0,T]\}$ will be discussed in Lemma 3.10.

DEFINITION 3.5. For a set-valued stochastic process $\{F_t, \mathcal{F}_t\,;\,t\in[0,T]\}\in\mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$, the set-valued random variable $I_{s,t}(F)$ defined as above is called the set-valued integral of $\{F_t, \mathcal{F}_t\,;\,t \in [0,T]\}$ with respect to the Lebesgue measure on the interval $[s,t]$. We denote it by $\int_s^t F_u du := I_{s,t}(F)$.

THEOREM 3.6. *For a set-valued stochastic process $\{F_t, \mathcal{F}_t\,;\,t \in [0,T]\} \in \mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$, the set-valued integral $\int_0^T F_s(\omega)ds$ is convex a.s.*

PROOF. Obviously, $S^p_{I_T(F)}(\mathcal{F}_T)$ is nonempty. According to [6, Corollary 1.6], it suffices to prove that

$$S^p_{I_T(F)}(\mathcal{F}_T) = \overline{\mathrm{de}}\Big\{ \int_0^T f_s ds\,;\, f \in S^p(F(\cdot)) \Big\}$$

is convex. It is noticed that if $\mathrm{cl}\{\int_0^T f_s ds\,;\, f \in S^p(F(\cdot))\}$ is convex, then $S^p_{I_T(F)}(\mathcal{F}_T)$ is convex, where cl denotes the closure in $L^p(\Omega\,;\mathfrak{X})$. In the following, we will show that $\mathrm{cl}\{\int_0^T f_s ds\,;\, f \in S^p(F(\cdot))\}$ is a convex subset of $L^p(\Omega\,;\mathfrak{X})$. It suffices to prove that, for any $g, h \in S^p(F(\cdot))$, for any $\alpha \in [0,1]$ and any $\varepsilon > 0$, there exists an $f \in S^p(F(\cdot))$ such that

$$\left\|\alpha \int_0^T g_s ds + (1-\alpha)\int_0^T h_s ds - \int_0^T f_s ds\right\|_{L^p(\Omega\,;\mathfrak{X})} < \varepsilon\,.$$

Define an $(L^p(\Omega\,;\mathfrak{X}), L^p(\Omega\,;\mathfrak{X}))$-valued measure μ on $\mathcal{B}([0,T])$ by

$$\mu(A) = \left(\int_A g_s ds, \int_A h_s ds\right), \quad A \in \mathcal{B}([0,T])\,.$$

The space $([0,T], \mathcal{B}([0,T]), \lambda)$ is non-atomic. Hence by the result in [24, p. 162], the closure of the range of μ is convex in $(L^p(\Omega\,;\mathfrak{X}), L^p(\Omega\,;\mathfrak{X}))$. Since $\mu(\emptyset) = (0,0)$ and $\mu([0,T]) = (\int_0^T g_s ds, \int_0^T h_s ds)$, for any $\alpha \in [0,1]$ and any $\varepsilon > 0$, there exists an $A \in \mathcal{B}([0,T])$ such that

$$\left\|\alpha \int_0^T g_s ds - \int_A g_s ds\right\|_{L^p(\Omega\,;\mathfrak{X})} < \frac{\varepsilon}{2}$$

and

$$\left\|\alpha \int_0^T h_s ds - \int_A h_s ds\right\|_{L^p(\Omega\,;\mathfrak{X})} < \frac{\varepsilon}{2}\,.$$

We take

$$f_s(\omega) = \chi_{\Omega\times A}(\omega, s) g_s(\omega) + \chi_{\Omega\times A^c}(\omega, s) h_s(\omega) \quad \text{for all } (\omega, s) \in \Omega \times [0,T]\,,$$

where χ_A denotes the indicator function of the set A. Then we have $f \in S^p(F(\cdot))$ since $\{\Omega \times A, \Omega \times A^c\}$ is an $\mathcal{F}_0 \otimes \mathcal{B}([0,T])$-measurable partition of the product space $\Omega \times [0,T]$. Furthermore,

$$\begin{aligned}
&\left\|\alpha \int_0^T g_s ds + (1-\alpha)\int_0^T h_s ds - \int_0^T f_s ds\right\|_{L^p(\Omega\,;\mathfrak{X})} \\
&\quad\le \left\|\alpha \int_0^T g_s ds - \int_0^T \chi_{\Omega\times A} g_s ds\right\|_{L^p(\Omega\,;\mathfrak{X})} \\
&\qquad + \left\|(1-\alpha)\int_0^T h_s(\omega) ds - \int_0^T \chi_{\Omega\times A^c} h_s ds\right\|_{L^p(\Omega\,;\mathfrak{X})} \\
&\quad< \frac{\varepsilon}{2} + \left\|\alpha \int_0^T h_s ds - \int_0^T \chi_{\Omega\times A} h_s ds\right\|_{L^p(\Omega\,;\mathfrak{X})} < \varepsilon\,,
\end{aligned}$$

which yields that the set $S^p_{I_T(F)}(\mathcal{F}_T)$ is convex. Then the integral $I_T(F)(\omega)$ is convex a.s. $\square$

REMARK 3.7. For $0 \le s \le t \le T$, the integral $I_{s,t}(F)(\omega)$ is also convex a.s.

Theorem 3.6 is similar to [6, Theorem 4.2], but it is not a direct result of [6, Theorem 4.2], since in Theorem 3.6, the full space is the product space $[0,T] \times \Omega$, but the domain of the integral is just the time interval $[0,T]$ and not $[0,T] \times \Omega$.

If $\mathcal{F}$ is separable with respect to the probability measure P, then the space $L^p(([0,T] \times \Omega), \mathcal{B}([0,T]) \otimes \mathcal{F}, \lambda \times P\,;\mathfrak{X})$ is separable by the same reason as in the proof of the separability of $L^p(\Omega\,;\mathfrak{X})$ in [25]. Therefore $S^p(F(\cdot))$ is separable since it is a closed subset of

$L^p([0,T]\times\Omega;\mathfrak{X})$. Hence we can find a sequence $\{f^n=(f_t^n)_{t\in[0,T]};n\in \boldsymbol{N}\}\subset S^p(F(\cdot))$ such that

$$S^p(F(\cdot))=\mathrm{cl}\{f^n;n\in \boldsymbol{N}\},$$

where cl stands for the closure in $L^p([0,T]\times\Omega;\mathfrak{X})$. Moreover, for $0\le s\le t\le T$, the equality

$$S^p_{I_{s,t}(F)}(\mathcal{F}_t)=\overline{\mathrm{de}}\Big\{\int_s^t f_u^n du;n\in \boldsymbol{N}\Big\}$$

holds.

THEOREM 3.8. *Assume $\mathcal{F}$ is separable with respect to the probability measure P. Then for a set-valued stochastic process $\{F_t,\mathcal{F}_t;t\in[0,T]\}\in\mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$, there exists a sequence $\{f^n;n\in \boldsymbol{N}\}\subset S^p(F(\cdot))$ such that*

$$F_t(\omega)=\mathrm{cl}\{f_t^n(\omega);n\in \boldsymbol{N}\}\quad for\ a.e.\ (t,\omega),$$

and, for $0\le s\le t\le T$,

$$I_{s,t}(F)(\omega)=\mathrm{cl}\Big\{\int_s^t f_u^n(\omega)du;n\in \boldsymbol{N}\Big\}\ a.s.,$$

where cl *denotes the closure in $\mathfrak{X}$.*

PROOF. For $0\le s\le t\le T$, $I_{s,t}(F)$ is in $\mathcal{M}(\Omega;\mathbf{K}(\mathfrak{X}))$ and $S^p_{I_{s,t}(F)}(\mathcal{F}_t)$ is nonempty. Then by [6, Theorem 1.0], there exists a sequence $\{g^i_{s,t};i\in \boldsymbol{N}\}\subset S^p_{I_{s,t}(F)}(\mathcal{F}_t)$ such that

$$I_{s,t}(F)(\omega)=\mathrm{cl}\{g^i_{s,t}(\omega);i\in \boldsymbol{N}\}\quad\text{for all }\ \omega\in\Omega.$$

Note that in the above equation, the sequence depends on s and t.

Since

$$S^p_{I_{s,t}(F)}(\mathcal{F}_t)=\overline{\mathrm{de}}\Big\{\int_s^t f_u du;f\in S^p(F(\cdot))\Big\},$$

by the separability of $S^p(F(\cdot))$, there exists a dense sequence $\{f^n;n\in N\}$ in $S^p(F(\cdot))$ such that

$$S^p_{I_{s,t}(F)}(\mathcal{F}_t)=\overline{\mathrm{de}}\Big\{\int_s^t f_u^n du;n\in \boldsymbol{N}\Big\}\quad for\ \ 0\le s\le t\le T.$$

Since $g^i_{s,t}\in S^p_{I_{s,t}(F)}(\mathcal{F}_t)$ for every $i\ge 1$, we have

$$g^i_{s,t}(\omega)\in\mathrm{cl}\Big\{\int_s^t f_u^n(\omega)du;n\in \boldsymbol{N}\Big\}\ \text{a.s.},$$

where cl stands for the closure in $\mathfrak{X}$. By the countability of the sequence, we can find an exceptional P-null set N such that for $\omega\in\Omega\setminus N$, we have

$$\{g^i_{s,t}(\omega);i\in \boldsymbol{N}\}\subset\mathrm{cl}\Big\{\int_s^t f_u^n(\omega)du;n\in \boldsymbol{N}\Big\}.$$

Then

$$I_{s,t}(F)(\omega) = \mathrm{cl}\{g^i_{s,t}(\omega)\,;\, i \in \boldsymbol{N}\} \subset \mathrm{cl}\Big\{\int_s^t f^n_u(\omega)du\,;\, n \in \boldsymbol{N}\Big\} \text{ a.s.}$$
$$\subset I_{s,t}(F)(\omega) \text{ a.s.},$$

i.e.,

$$I_{s,t}(F)(\omega) = \mathrm{cl}\Big\{\int_s^t f^n_u(\omega)du\,;\, n \in \boldsymbol{N}\Big\} \text{ a.s.}$$

Since $f^n(t,\omega)$ is in $F_t(\omega)$ for a.e. (t,ω), by the countability of the sequence, we have

$$\mathrm{cl}\{f^n_t(\omega)\,;\, n \in \boldsymbol{N}\} \subset F_t(\omega) \quad \text{for a.e. } (t,\omega)\,. \tag{3.5}$$

On the other hand, by Proposition 2.5, there exists a sequence $\{g^m \in S^p(F(\cdot))\,;\, m \in \boldsymbol{N}\}$ such that

$$F_t(\omega) = \mathrm{cl}\{g^m_t(\omega)\,;\, m \in \boldsymbol{N}\} \quad \text{for all } t \text{ and } \omega\,.$$

Since every $g^m \in \mathrm{cl}\{f^n\,;\, n \in N\}$, we have $g^m(t,\omega) \in \mathrm{cl}\{f^n(t,\omega)\,;\, n \in \boldsymbol{N}\}$ for a.e. (t,ω). Owing to the countability of the sequence, we obtain

$$F_t(\omega) \subset \mathrm{cl}\{f^n_t(\omega)\,;\, n \in \boldsymbol{N}\} \quad \text{for a.e. } (t,\omega)\,,$$

which, together with (3.5), yields

$$F_t(\omega) = \mathrm{cl}\{f^n_t(\omega)\,;\, n \in \boldsymbol{N}\} \quad \text{for a.e. } (t,\omega)\,.$$

□

THEOREM 3.9. *Assume $\mathcal{F}$ is separable with respect to P. Then for a set-valued stochastic process $\{F_t, \mathcal{F}_t\,;\, t \in [0,T]\} \in \mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$, $S^p_{I_{s,t}(F)}(\mathcal{F}_t)$ is nonempty and bounded in $L^p(\Omega, \mathcal{F}_t, P\,;\, \mathfrak{X})$ for $0 \le s \le t \le T$. Furthermore, if $\mathfrak{X}$ is reflexive, then for $0 \le s \le t \le T$, $I_{s,t}(F)(\omega)$ is almost surely weakly compact in $\mathfrak{X}$ and $S^p_{I_{s,t}(F)}(\mathcal{F}_t)$ is weakly compact in $L^p(\Omega, \mathcal{F}_t, P\,;\, \mathfrak{X})$.*

PROOF. Let $\{A_k\,;\, k = 1,\dots,m\}$ be an $\mathcal{F}_t$-measurable partition of Ω, and $\{g^k\,;\, k = 1,\dots,m\} \subset S^p(F(\cdot))$. Then for $0 \le s \le t \le T$, $\int_s^t g^k_u du$ is $\mathcal{F}_t$-measurable, and

$$E\Big[\Big\|\int_s^t g^k_u du\Big\|^p\Big] \le t^{p-1} E\Big[\int_s^t \|g^k_u\|^p du\Big] \le T^{p-1} E\Big[\int_0^T \|F_u\|^p_{\mathbf{K}} du\Big] < +\infty\,,$$

which implies that $S^p_{I_{s,t}(F)}(\mathcal{F}_t)$ is nonempty.

In the following, we will show that $S^p_{I_{s,t}(F)}(\mathcal{F}_t)$ is bounded in $L^p(\Omega, \mathcal{F}_t, P\,;\, \mathfrak{X})$. By Theorem 3.8, there exists a sequence $\{f^n\,;\, n \in \boldsymbol{N}\} \subset S^p(F(\cdot))$ such that

$$F_t(\omega) = \mathrm{cl}\{f^n_t(\omega)\,;\, n \in \boldsymbol{N}\} \quad \text{a.e. } (t,\omega)$$

and for $0 \le s \le t \le T$,

$$I_{s,t}(F)(\omega) = \mathrm{cl}\Big\{\int_s^t f^n_u(\omega)du\,;\, n \in \boldsymbol{N}\Big\} \text{ a.s.}$$

Therefore we have

$$\begin{aligned} E[\|I_{s,t}(F)\|_{\mathbf{K}}^{p}] &= E\Big[\sup_{x\in I_{s,t}(F)(\omega)}\|x\|^{p}\Big] = E\Big[\sup_{n\in \boldsymbol{N}}\Big\|\int_s^t f_u^n du\Big\|^{p}\Big] \\ &\le E\Big[\sup_{n\in \boldsymbol{N}} t^{p-1}\int_s^t \|f_u^n\|^p du\Big] \le T^{p-1}E\Big[\sup_{n\in \boldsymbol{N}}\int_s^t \|f_u^n\|^p du\Big] \\ &\le T^{p-1}E\Big[\int_s^t \sup_{n\in \boldsymbol{N}}\|f_u^n\|^p du\Big] = T^{p-1}E\Big[\int_0^T \|F_u\|_{\mathbf{K}}^p du\Big] < +\infty\,, \end{aligned}$$

which shows $I_{s,t}(F)$ is L^p-integrably bounded. Therefore, by Lemma 2.4, the family of all $\mathcal{F}_t$-measurable L^p-integrable selections $S^p_{I_{s,t}(F)}(\mathcal{F}_t)$ is bounded in $L^p(\Omega, \mathcal{F}_t, P\,; \mathfrak{X})$.

By Theorem 3.6, $I_{s,t}(F)(\omega)$ is convex for almost sure ω. If $\mathfrak{X}$ is reflexive, then $I_{s,t}(F)(\omega)$ is weakly compact for almost sure ω since it is almost surely a closed, bounded and convex subset of $\mathfrak{X}$. Similarly, when $p > 1$, since $L^p(\Omega, \mathcal{F}_t, P\,; \mathfrak{X})$ is reflexive too, $S^p_{I_{s,t}(F)}(\mathcal{F}_t)$ is weakly compact in $L^p(\Omega, \mathcal{F}_t, P\,; \mathfrak{X})$. When $p = 1$, $S^1_{I_{s,t}(F)}(\mathcal{F}_t)$ is also weakly compact in $L^1(\Omega, \mathcal{F}_t, P\,; \mathfrak{X})$ from [6, Theorem 3.7]. □

LEMMA 3.10. *Assume $\mathcal{F}$ is separable with respect to P. Let a set valued stochastic process $\{F_t, \mathcal{F}_t\,; t \in [0, T]\}$ be in $\mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$. Then there exists a $\mathcal{B}([0, T]) \otimes \mathcal{F}$-measurable version $\{\widetilde{I_{s,t}}(F)\,; t \in [s, T]\}$ of $\{I_{s,t}(F)\,; t \in [s, T]\}$ such that $I_{s,t}(F)(\omega) = \widetilde{I_{s,t}}(F)(\omega)$ a.s. and $\widetilde{I_{s,t}}(F)(\omega) \in \mathbf{K}_b(\mathfrak{X})$ for all $s \le t \le T$ and almost sure ω, where $s \in [0, T)$ is arbitrarily fixed.*

PROOF. If $\mathcal{F}$ is separable, by Theorem 3.8, there exists a sequence $\{f^n\,; n \in \boldsymbol{N}\} \subset S^p(F(\cdot))$, such that

$$F_t(\omega) = \mathrm{cl}\big\{f_t^n(\omega)\,; n \in \boldsymbol{N}\big\} \text{ a.e. } (t, \omega)\,,$$

and for $s \in [0, T)$ being arbitrarily fixed, $t \in [s, T]$,

$$I_{s,t}(F)(\omega) = \mathrm{cl}\Big\{\int_s^t f_u^n(\omega)du\,; n \in \boldsymbol{N}\Big\} \text{ a.s.}$$

For every $f^n \in S^p(F(\cdot))$, there exists a P-null set N_n (independent of s and t) such that

$$\int_0^T \|f_s^n(\omega)\|^p ds \le \int_0^T \|F_s(\omega)\|_{\mathbf{K}}^p ds \quad \text{for } \omega \in \Omega \setminus N_n\,.$$

Set $N := \bigcup_n N_n$, then $P(N) = 0$, so that, for $0 \le s \le t \le T$ and $\omega \in \Omega \setminus N$, we have

$$\int_s^t \|f_u^n(\omega)\|^p du \le \int_0^T \|f_u^n(\omega)\|^p du \le \int_0^T \|F_u(\omega)\|_{\mathbf{K}}^p du\,.$$

For $0 \le s \le T$, set

$$\widetilde{I_{s,t}}(F)(\omega) = \mathrm{cl}\Big\{\int_s^t f_u^n(\omega)du\,; n \in \boldsymbol{N}\Big\} \quad \text{for all } t \in [s, T] \text{ and } \omega\,.$$

Then $I_{s,t}(F)(\omega) = \widetilde{I_{s,t}}(F)(\omega)$ a.s. and $\widetilde{I_{s,t}}(F)(\omega)$ is in $\mathbf{K}_b(\mathfrak{X})$ for all $t \in [s, T]$ and $\omega \in \Omega \setminus N$. By using the Castaing representation theorem (see e.g., [6, Theorem 1.0]), the set-valued process $\{\widetilde{I_{s,t}}(F)\,;\, t \in [s, T]\}$ is $\mathcal{B}([0, T]) \otimes \mathcal{F}$-measurable since, for every n, $\int_s^t f_u^n du$ is $\mathcal{B}([0, T]) \otimes \mathcal{F}$-measurable. □

From now on, if $\mathcal{F}$ is separable, we will always assume that the set-valued integral of $\{F_t, \mathcal{F}_t\,;\, t \in [0, T]\} \in \mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$ means the $\mathcal{B}([0, T]) \otimes \mathcal{F}$-measurable version $\{\widetilde{I_{s,t}}(F)\,;\, t \in [s, T]\}$. For convenience, we still denote $\widetilde{I_{s,t}}(F)(\omega)$ by $I_{s,t}(F)(\omega)$ or $\int_s^t F_u(\omega)du$.

THEOREM 3.11. *Assume $\mathcal{F}$ is separable with respect to P. For set-valued stochastic processes $\{F_t, \mathcal{F}_t\,;\, t \in [0, T]\}$ and $\{G_t, \mathcal{F}_t\,;\, t \in [0, T]\} \in \mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$, set*

$$\phi(t, \omega) := H\Big(\int_0^t F_s(\omega)ds, \int_0^t G_s(\omega)ds\Big) : [0, T] \times \Omega \to \boldsymbol{R}.$$

Then $\phi(\cdot, \cdot)$ is $\mathcal{B}([0, T]) \otimes \mathcal{F}$-measurable.

PROOF. If $\mathcal{F}$ is separable with respect to P, assume that $\{f^i\,;\, i \in \boldsymbol{N}\}$ is the dense subset of $S^p(F(\cdot))$ and $\{g^j\,;\, j \in \boldsymbol{N}\}$ the dense subset of $S^p(G(\cdot))$. By the definition of the Hausdorff distance and Lemma 3.10 (and its proof), we have

$$\phi(t, \omega) = \max\left\{ \sup_i \inf_j \left\| \int_0^t (f_s^i(\omega) - g_s^j(\omega))ds \right\|, \sup_j \inf_i \left\| \int_0^t (f_s^i(\omega) - g_s^j(\omega))ds \right\| \right\}$$

for all $t \in [0, T]$ and ω. For every i, j, $\int_0^t f_s^i(\omega)ds$ and $\int_0^t g_s^j(\omega)ds$ are $\mathcal{B}([0, T]) \otimes \mathcal{F}$-measurable. Then $\phi(\cdot, \cdot)$ is $\mathcal{B}([0, T]) \otimes \mathcal{F}$-measurable. □

PROPOSITION 3.12. *Assume $\mathcal{F}$ is separable with respect to P. Then for a set-valued stochastic process $\{F_t, \mathcal{F}_t\,;\, t \in [0, T]\} \in \mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$, the equality*

$$I_t(F)(\omega) = \mathrm{cl}\{I_s(F)(\omega) + I_{s,t}(F)(\omega)\}$$

holds for $0 \le s < t \le T$ and almost sure ω, where cl *stands for the closure in $\mathfrak{X}$.*

PROOF. By Theorem 3.8 and Lemma 3.10, it is not difficult to get the desired result. □

LEMMA 3.13. *Assume $\mathcal{F}$ is separable with respect to P. Then for a set-valued stochastic process $\{F_t, \mathcal{F}_t\,;\, t \in [0, T]\} \in \mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$, the set-valued integral $\{I_t(F)\,;\, t \in [0, T]\}$ is H-continuous in t a.s.*

PROOF. By Theorem 3.8, for a set-valued stochastic process $\{F_t, \mathcal{F}_t\,;\, t \in [0, T]\} \in \mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$, there exists a sequence $\{f^n\,;\, n \in \boldsymbol{N}\} \subset S^p(F(\cdot))$ such that

$$F_t(\omega) = \mathrm{cl}\{f_t^n(\omega)\,;\, n \in \boldsymbol{N}\} \quad \text{a.e. } (t, \omega)$$

and, for each $t \in [0, T]$,

$$I_t(F)(\omega) = \mathrm{cl}\left\{ \int_0^t f_s^n(\omega)ds\,;\, n \in \boldsymbol{N}\right\},$$

so that, by properties of Hausdorff distance and Proposition 3.12, we have

$$H\Big(\int_0^t F_u du, \int_0^s F_u du\Big) = H\Big(\int_0^s F_u du + \int_s^t F_u du, \int_0^s F_u du\Big)$$
$$\leq H\Big(\int_s^t F_u du, \{0\}\Big) = \sup_n \Big\| \int_s^t f_u^n du \Big\| \leq \int_s^t \sup_n \|f_u^n\| du .$$

Since $\int_s^t \sup_n \|f_u^n\| du = \int_s^t \|F_u\|_{\mathbf{K}} du$ a.s., the latter converges to 0 a.s. as $(t-s)$ goes to zero, which yields the desired result. □

LEMMA 3.14. *Assume $\mathcal{F}$ is separable with respect to P. For set-valued stochastic processes $\{F_t\}_{t\in[0,T]}, \{G_t\}_{t\in[0,T]} \in \mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$, and for all t, we have*

$$H^p\Big(\int_0^t F_s(\omega) ds, \int_0^t G_s(\omega) ds\Big) \leq t^{p-1} \int_0^t H^p(F_s(\omega), G_s(\omega)) ds \ \ a.s.$$

PROOF. When $\mathcal{F}$ is separable with respect to P, by Theorem 3.8, there exists a sequence $\{f^i\,; i \in \boldsymbol{N}\} \subset S^p(F(\cdot))$, such that

$$F_t(\omega) = \mathrm{cl}\big\{f_t^i(\omega)\,; i \in \boldsymbol{N}\big\} \ \text{ a.e. } \ (t, \omega)$$

and, for each $t \in [0, T]$,

$$\int_0^t F_s(\omega) ds = \mathrm{cl}\Big\{\int_0^t f_s^i(\omega) ds\,; i \in \boldsymbol{N}\Big\}.$$

For each $i \geq 1$, we can choose a sequence $\{g^{ij}\,; j \in \boldsymbol{N}\} \subset S^p\,(G(\cdot))$ (this sequence depends on i), such that

$$\|f^i - g^{ij}\|_{L^p([0,T]\times\Omega\,;\mathfrak{X})} \downarrow d(f^i, S^p(G(\cdot)))\ (j \to +\infty).$$

By (3.2) and [6, Theorem 2.2], we have

$$\begin{aligned}
d(f^i, S^p\,(G(\cdot))) &= \inf_{g\in S^p(G(\cdot))} \|f^i - g\|_{L^p([0,T]\times\Omega\,;\mathfrak{X})} \\
&= \inf_{g\in S^p(G(\cdot))} \Big(\int_\Omega \int_0^T \|f_s^i(\omega) - g_s(\omega)\|^p ds dP\Big)^{1/p} \\
&= \Big(\inf_{g\in S^p(G(\cdot))} \int_\Omega \int_0^T \|f_s^i(\omega) - g_s(\omega)\|^p ds dP\Big)^{1/p} \\
&= \Big(\int_\Omega \int_0^T \inf_{y\in G_s(\omega)} \|f_s^i(\omega) - y\|^p ds dP\Big)^{1/p} \\
&= \Big(\int_\Omega \int_0^T d^p(f_s^i(\omega), G_s(\omega)) ds dP\Big)^{1/p},
\end{aligned}$$

so that

$$\int_\Omega \int_0^T \|f_s^i(\omega) - g_s^{ij}(\omega)\|^p ds dP \downarrow \int_\Omega \int_0^T d^p(f_s^i(\omega), G_s(\omega)) ds dP\ (j \to \infty).$$

Namely, noticing that $\|f^i - g^{ij}\|_{L^p([0,T]\times\Omega\,;\mathfrak{X})} \geq d\left(f^i, S^p\left(G(\cdot)\right)\right)$ and $\|f_s^i(\omega) - g_s^{ij}(\omega)\| \geq d(f_s^i(\omega), G_s(\omega))$ for a.e. (s,ω), we have for any $\varepsilon > 0$, there exists a natural number J such that for any $j \geq J$,

$$\begin{aligned}
\varepsilon &> \Big| \int_\Omega \int_0^T \|f_s^i(\omega) - g_s^{ij}(\omega)\|^p ds dP - \int_\Omega \int_0^T d^p(f_s^i(\omega), G_s(\omega)) ds dP \Big| \\
&= \int_\Omega \int_0^T \|f_s^i(\omega) - g_s^{ij}(\omega)\|^p ds dP - \int_\Omega \int_0^T d^p(f_s^i(\omega), G_s(\omega)) ds dP \\
&= \int_\Omega \int_0^T (\|f_s^i(\omega) - g_s^{ij}(\omega)\|^p - d^p(f_s^i(\omega), G_s(\omega))) ds dP \\
&= \int_\Omega \int_0^T |\|f_s^i(\omega) - g_s^{ij}(\omega)\|^p - d^p(f_s^i(\omega), G_s(\omega))| ds dP\,.
\end{aligned}$$

Hence there exists a subsequence of $\{g^{ij}\,;\, j \in \boldsymbol{N}\}$, denoted as $\{g^{ij_k}\,;\, k \in \boldsymbol{N}\}$ such that

$$\|f_s^i(\omega) - g_s^{ij_k}(\omega)\|^p \to d^p(f_s^i(\omega), G_s(\omega)) \ (k \to +\infty) \ \text{ a.e. } (s,\omega)\,.$$

Since $\{F_t\}_{t\in[0,T]}$ and $\{G_t\}_{t\in[0,T]}$ are in $\mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$, we have

$$E\Big[\int_0^T (\|F_s(\omega)\|_{\mathbf{K}}^p + \|G_s(\omega)\|_{\mathbf{K}}^p) ds\Big] < \infty\,. \tag{3.6}$$

Since

$$\|f_s^i(\omega) - g_s^{ij_k}(\omega)\|^p \leq 2^p(\|F_s(\omega)\|_{\mathbf{K}}^p + \|G_s(\omega)\|_{\mathbf{K}}^p) \quad \text{for a.e. } (s,\omega)$$

and (3.6) yields

$$\int_0^T (\|F_s(\omega)\|_{\mathbf{K}}^p + \|G_s(\omega)\|_{\mathbf{K}}^p) ds < \infty \ \text{ a.s.},$$

by the Lebesgue dominated convergence theorem, for all t and almost sure ω, we have

$$\int_0^t \|f_s^i(\omega) - g_s^{ij_k}(\omega)\|^p ds \to \int_0^t d^p(f_s^i(\omega), G_s(\omega)) ds \quad (k \to +\infty)\,.$$

Therefore, for all t and almost sure ω

$$\inf_k \int_0^t \|f_s^i(\omega) - g_s^{ij_k}(\omega)\|^p ds \leq \int_0^t d^p(f_s^i(\omega), G_s(\omega)) ds\,.$$

Hence, for all t and almost sure ω, we have

$$\begin{aligned}\sup_{x\in\int_0^t F_s(\omega)ds} d^p\Big(x, \int_0^t G_s(\omega)ds\Big) &\le \sup_i \inf_j \Big\| \int_0^t f_s^i(\omega)ds - \int_0^t g_s^{ij}(\omega)ds\Big\|^p \\ &\le \sup_i \inf_k \Big\| \int_0^t f_s^i(\omega)ds - \int_0^t g_s^{ij_k}(\omega)ds\Big\|^p \\ &\le t^{p-1}\sup_i \inf_k \int_0^t \|f_s^i(\omega) - g_s^{ij_k}(\omega)\|^p ds \\ &\le t^{p-1}\sup_i \int_0^t d^p(f_s^i(\omega), G_s(\omega))ds \\ &\le t^{p-1}\int_0^t \sup_i d^p(f_s^i(\omega), G_s(\omega))ds\,.\end{aligned}$$

Similarly, by Theorem 3.8, there exists a sequence $\{g^m\,; m\in \boldsymbol{N}\}\subset S^p(G(\cdot))$ such that

$$G_t(\omega) = \mathrm{cl}\big\{g_t^m(\omega)\,; m\in\boldsymbol{N}\big\}\ \text{ a.e. } (t,\omega)$$

and, for each $t\in[0,T]$,

$$\int_0^t G_s(\omega)ds = \mathrm{cl}\Big\{\int_0^t g_s^m(\omega)ds\,; m\in\boldsymbol{N}\Big\}.$$

In the same way as above, we obtain that for all t and almost sure ω,

$$\sup_{y\in\int_0^t G_s(\omega)ds} d^p\Big(y, \int_0^t F_s(\omega)ds\Big) \le t^{p-1}\int_0^t \sup_m d^p(g_s^m(\omega), F_s(\omega))ds\,.$$

Therefore, the inequality

$$H^p\Big(\int_0^t F_s(\omega)ds, \int_0^t G_s(\omega)ds\Big) \le t^{p-1}\int_0^t H^p(F_s(\omega), G_s(\omega))ds$$

holds for all t and almost sure ω. □

THEOREM 3.15. *Assume $\mathcal{F}$ is separable with respect to P. Let $\{F_t\}_{t\in[0,T]}$ and $\{G_t\}_{t\in[0,T]}$ be set-valued stochastic processes in $\mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$. Then for $1\le r\le p$ and all t, it follows that*

$$H^r\Big(\int_0^t F_s(\omega)ds, \int_0^t G_s(\omega)ds\Big) \le t^{r-1}\int_0^t H^r(F_s(\omega), G_s(\omega))ds\ \ a.s.$$

and then

$$E\Big[H^r\Big(\int_0^t F_s ds, \int_0^t G_s ds\Big)\Big] \le t^{r-1}E\Big[\int_0^t H^r(F_s, G_s)ds\Big] < +\infty\,.$$

PROOF. For $1\le r\le p$, $\{F_t\}_{t\in[0,T]}$ and $\{G_t\}_{t\in[0,T]}$ are in $\mathcal{L}^r(\mathbf{K}(\mathfrak{X}))$. Hence Lemma 3.14 yields the first inequality immediately. Since we have

$$\begin{aligned}H^r(F_s(\omega), G_s(\omega)) &\le (H(F_s(\omega),\{0\}) + H(\{0\}, G_s(\omega)))^r \\ &= 2^r(\|F_s(\omega)\|_{\mathbf{K}}^r + \|G_s(\omega)\|_{\mathbf{K}}^r) \in L^r(\Omega\times[0,T]; \boldsymbol{R})\,,\end{aligned}$$

for each $t \in [0, T]$, $E[\int_0^t H^r(F_s, G_s)ds]$ is finite. Hence the second result holds. □

4. Set-valued stochastic differential equation. In this section, we mainly study the solution to a set-valued stochastic differential equation. Assume $\mathfrak{X}$ is a separable M-type 2 Banach space. Let the functions

$a(\cdot,\cdot) : [0, T] \times \mathbf{K}(\mathfrak{X}) \to \mathbf{K}(\mathfrak{X})$ be $(\mathcal{B}([0, T]) \otimes \sigma(\mathcal{C}))\,/\sigma(\mathcal{C})$-measurable, and

$b(\cdot,\cdot) : [0, T] \times \mathbf{K}(\mathfrak{X}) \to \mathfrak{X}$ be $(\mathcal{B}([0, T]) \otimes \sigma(\mathcal{C}))\,/\mathcal{B}(\mathfrak{X})$-measurable.

LEMMA 4.1. *Let $\{X_t\,;\ t \in [0, T]\}$ be an $\mathcal{F}_t$-adapted, measurable set-valued stochastic process, then the following statements hold*:

(i) $a(t, X_t(\omega)) : [0, T] \times \Omega \to \mathbf{K}(\mathfrak{X})$ *is* $(\mathcal{B}([0, T]) \otimes \mathcal{F})\,/\sigma(\mathcal{C})$*-measurable and for fixed* $t \in [0, T]$, $a(t, X_t(\cdot))$ *is* $\mathcal{F}_t/\sigma(\mathcal{C})$*-measurable, and*

(ii) $b(t, X_t(\omega))\,; [0, T] \times \Omega \to \mathfrak{X}$ *is* $(\mathcal{B}([0, T]) \otimes \mathcal{F})\,/\mathcal{B}(\mathfrak{X})$*-measurable and for fixed* $t \in [0, T]$, $b(t, X_t(\cdot))$ *is* $\mathcal{F}_t/\mathcal{B}(\mathfrak{X})$*-measurable.*

PROOF. Here we only prove the results for the function $a(\cdot,\cdot)$, since for the function $b(\cdot,\cdot)$, we can prove similarly.

For fixed $t \in [0, T]$, $a(t,\cdot)$ is $\sigma(\mathcal{C})/\sigma(\mathcal{C})$-measurable. It is noticed that $\{X_t\,; t \in [0, T]\}$ is $\mathcal{F}_t$-adapted. Hence, for every fixed $t \in [0, T]$, the composite function $a(t, X_t(\cdot))$ is $\mathcal{F}_t/\sigma(\mathcal{C})$-measurable.

By Proposition 2.1, $X : [0, T] \times \Omega \to \mathbf{K}(\mathfrak{X})$ can be considered as a $(\mathcal{B}([0, T]) \otimes \mathcal{F})\,/\,\sigma(\mathcal{C})$- measurable function. Then the composite function $\hat{a}(\cdot,\cdot,\cdot)$ mapping $[0, T] \times ([0, T] \times \Omega)$ to $\mathbf{K}(\mathfrak{X})$ with $\hat{a}(t, s, \omega) = a(t, X_s(\omega))$ is $(\mathcal{B}([0, T]) \otimes (\mathcal{B}([0, T]) \otimes \mathcal{F}))\,/\sigma(\mathcal{C})$-measurable. Indeed, let $(\Omega_0, \mathcal{B}_0)$, $(\Omega_1, \mathcal{B}_1)$ and $(\Omega_2, \mathcal{B}_2)$ be measurable spaces. Let Id denote the identity measurable mapping from $(\Omega_1, \mathcal{B}_1)$ to itself. Let φ be a measurable function from $(\Omega_1, \mathcal{B}_1)$ to $(\Omega_2, \mathcal{B}_2)$ and set

$$\psi := (\mathrm{Id}, \varphi) : \Omega_0 \times \Omega_1 \to \Omega_0 \times \Omega_2\,.$$

Taking any $A \times B \in \mathcal{B}_0 \otimes \mathcal{B}_2$, we have

$$\psi^{-1}(A \times B) = A \times \varphi^{-1}(B) \in \mathcal{B}_0 \otimes \mathcal{B}_1\,.$$

Hence,

$$\psi^{-1}(\mathcal{B}_0 \otimes \mathcal{B}_2) = \mathcal{B}_0 \otimes \varphi^{-1}(\mathcal{B}_2) \subset \mathcal{B}_0 \otimes \mathcal{B}_1\,,$$

which implies that ψ is $(\mathcal{B}_0 \otimes \mathcal{B}_1)/\mathcal{B}_0 \otimes \mathcal{B}_2$-measurable. Now let $(\Omega_0, \mathcal{B}_0) = ([0, T], \mathcal{B}([0, T]))$, $(\Omega_1, \mathcal{B}_1) = ([0, T] \times \Omega, \mathcal{B}([0, T]) \otimes \mathcal{F})$, $(\Omega_2, \mathcal{B}_2) = (\mathbf{K}(\mathfrak{X}), \sigma(\mathcal{C}))$ and $\varphi = X$, then ψ is $(\mathcal{B}([0, T]) \otimes (\mathcal{B}([0, T]) \otimes \mathcal{F}))\,/\,(\mathcal{B}([0, T]) \otimes \sigma(\mathcal{C}))$-measurable, so the composition $(\hat{a} \circ \psi)(t, s, \omega) = a(t, X_s(\omega))$ is $(\mathcal{B}([0, T]) \otimes (\mathcal{B}([0, T]) \otimes \mathcal{F}))\,/\sigma(\mathcal{C})$-measurable.

Let $t = s$ for $\hat{a}(t, s, \omega)$. Then it is not difficult to obtain that $a(t, X_t(\omega)) : [0, T] \times \Omega \to \mathbf{K}(\mathfrak{X})$ is $(\mathcal{B}([0, T]) \otimes \mathcal{F})\,/\sigma(\mathcal{C})$-measurable. □

Assume the above functions $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ also satisfy the following conditions:

(4.1) $$H(\{0\}, a(t, X)) + \|b(t, X)\| \le C(1 + H(\{0\}, X))\,, \quad X \in \mathbf{K}(\mathfrak{X})\,,\ \ t \in [0, T]$$

for some constant C, and

$$H(a(t,X),a(t,Y))+\|b(t,X)-b(t,Y)\|\leq DH(X,Y)\,,\quad X,Y\in\mathbf{K}(\mathfrak{X})\,,\quad t\in[0,T] \tag{4.2}$$

for some constant D.

Let X_0 be an L^2-integrably bounded, weakly compact (in the sense of weak topology $\sigma(\mathfrak{X},\mathfrak{X}^*)$) set-valued random variable. Assume $a(\cdot,\cdot)$ is $(\mathcal{B}([0,T])\otimes\sigma(\mathcal{C}))\,/\sigma(\mathcal{C})$-measurable, $b(\cdot,\cdot)$ is $(\mathcal{B}([0,T])\otimes\sigma(\mathcal{C}))\,/\mathcal{B}(\mathfrak{X})$-measurable and both $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ satisfy the conditions (4.1) and (4.2). Then, by Lemma 4.1, it is reasonable to define the set-valued stochastic differential equation as follows:

DEFINITION 4.2.

$$X_t=X_0+\int_0^t a(s,X_s)ds+\int_0^t b(s,X_s)dB_s\quad\text{for } t\in[0,T]\text{ a.s.} \tag{4.3}$$

An $\mathcal{F}_t$-adapted, H-continuous in t almost surely and measurable set-valued process $\{X_t\,;t\in[0,T]\}$ is called a strong solution if it satisfies the equation (4.3).

REMARK 4.3. There are three terms on the right hand side of equation (4.3). The first term is weakly compact a.s. By Theorem 3.6, the second one is convex and closed a.s., then weakly closed a.s. The third one is a single valued set, then weakly compact a.s. Therefore the sum is weakly closed a.s., and hence closed a.s. Also, the left hand side X_t is closed. So it is reasonable and possible to consider a solution to equation (4.3).

In the following, we will study the existence and uniqueness of the solutions to (4.3).

THEOREM 4.4. *Assume $\mathcal{F}$ is separable with respect to P. Let $T>0$, and let $a(\cdot,\cdot)$: $[0,T]\times\mathbf{K}(\mathfrak{X})\to\mathbf{K}(\mathfrak{X})$ and $b(\cdot,\cdot):[0,T]\times\mathbf{K}(\mathfrak{X})\to\mathfrak{X}$ be measurable functions satisfying conditions* (4.1) *and* (4.2). *Then for any given L^2-integrably bounded, weakly compact initial value X_0, there exists a strong solution to* (4.3).

PROOF. As a manner similar to that of solving the ordinary stochastic differential equation, we can use the successive approximation method to construct a solution to equation (4.3).

Define $Y_t^0=X_0$. Then we can define $Y_t^k=Y_t^k(\omega)$ inductively as follows:

$$Y_t^{k+1}=X_0+\int_0^t a(s,Y_s^k)ds+\int_0^t b(s,Y_s^k)dB_s\,. \tag{4.4}$$

By Theorem 3.15 and condition (4.2), we have

$$\begin{aligned}E[H^2(Y_t^{k+1},Y_t^k)]&=E\Big[H^2\Big(X_0+\int_0^t a(s,Y_s^k)ds+\int_0^t b(s,Y_s^k)dB_s,\\&\qquad X_0+\int_0^t a(s,Y_s^{k-1})ds+\int_0^t b(s,Y_s^{k-1})dB_s\Big)\Big]\\&\leq E\Big[\Big(H(X_0,X_0)+H\Big(\int_0^t a(s,Y_s^k)ds,\int_0^t a(s,Y_s^{k-1})ds\Big)\end{aligned}$$

$$
\begin{aligned}
&\quad + H\Big(\int_0^t b(s, Y_s^k)dB_s, \int_0^t b(s, Y_s^{k-1})dB_s\Big)\Big)^2\Big] \\
&= E\Big[\Big(H\Big(\int_0^t a(s, Y_s^k)ds, \int_0^t a(s, Y_s^{k-1})ds\Big) \\
&\quad + \Big\| \int_0^t b(s, Y_s^k)dB_s - \int_0^t b(s, Y_s^{k-1})dB_s \Big\|\Big)^2\Big] \\
&\le 2E\Big[H^2\Big(\int_0^t a(s, Y_s^k)ds, \int_0^t a(s, Y_s^{k-1})ds\Big)\Big] \\
&\qquad + 2E\Big[\Big\| \int_0^t b(s, Y_s^k)dB_s - \int_0^t b(s, Y_s^{k-1})dB_s \Big\|^2\Big] \\
&\le 2tE\Big[\int_0^t H^2(a(s, Y_s^k), a(s, Y_s^{k-1}))ds\Big] \\
&\qquad + 2C_{\mathfrak{X}}E\Big[\int_0^t \|(b(s, Y_s^k) - b(s, Y_s^{k-1})\|^2 ds\Big] \\
&\le 2tD^2E\Big[\int_0^t H^2(Y_s^k, Y_s^{k-1})ds\Big] + 2C_{\mathfrak{X}}D^2E\Big[\int_0^t H^2(Y_s^k, Y_s^{k-1})ds\Big] \\
&\le 2D^2(T + C_{\mathfrak{X}})\int_0^t E[H^2(Y_s^k, Y_s^{k-1})]ds \quad \text{for } k \ge 1, \quad t \in [0, T],
\end{aligned}
$$

and

$$
\begin{aligned}
E[H^2(Y_t^1, Y_t^0)] &= E\Big[H^2\Big(Y_t^0 + \int_0^t a(s, Y_s^0)ds + \int_0^t b(s, Y_s^0)dB_s, Y_t^0\Big)\Big] \\
&\le E\Big[\Big(H(Y_t^0, Y_t^0) + H\Big(\int_0^t a(s, Y_s^0)ds, \{0\}\Big) \\
&\qquad + H\Big(\int_0^t b(s, Y_s^0)dB_s, \{0\}\Big)\Big)^2\Big] \\
&\le 2E\Big[H^2\Big(\int_0^t a(s, Y_s^0)ds, \{0\}\Big)\Big] + 2E\Big[\Big\| \int_0^t b(s, Y_s^0)dB_s \Big\|^2\Big] \\
&\le 2tE\Big[\int_0^t H^2(a(s, Y_s^0), \{0\})ds\Big] + 2C_{\mathfrak{X}}E\Big[\int_0^t \|b(s, Y_s^0)\|^2 ds\Big] \\
&= 2(t + C_{\mathfrak{X}})E\Big[\int_0^t (H^2(a(s, Y_s^0), \{0\}) + \|b(s, Y_s^0)\|^2)ds\Big] \\
&\le 2(t + C_{\mathfrak{X}})E\Big[\int_0^t C^2(1 + \|X_0\|_{\mathbf{K}})^2 ds\Big] \\
&\le 2C^2(T + C_{\mathfrak{X}})tE[(1 + \|X_0\|_{\mathbf{K}})^2] = A_1 t,
\end{aligned}
$$

where $A_1 := 2C^2(T+C_{\mathfrak{X}})E[(1+\|X_0\|_{\mathbf{K}})^2] < +\infty$ and is independent of t. So by induction on k we obtain

$$\begin{aligned}
E[H^2(Y_t^{k+1}, Y_t^k)] &\le 2D^2(T + C_{\mathfrak{X}}) \int_0^t E[H^2(Y_s^k, Y_s^{k-1})]ds \\
&\le (2D^2(T + C_{\mathfrak{X}}))^2 \int_0^t \int_0^s E[H^2(Y_\tau^{k-1}, Y_\tau^{k-2})]d\tau ds \\
&\le \cdots \le \frac{A_2^{k+1} t^{k+1}}{(k+1)!}, \quad k \ge 0, \quad t \in [0, T],
\end{aligned}$$

where $A_2 := \max\{1, A_1, (2D^2(T + C_{\mathfrak{X}}))\}$.

For $m > n > 0$, by the above inequality, we have

$$\begin{aligned}
&(E[H^2(Y_t^m, Y_t^n)])^{1/2} = \|H(Y_t^m, Y_t^n)\|_{L^2} \\
&\quad \le \|H(Y_t^m, Y_t^{m-1}) + H(Y_t^{m-1}, Y_t^{m-2}) + \cdots + H(Y_t^{n+1}, Y_t^n)\|_{L^2} \\
&\quad = \Big\| \sum_{k=n}^{m-1} H(Y_t^{k+1}, Y_t^k) \Big\|_{L^2} \le \sum_{k=n}^{m-1} \|H(Y_t^{k+1}, Y_t^k)\|_{L^2} \\
&\quad \le \sum_{k=n}^{\infty} \|H(Y_t^{k+1}, Y_t^k)\|_{L^2} \le \sum_{k=n}^{\infty} \Big(\frac{A_2^{k+1} t^{k+1}}{(k+1)!} \Big)^{1/2} \to 0 \quad \text{as } n \to +\infty,
\end{aligned}$$

which means $\{Y_t^n; n \in \boldsymbol{N}\}$ is a Cauchy sequence in the complete metric space $L^2(\Omega\,; (K_b(\mathfrak{X}), H))$, so that the sequence $\{Y_t^n\,; n \in \boldsymbol{N}\}$ converges to a limit $\tilde{Y}_t$ in the sense that $\lim_{n\to+\infty} E\big[H^2(Y_t^n, \tilde{Y}_t)\big] = 0$ for every $t \in [0, T]$. The convergence is also uniform in $t \in [0, T]$. Indeed, by Theorem 3.15, we have

$$\begin{aligned}
\sup_{0\le t\le T} H(Y_t^{k+1}, Y_t^k) &= \sup_{0\le t\le T} H\Big(X_0 + \int_0^t a(s, Y_s^k)ds + \int_0^t b(s, Y_s^k)dB_s, X_0 \\
&\qquad + \int_0^t a(s, Y_s^{k-1})ds + \int_0^t b(s, Y_s^{k-1})dB_s\Big) \\
&\le \sup_{0\le t\le T} \Big(H\Big(\int_0^t a(s, Y_s^k)ds, \int_0^t a(s, Y_s^{k-1})ds \Big) \\
&\qquad + \Big\| \int_0^t b(s, Y_s^k)dB_s - \int_0^t b(s, Y_s^{k-1})dB_s \Big\| \Big) \\
&\le \int_0^T H(a(s, Y_s^k), a(s, Y_s^{k-1}))ds \\
&\quad + \sup_{0\le t\le T} \Big\| \int_0^t b(s, Y_s^k) - b(s, Y_s^{k-1})dB_s \Big\| \quad \text{a.s.}
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&P\Big(\sup_{0\le t\le T} H(Y_t^{k+1}, Y_t^k) > 2^{-k}\Big)\\
&\quad \le P\Big(\Big(\int_0^T H(a(s, Y_s^k), a(s, Y_s^{k-1}))ds\Big)^2 > 2^{-2k-2}\Big)\\
&\qquad + P\Big(\sup_{0\le t\le T}\Big\|\int_0^t (b(s, Y_s^k) - b(s, Y_s^{k-1}))dB_s\Big\| > 2^{-k-1}\Big).
\end{aligned}$$

The first term of the right hand side of the above inequality is dominated as follows:

$$\begin{aligned}
&P\Big(\Big(\int_0^T H(a(s, Y_s^k), a(s, Y_s^{k-1}))ds\Big)^2 > 2^{-2k-2}\Big)\\
&\quad \le 2^{2k+2}E\Big[\Big(\int_0^T H(a(s, Y_s^k, a(s, Y_s^{k-1}))ds\Big)^2\Big] \text{ (by Markov's inequality)}\\
&\quad \le 2^{2k+2}T E\Big[\int_0^T H^2(a(s, Y_s^k), a(s, Y_s^{k-1}))ds\Big] \text{ (by Schwarz's inequality)}.
\end{aligned}$$

Now we consider the second term. By Proposition 3.4, $\{\int_0^t b(s, Y_s)dB_s\,;\, t \in [0, T]\}$ is a t-continuous L^2-bounded Banach-valued martingale, then the norm process $\{\|\int_0^t b(s, Y_s)dB_s\|\,;\, t \in [0, T]\}$ is t-continuous L^2-bounded real valued submartingale, so that, by Doob's inequality, the second term is dominated as follows:

$$\begin{aligned}
&P\Big(\sup_{0\le t\le T}\Big\|\int_0^t (b(s, Y_s^k) - b(s, Y_s^{k-1}))dB_s\Big\| > 2^{-k-1}\Big)\\
&\quad \le 2^{2k+2}E\Big[\Big\|\int_0^T (b(s, Y_s^k) - b(s, Y_s^{k-1}))dB_s\Big\|^2\Big]\\
&\quad \le 2^{2k+2}C_{\mathfrak{X}}E\Big[\int_0^T \|b(s, Y_s^k) - b(s, Y_s^{k-1})\|^2 ds\Big].
\end{aligned}$$

Then we have

$$\begin{aligned}
&P\Big(\sup_{0\le t\le T} H(Y_t^{k+1}, Y_t^k) > 2^{-k}\Big)\\
&\le 2^{2k+2}T E\Big[\int_0^T H^2(a(s, Y_s^k), a(s, Y_s^{k-1}))ds\Big]\\
&\quad + 2^{2k+2}C_{\mathfrak{X}}E\Big[\int_0^T \|b(s, Y_s^k) - b(s, Y_s^{k-1})\|^2 ds\Big]\\
&= 2^{2k+2}(T + C_{\mathfrak{X}})\int_0^T E[H^2(a(s, Y_s^k), a(s, Y_s^{k-1})) + \|b(s, Y_s^k) - b(s, Y_s^{k-1})\|^2]ds\\
&\le 2^{2k+2}(T + C_{\mathfrak{X}})\int_0^T E[(H(a(s, Y_s^k), a(s, Y_s^{k-1})) + \|b(s, Y_s^k) - b(s, Y_s^{k-1})\|)^2]ds\\
&\le 2^{2k+2}(T + C_{\mathfrak{X}})D^2\int_0^T E[H^2(Y_s^k, Y_s^{k-1})]ds
\end{aligned}$$

$$\leq 2^{2k+2}(T + C_{\mathfrak{X}})D^2 \int_0^T \frac{A_2^k s^k}{k!} ds = 2^{2k+2}(T + C_{\mathfrak{X}})D^2 \frac{A_2^k T^{k+1}}{(k+1)!},$$

which yields that

$$\sum_{k=1}^{\infty} P\Big(\sup_{0 \leq t \leq T} H(Y_t^{k+1}, Y_t^k) > 2^{-k} \Big) < +\infty .$$

By the Borel-Cantelli Lemma, it follows that

$$P\Big(\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} \Big\{ \omega ; \sup_{0 \leq t \leq T} H(Y_t^{k+1}, Y_t^k) > 2^{-k} \Big\} \Big) = 0 ,$$

which implies, for almost sure ω, there is an integer $k_0 = k_0(\omega)$ such that

$$\sup_{0 \leq t \leq T} H(Y_t^{k+1}, Y_t^k) \leq 2^{-k} \quad \text{for } k \geq k_0 .$$

Therefore the sequence $\{Y_t^k ; k \in \boldsymbol{N}\}$ converges to Y_t uniformly in $[0, T]$ a.s. By Lemma 3.13 and the definition of the stochastic integral, every Y_t^k is H-continuous in t, so that the limit Y_t is also H-continuous in t a.s. Further, it is clear that, for any fixed t, $Y_t = \tilde{Y}_t$ a.s., so that we can replace $\tilde{Y}_t$ by Y_t.

It remains to show that Y_t satisfies equation (4.3). Indeed, for each integer $n \geq 0$,

$$Y^{n+1} = X_0 + \int_0^t a(s, Y_s^n) ds + \int_0^t b(s, Y_s^n) dB_s . \tag{4.5}$$

Similarly, we have

$$\lim_{n \to \infty} E\Big[H^2 \Big(\int_0^t a(s, Y_s^n) ds, \int_0^t a(s, Y_s) ds \Big) \Big] = 0$$

and

$$\lim_{n \to \infty} E\Big[H^2 \Big(\int_0^t b(s, Y_s^n) dB_s, \int_0^t b(s, Y_s) dB_s \Big) \Big] = 0 ,$$

which, together with (4.5), verify the existence of a solution to (4.3). □

THEOREM 4.5. *Under the same condition as that in Theorem* 4.4, *the solution to equation* (4.3) *is strongly unique in the sense that* $P(H(X_t, \hat{X}_t) = 0$ *for all* $t \in [0, T]) = 1$ *if* X_t *and* $\hat{X}_t$ *are solutions to* (4.3) *with the same initial value* X_0.

PROOF. Assume that X_t and $\hat{X}_t$ are two solutions to equation (4.3) with the same initial value X_0. Then we have

$$\begin{aligned}
E[H^2(X_t, \hat{X}_t)] &\le E\Big[\Big(H(X_0, X_0) + H\Big(\int_0^t a(s, X_s)ds, \int_0^t a(s, \hat{X}_s)ds\Big) \\
&\qquad + \Big\| \int_0^t (b(s, X_s) - b(s, \hat{X}_s))dB_s \Big\|\Big)^2\Big] \\
&\le 2E\Big[H^2\Big(\int_0^t a(s, X_s)ds, \int_0^t a(s, \hat{X}_s)ds\Big)\Big] \\
&\quad + 2E\Big[\Big\| \int_0^t (b(s, X_s) - b(s, \hat{X}_s))dB_s \Big\|^2\Big] \\
&\le 2tE\Big[\int_0^t H^2(a(s, X_s), a(s, \hat{X}_s))ds\Big] \\
&\quad + 2C_{\mathfrak{X}}E\Big[\int_0^t \|b(s, X_s) - b(s, \hat{X}_s)\|^2 ds\Big] \\
&\le 2tD^2E\Big[\int_0^t H^2(X_s, \hat{X}_s)ds\Big] + 2C_{\mathfrak{X}}D^2E\Big[\int_0^t H^2(X_s, \hat{X}_s)ds\Big] \\
&= 2D^2(t + C_{\mathfrak{X}})E\Big[\int_0^t H^2(X_s, \hat{X}_s)ds\Big] \\
&= 2D^2(t + C_{\mathfrak{X}})\int_0^t E[H^2(X_s, \hat{X}_s)]ds\,.
\end{aligned}$$

Set

$$\nu(t) := E[H^2(X_t, \hat{X}_t)] \quad \text{for } t \in [0, T]\,,$$

then

$$\nu(t) \le 2D^2(T + C_{\mathfrak{X}})\int_0^t \nu(s)ds\,,$$

which, together with Gronwall's inequality, implies

$$\nu(t) = E[H^2(X_t, \hat{X}_t)] = 0 \quad \text{for all } t \in [0, T]\,.$$

Therefore $H^2(X_t, \hat{X}_t) = 0$ a.s. Since X_t and $\hat{X}_t$ are H-continuous in t with probability 1, we have

$$P\big(H(X_t, \hat{X}_t) = 0 \text{ for all } t \in [0, T]\big) = 1\,,$$

which completes the proof of the uniqueness. □

REMARK 4.6. If the separable M-type 2 Banach space $\mathfrak{X}$ is reflexive, and if the initial value X_0 is L^2-integrably bounded and weakly closed, then equation (4.3) also can be defined well by Theorem 3.9. Similarly, Theorems 4.4 and 4.5 hold too.

Acknowledgment. The authors would like to express their gratitude to Professor Yukio Ogura for his constant encouragement and his valuable comments, and also would like to thank the referee for valuable suggestions.

REFERENCES

[1] J. P. AUBIN AND H. FRANKOWSKA, Set-valued analysis, Birkhäuser, Boston, Basel, Berlin, 1990.

[2] J. P. AUBIN AND G. DA PRATO, The viability theorem for stochastic differential inclusions, Stochastic Anal. Appl. 16 (1998), 1–15.

[3] Z. BRZEŹNIAK AND A. CARROLL, Approximations of the Wong-Zakai type for stochastic differential equations in M-type 2 Banach spaces with applications to loop spaces, Séminaire de Probabilitiés, XXXVII: 251–289, Lecture Notes in Math. 1832, Springer-Verlag, Berlin, 2003.

[4] C. CASTAING AND M. VALADIER, Convex analysis and measurable multifunctions, Lecture Notes in Math. 580, Springer-Verlag, Berlin, 1977.

[5] D. A. CHARALAMBOS AND C. B. KIM, Infinite dimensional analysis, Springer-Verlag, Berlin, 1994.

[6] F. HIAI AND H. UMEGAKI, Integrals, conditional expectations and martingales of multivalued functions, J. Multivariate. Anal. 7 (1977), 149–182.

[7] F. HIAI, Convergence of conditional expectations and strong laws of large numbers for multivalued random variables, Trans. Amer. Math. Soc. 291 (1985), 613–627.

[8] N. IKEDA AND S. WATANABE, Stochastic differential equations and diffusion processes, North-Holland Math. Library, 24, North-Holland Publishing Co. Amsterdam-New York, Kodansha, Ltd., Tokyo, 1981.

[9] D. A. KANDILAKIS AND N. S. PAPAGEORGIOU, On the existence of solutions for random differential inclusions in a Banach space, J. Math. Anal. Appl. 126 (1987), 11–23.

[10] M. KISIELEWICZ, M. MICHTA AND J. MOTYL, Set valued approach to stochastic control, part I, Existence and regularity properties, Dynam. Systems Appl. 12 (2003), 405–432.

[11] M. KISIELEWICZ, M. MICHTA AND J. MOTYL, Set valued approach to stochastic control, part II, Viability and semimartingale issues, Dynam. Systems Appl. 12 (2003), 433–466.

[12] M. KISIELEWICZ, Set-valued stochastic integrals and stochastic inclusions, Stochastic Anal. Appl. 15 (1997), 780–800.

[13] M. KISIELEWICZ, Properties of solution set of stochastic inclusions, J. Appl. Math. Stochastic Anal. 6 (1993), 217–236.

[14] M. KISIELEWICZ, Existence theorem for nonconvex stochastic inclusions, J. Appl. Math. Stochastic Anal. 7 (1994), 151–159.

[15] M. KISIELEWICZ, Quasi-retractive representation of solution sets to stochastic inclusions, J. Appl. Math. Stochastic Anal. 10 (1997), 227–238.

[16] H. KUNITA, Stochastic differential equations and stochastic flows of diffeomorphisms, École d'été de Probabilités de Saint-Flour, XII–1982, 143–303, Lecture Notes in Math. 1097, Springer-Verlag, Berlin, 1984.

[17] S. LI, Y. OGURA AND V. KREINOVICH, Limit theorems and applications of set-valued and fuzzy set-valued random variables, Kluwer Academic Publishers, Dordrecht, 2002.

[18] R. E. MEGGINSON, An introduction to Banach space theory, Springer, New York, 1998.

[19] M. MICHTA, Set-valued random differential equations in Banach space, Discuss. Math. Differential Incl. 15 (1995), 191–200.

[20] J. MOTYL, On the solution of stochastic differential inclusion, J. Math. Anal. Appl. 192 (1995), 117–132.

[21] Y. OGURA, On stochastic differential equations with set coefficients and the Black-Scholes model, Proceedings of the Eighth International Conference on Intelligent Technologies, 300–304, Sydney, 2007.

[22] B. ØKSENDAL, Stochastic differential equations, Springer-Verlag, 1998.

[23] G. Pisier, Martingales with values in uniformly convex spaces, Israel J. Math. 20 (1976), 326–350.

[24] J. J. UHL, JR., The range of a vector-valued measure, Proc. Amer. Math. Soc. 23 (1969), 158–163.

[25] J. ZHANG, S. LI, I. MITOMA AND Y. OKAZAKI, On set-valued stochastic integrals in an M-type 2 Banach space, J. Math. Anal. Appl. 350 (2009), 216–233.

MATHEMATICAL DEPARTMENT
SAGA UNIVERSITY
SAGA 840–8502
JAPAN

E-mail address: zhangjinping@bjut.edu.cn

DEPARTMENT OF APPLIED MATHEMATICS
BEIJING UNIVERSITY OF TECHNOLOGY
BEIJING 100022
P.R. CHINA

E-mail address: lisma@bjut.edu.cn

MATHEMATICAL DEPARTMENT
SAGA UNIVERSITY
SAGA 840–8502
JAPAN

E-mail address: mitoma@ms.saga-u.ac.jp

DEPARTMENT OF SYSTEMS DESIGN
AND INFORMATICS
KYUSHU INSTITUTE OF TECHNOLOGY
IIZUKA 820–8502
JAPAN

E-mail address: okazaki@ces.kyutech.ac.jp