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A ROUGH MULTIPLE MARCINKIEWICZ INTEGRAL ALONG CONTINUOUS SURFACES

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Abstract. By means of the method of block decompositions for kernel functions and some delicate estimates on Fourier transforms, the $L^p(\boldsymbol{R}^m \times \boldsymbol{R}^n \times \boldsymbol{R})$-boundedness of the multiple Marcinkiewicz integral is established along a continuous surface with rough kernel for some $p > 1$. The condition on the integral kernel is the best possible for the L^2-boundedness of the multiple Marcinkiewicz integral operator.

1. Introduction. Let $\boldsymbol{R}^N$ ($N = m$ or n), $N \geq 2$, be the N-dimensional Euclidean space and S^{N-1} the unit sphere in $\boldsymbol{R}^N$. For nonzero points $x \in \boldsymbol{R}^m$ and $y \in \boldsymbol{R}^n$, we denote $x' = x/|x|$ and $y' = y/|y|$. For $m \geq 2$ and $n \geq 2$, let $\Omega(x', y') \in L^1(S^{m-1} \times S^{n-1})$ be a homogeneous function of degree zero satisfying

$$\int_{S^{m-1}} \Omega(x', y')dx' = \int_{S^{n-1}} \Omega(x', y')dy' = 0\,. \tag{1.1}$$

Then the Marcinkiewicz integral operator μ_Ω on the product space $\boldsymbol{R}^m \times \boldsymbol{R}^n$ is defined by

$$\mu_\Omega(f)(x, y) = \left(\int_0^\infty\int_0^\infty |F_{s,t}(x, y)|^2 \frac{dsdt}{s^3t^3}\right)^{1/2}$$

for all $f \in \mathcal{S}(\boldsymbol{R}^m \times \boldsymbol{R}^n)$, where

$$F_{s,t}(x, y) = \iint_{|\xi|<s,|\eta|<t} \frac{\Omega(\xi', \eta')}{|\xi|^{m-1}|\eta|^{n-1}} f(x-\xi, y-\eta)d\xi d\eta\,.$$

Obviously, the operator μ_Ω is a natural analogy of the higher-dimensional Marcinkiewicz integral introduced by Stein [24]. It is well known that the Marcinkiewicz integral operator is an important special case of the Littlewood-Paley-Stein functions and plays a key role in harmonic analysis. In the one-parameter case, it is shown that if Ω satisfies some regularity conditions, then μ_Ω is bounded on $L^p(\boldsymbol{R}^N)$, $1 < p < \infty$ (see [2, 24]). Subsequently, the result mentioned above was improved by many authors. One can consult [7, 8, 14, 21, 22, 24, 25, 28], among numerous references, for its development and applications.

For the multiple Marcinkiewicz integral operator μ_Ω, the following results have been known.

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THEOREM A. *If Ω satisfies* (1.1) *and one of the following conditions holds, then μ_Ω is bounded on $L^p(\boldsymbol{R}^m \times \boldsymbol{R}^n)$.*

(i) $\Omega \in L^q(S^{m-1} \times S^{n-1})$ *for* $q > 1$, $1 < p < \infty$ (*cf.* [3, 4]).

(ii) $\Omega \in L(\log^+ L)^2(S^{m-1} \times S^{n-1})$, $1 < p < \infty$ (*cf.* [5, 10]).

(iii) $\Omega \in L\log^+ L(S^{m-1} \times S^{n-1})$, $p = 2$ (*cf.* [6, 9]).

(iv) $\Omega \in B_q^{0,1}(S^{m-1} \times S^{n-1})$ *for* $q > 1$, $1 < p < \infty$ (*cf.* [11, 29, 30]).

(v) $\alpha > 1/2, 1 + 1/(2\alpha) < p < 1 + 2\alpha$, *and Ω satisfies*

$$\iint_{S^{m-1}\times S^{n-1}} |\Omega(x', y')| \left(\log \frac{1}{|\xi' \cdot x'|} \log \frac{1}{|\eta' \cdot y'|} \right)^\alpha dx'\, dy' \in L^\infty(S^{m-1} \times S^{n-1})$$

(*cf.* [13, 32]).

Here $B_q^{0,\upsilon}(S^{m-1} \times S^{n-1})$ denotes a special class of block spaces on $S^{m-1} \times S^{n-1}$ (see Section 2 for the definition), which were introduced by Jiang and Lu [15] in the study of the L^p-boundedness of Calderón-Zygmund singular integral operator. Employing the ideas of [16], the present author [31] pointed out that, for $q > 1$ and $\upsilon_1 > \upsilon_2 > -1$,

$$\bigcup_{r>1} L^r(S^{m-1} \times S^{n-1}) \subset B_q^{0,\upsilon_1}(S^{m-1} \times S^{n-1}) \subset B_q^{0,\upsilon_2}(S^{m-1} \times S^{n-1}), \tag{1.3}$$

which are proper inclusions, and $B_q^{0,\upsilon}$ can not be contained in $L(\log^+ L)^{\upsilon+\varepsilon}$ for any $\upsilon > -1$ and $\varepsilon > 1$.

On the other hand, for the one-parameter case, Al-Qassem and Al-Salman [1] established the following.

THEOREM B ([1]). *If Ω has the mean value zero on the unit sphere S^{N-1} and belongs to $B_q^{0,-1/2}(S^{N-1})$ for $q > 1$, then μ_Ω is of type (p, p) for $1 < p < \infty$. Moreover, for all $-1 < \upsilon < -1/2$, there exists an $\Omega \in B_q^{0,\upsilon}(S^{N-1})$ such that μ_Ω is not bounded on $L^2(\boldsymbol{R}^N)$.*

A natural problem, which arises on the above results, is the following.

QUESTION. For the $L^p(\boldsymbol{R}^m \times \boldsymbol{R}^n)$-boundedness of μ_Ω, is it sufficient that $\Omega \in B_q^{0,0}(S^{m-1} \times S^{n-1})$ for some $q > 1$, $1 < p < \infty$?

In this paper, we give an affirmative answer to the above question by studying the operator $\mu_{\Omega,\gamma}$ along a continuous surface γ. Precisely, suppose that $\gamma(u, v)$ is a continuous surface in $\boldsymbol{R}^+ \times \boldsymbol{R}^+$. For $(x, y, z) \in \boldsymbol{R}^m \times \boldsymbol{R}^n \times \boldsymbol{R}$, we define

$$\mu_{\Omega,\gamma}(f)(x, y, z) = \left(\int_0^\infty \int_0^\infty |F_{s,t}^\gamma(x, y, z)|^2 \frac{ds\,dt}{s^3 t^3} \right)^{1/2},$$

where

$$F_{s,t}^\gamma(x, y, z) = \iint_{|\xi|<s, |\eta|<t} \frac{\Omega(\xi', \eta')}{|\xi|^{m-1} |\eta|^{n-1}} f(x - \xi, y - \eta, z - \gamma(|\xi|, |\eta|))\, d\xi\, d\eta\,.$$

For $\iota, \tau, z \in \boldsymbol{R}$, we also define the following two maximal functions

$$M^1_\gamma h(\iota, \tau, z) = \sup_{R>0, S>0} R^{-1}S^{-1} \int_0^R \int_0^S |h(\iota - u, \tau - v, z - \gamma(u, v))| du dv \,,$$

$$M^2_\gamma g(\iota, z) = \sup_{R>0, S>0} R^{-1}S^{-1} \int_0^R \int_0^S |g(\iota - u, z - \gamma(u, v))| du dv \,.$$

For the operator $\mu_{\Omega,\gamma}$, Ding et al. [11] obtained the following.

THEOREM C. *Suppose that Ω is a homogeneous function of degree zero satisfying* (1.1). *Then $\mu_{\Omega,\gamma}$ is bounded on $L^p(\boldsymbol{R}^m \times \boldsymbol{R}^n \times \boldsymbol{R})$, provided that $\Omega \in B_q^{0,1}(S^{m-1} \times S^{n-1})$ for some $q > 1$, and for $r > 1$*:

(i) $\|M^1_\gamma h\|_{L^r(\boldsymbol{R}^3)} \leq C\|h\|_{L^r(\boldsymbol{R}^3)}$;

(ii) $\|M^2_\gamma g\|_{L^r(\boldsymbol{R}^2)} \leq C\|g\|_{L^r(\boldsymbol{R}^2)}$.

It is clear that the above two maximal functions are natural extensions of the maximal functions

$$M^1_\Gamma(h)(\iota, \tau) = \sup_{R>0} R^{-1} \int_0^R |h(\iota - u, \tau - \Gamma(u))| du$$

and

$$M^2_\Gamma(g)(\iota) = \sup_{R>0} R^{-1} \int_0^R |g(\iota - \Gamma(u))| du \,.$$

The maximal functions M^1_Γ and M^2_Γ play an important role in harmonic analysis and they are extensively studied by many authors (see [26]). Also, the surfaces γ satisfying (i) and (ii) are easily available. A simple example is $\gamma(u, v) = u^\alpha v^\beta$ with $\alpha > 0$ and $\beta > 0$ (see [12, Corollary 3]). In this paper, we prove our main theorem as follows.

THEOREM 1. *Suppose that Ω is a homogeneous function of degree zero satisfying* (1.1), *and for $r \in (1, \infty)$*,

$$\|M^1_\gamma h\|_{L^r(\boldsymbol{R}^3)} \leq C\|h\|_{L^r(\boldsymbol{R}^3)} \,. \tag{1.2}$$

If one of the following conditions holds, then $\mu_{\Omega,\gamma}$ is bounded on $L^p(\boldsymbol{R}^m \times \boldsymbol{R}^n \times \boldsymbol{R})$:

(i) $\Omega \in B_q^{0,0}(S^{m-1} \times S^{n-1})$ *for some* $q > 1$, $p = 2$;

(ii) $\Omega \in B_q^{0,\upsilon}(S^{m-1} \times S^{n-1})$ *for some* $q > 1$, $0 < \upsilon < 1$, *and* $p \in (2/(1+\upsilon), 2/(1-\upsilon))$.

By Theorem 1, we also obtain the following result.

THEOREM 2. *Suppose that Ω is a homogeneous function of degree zero satisfying* (1.1). *If one of the following conditions holds, then the multiple Marcinkiewicz integral operator μ_Ω is bounded on $L^p(\boldsymbol{R}^m \times \boldsymbol{R}^n)$*:

(i) $\Omega \in B_q^{0,0}(S^{m-1} \times S^{n-1})$ *for some* $q > 1$, $p = 2$;

(ii) $\Omega \in B_q^{0,\upsilon}(S^{m-1} \times S^{n-1})$ *for some* $q > 1$, $0 < \upsilon < 1$, *and* $p \in (2/(1+\upsilon), 2/(1-\upsilon))$.

REMARK 1. Since

$$\bigcup_{r>1} L^r(S^{m-1}\times S^{n-1}) \subset B_q^{0,1}(S^{m-1}\times S^{n-1}) \subset B_q^{0,\upsilon}(S^{m-1}\times S^{n-1})$$

are proper inclusions, and $L(\log^+ L)^2(S^{m-1}\times S^{n-1})$ does not contain $B_q^{0,\upsilon}(S^{m-1}\times S^{n-1})$ for $0\le \upsilon < 1$, Theorem 1 essentially improves Theorem C, and Theorem 2 is an improvement or extension of Theorem A or B. In addition, condition (ii) of Theorem C is not necessary.

REMARK 2. By an argument similar to that used in [1], we remark that $\Omega \in B_q^{0,0}(S^{m-1}\times S^{n-1})$ for $q>1$ is the best possible condition such that $\mu_{\Omega,\gamma}$ is bounded on $L^2(\boldsymbol{R}^m\times \boldsymbol{R}^n\times \boldsymbol{R})$. Namely, there exists an Ω which lies in $B_q^{0,\upsilon}(S^{m-1}\times S^{n-1})$ for all $-1<\upsilon<0$ and satisfies (1.1) such that $\mu_{\Omega,\gamma}$ is not bounded on $L^2(\boldsymbol{R}^m\times \boldsymbol{R}^n\times \boldsymbol{R})$.

This paper is organized as follows. In Section 2 we review the definition of block spaces on $S^{m-1}\times S^{n-1}$. Next, we introduce some notation and establish some estimates which will play a key role in our proofs in Section 3. Finally, the proofs of our theorems are given in Section 4. We would like to remark that we are especially motivated by [11–13].

Throughout this paper, C always denotes positive constants that are independent of the essential variables but whose value may vary at each occurrence.

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2. Block spaces. In this section, we review the definitions of block spaces $B_q^{0,\upsilon}(S^{m-1}\times S^{n-1})$ (see [15]).

DEFINITION 1. For $1<q\le\infty$, we say that a Lebesgue measurable function $b(\cdot,\cdot)$ defined on $S^{m-1}\times S^{n-1}$ is a q-block, if it satisfies the following conditions:

(i) $\operatorname{supp}(b)\subseteq Q$;

(ii) $\|b\|_{L^q(S^{m-1}\times S^{n-1})}\le |Q|^{1/q-1}$;

where Q is an interval on $S^{m-1}\times S^{n-1}$, i.e., there exist $x_0'\in S^{m-1}$, $y_0'\in S^{n-1}$, and $\rho>0$, $\varrho>0$ such that

$$Q=\{x'\in S^{m-1};\,|x'-x_0'|<\rho\}\times\{y'\in S^{n-1};\,|y'-y_0'|<\varrho\},$$

where $|Q|$ denotes the volume of Q.

DEFINITION 2. The block space $B_q^{0,\upsilon}(S^{m-1}\times S^{n-1})$ is defined by

$$B_q^{0,\upsilon}(S^{m-1}\times S^{n-1}) = \left\{\Omega\in L^1(S^{m-1}\times S^{n-1});\,\Omega(\xi',\eta')=\sum_\alpha C_\alpha b_\alpha(\xi',\eta'),\ M_q^{0,\upsilon}(\{C_\alpha\})<\infty\right\},$$

where each b_α is a q-block supported in Q_α, and

$$M_q^{0,\upsilon}(\{C_\alpha\})=\sum_\alpha |C_\alpha|\left\{1+\left(\log^+\frac{1}{|Q_\alpha|}\right)^{\upsilon+1}\right\}.$$

When $\Omega \in B_q^{0,\upsilon}(S^{m-1} \times S^{n-1})$, the norm of Ω is defined by $M_q^{0,\upsilon}(\Omega) = \inf\{M_q^{0,\upsilon}(\{C_\alpha\})\}$, where the infimum is taken over all q-block decompositions of Ω.

The method of block decomposition of functions was originated by Taibleson and Weiss in the study of the convergence of Fourier series (see [27]). Later on, many applications of the block decomposition to harmonic analysis were discovered (see [17, 18, 20, 23]). For details, one may see the survey book [19]. In particular, it was noted by Keitoku and Sato [16] that for any fixed $q > 1$, $\bigcup_{r>1} L^r(S^{N-1}) \subset B_q^{0,\upsilon_1}(S^{N-1}) \subset B_q^{0,\upsilon_2}(S^{N-1})$, for any $\upsilon_1 > \upsilon_2 > -1$, which are proper inclusions. Similarly,

$$\bigcup_{r>1} L^r(S^{m-1} \times S^{n-1}) \subset B_q^{0,\upsilon_1}(S^{m-1} \times S^{n-1}) \subset B_q^{0,\upsilon_2}(S^{m-1} \times S^{n-1}), \quad -1 < \upsilon_2 < \upsilon_1 ,$$

which are proper inclusions, and $L(\log^+ L)^{\upsilon+\varepsilon}$ does not contain $B_q^{0,\upsilon}$ on $(S^{m-1} \times S^{n-1})$ for any $\upsilon > -1$ and $\varepsilon > 1$, although, so far, the relationship between the spaces $B_q^{0,\upsilon}$ and $L(\log^+ L)^{\upsilon+1}$ on $S^{m-1} \times S^{n-1}$ is still not clear (see [31]).

3. Notation and lemmas. Let Ω be as in Theorem 1. It follows from Definition 2 that $\Omega(x', y') = \sum_\alpha C_\alpha b_\alpha(x', y')$, where each b_α is a q-block supported in Q_α, i.e.,

$$\operatorname{supp}(b_\alpha) \subseteq Q_\alpha \quad \text{and} \quad \|b_\alpha\|_{L^q(S^{m-1}\times S^{n-1})} \le |Q_\alpha|^{1/q-1} .$$

The lack of the mean zero property of the block function b_α will make our work difficult. Therefore, for each function b_α, we define a function $\tilde{b}_\alpha$ as follows:

$$\begin{aligned} \tilde{b}_\alpha(x', y') = b_\alpha(x', y') &- \frac{1}{|S^{m-1}|}\int_{S^{m-1}} b_\alpha(\xi', y')d\xi' - \frac{1}{|S^{n-1}|}\int_{S^{n-1}} b_\alpha(x', \eta')d\eta' \\ &+ \frac{1}{|S^{m-1}||S^{n-1}|}\iint_{S^{m-1}\times S^{n-1}} b_\alpha(\xi', \eta')d\xi' d\eta' , \end{aligned} \tag{3.1}$$

where $|S^{m-1}|$ and $|S^{n-1}|$ denote the Lebesgue measures of S^{m-1} and S^{n-1}, respectively. It is easy to verify that

$$\int_{S^{m-1}} \tilde{b}_\alpha(x', y')dx' = \int_{S^{n-1}} \tilde{b}_\alpha(x', y')dy' = 0 , \tag{3.2}$$

$$\|\tilde{b}_\alpha\|_{L^q(S^{m-1}\times S^{n-1})} \le 4|Q_\alpha|^{1/q-1} \quad \text{and} \quad \|\tilde{b}_\alpha\|_{L^1(S^{m-1}\times S^{n-1})} \le 4 . \tag{3.3}$$

Noting that $\Omega(x', y') = \sum_\alpha C_\alpha b_\alpha(x', y')$, we can deduce from (1.1) and (3.2) that

$$\Omega(x', y') = \sum_\alpha C_\alpha \tilde{b}_\alpha(x', y') . \tag{3.4}$$

Now our aim is to establish some Fourier transform estimates related to $\tilde{b}_\alpha$, which will play a key role in the proofs of our theorems. For $j, k \in \boldsymbol{Z}$, $s, t \in \boldsymbol{R}^+$, we write

$$B_{j,k}^{s,t} = \{(x, y) \in \boldsymbol{R}^m \times \boldsymbol{R}^n; 2^j s \le |x| < 2^{j+1}s, 2^k t \le |y| < 2^{k+1}t\} .$$

For each $\tilde{b}_\alpha$, we define measures $\sigma^{s,t}_{\alpha;j,k}$ by

$$\sigma^{s,t}_{\alpha;j,k} * f(x,y,z) = \frac{1}{2^{j+k}st}\int_{B^{s,t}_{j,k}} \frac{\tilde{b}_\alpha(\xi',\eta')}{|\xi|^{m-1}|\eta|^{n-1}} f(x-\xi, y-\eta, z-\gamma(|\xi|,|\eta|))d\xi d\eta\,.$$

It is easy to see that its Fourier transform is

$$\hat{\sigma}^{s,t}_{\alpha;j,k}(\xi,\eta,\zeta) = \frac{1}{2^{j+k}st}\int_{B^{s,t}_{j,k}} \frac{\tilde{b}_\alpha(x',y')}{|x|^{m-1}|y|^{n-1}}\, e^{-i\{x\cdot\xi+y\cdot\eta+\gamma(|x|,|y|)\zeta\}}dxdy\,. \tag{3.5}$$

Similarly, we define the measure $|\sigma^{s,t}_{\alpha;j,k}|$ by letting its Fourier transform be

$$|\hat{\sigma}^{s,t}_{\alpha;j,k}|(\xi,\eta,\zeta) = \frac{1}{2^{j+k}st}\int_{B^{s,t}_{j,k}} \frac{|\tilde{b}_\alpha(x',y')|}{|x|^{m-1}|y|^{n-1}} e^{-i\{x\cdot\xi+y\cdot\eta+\gamma(|x|,|y|)\zeta\}}dxdy\,. \tag{3.6}$$

Then

$$|\sigma^{s,t}_{\alpha;j,k}| * f(x,y,z) = \frac{1}{2^{j+k}st}\int_{B^{s,t}_{j,k}} \frac{|\tilde{b}_\alpha(\xi',\eta')|}{|\xi|^{m-1}|\eta|^{n-1}} f(x-\xi, y-\eta, z-\gamma(|\xi|,|\eta|))d\xi d\eta\,.$$

By (3.4), we have

$$F^\gamma_{s,t}(x,y,z) = \sum_\alpha C_\alpha \sum_{j=-\infty}^{-1}\sum_{k=-\infty}^{-1} 2^{j+k}st\,\sigma^{s,t}_{\alpha;j,k} * f(x,y,z)\,. \tag{3.7}$$

Also, we define the maximal function σ^*_α by

$$\sigma^*_\alpha(f)(x,y,z) = \sup_{s,t\in \boldsymbol{R}^+;\, j,k\in \boldsymbol{Z}} ||\sigma^{s,t}_{\alpha;j,k}| * f(x,y,z)|\,.$$

It is easy to verify that the total variation of $\sigma^{s,t}_{\alpha;j,k}$ satisfies

$$\||\sigma^{s,t}_{\alpha;j,k}|\|_1 \le \frac{1}{2^{j+k}st}\int_{B^{s,t}_{j,k}} |\tilde{b}_\alpha(x',y')||x|^{1-m}|y|^{1-n}dxdy \le 1\,, \tag{3.8}$$

uniformly for $s,t\in \boldsymbol{R}^+$, $j,k\in \boldsymbol{Z}$ and b_α, and $|\sigma^{s,t}_{\alpha;j,k}|$ is positive.

LEMMA 1. *If the surface γ satisfies* (1.2) *in Theorem* 1, *then σ^*_α is bounded on $L^p(\boldsymbol{R}^m\times \boldsymbol{R}^n\times \boldsymbol{R})$, $1<p<\infty$, and the bound is independent of the block $b_\alpha(\cdot,\cdot)$.*

Proof. By definition, for any $f(x)\ge 0$, we have

$$\begin{aligned}
&\sigma^*_\alpha(f)(x,y,z)\\
&\le \sup_{s,t\in \boldsymbol{R}^+}\sup_{j,k\in \boldsymbol{Z}} \frac{1}{2^{j+k}st}\int_{2^js}^{2^{j+1}s}\int_{2^kt}^{2^{k+1}t} |\tilde{b}_\alpha(\xi',\eta')| f(x-u\xi', y-u\eta', z-\gamma(u,v))d\xi'd\eta'dudv\\
&\le C\iint_{S^{m-1}\times S^{n-1}} |\tilde{b}_\alpha(\xi',\eta')| M_{\xi',\eta'}(f)(x,y,z)d\xi'd\eta',
\end{aligned}$$

where

$$M_{\xi',\eta'}(f)(x,y,z) = \sup_{s,t\in \boldsymbol{R}^+} \frac{1}{st}\int_0^s\int_0^t f(x-u\xi', y-v\eta', z-\gamma(u,v))dudv$$

is the Hardy-Littlewood maximal function in the space $\boldsymbol{R} \times \boldsymbol{R}$ along γ in the direction (ξ', η'). Then

$$\|\sigma_\alpha^*(f)\|_p \le C \iint_{S^{m-1}\times S^{n-1}} |\tilde{b}_\alpha(\xi', \eta')| \|M_{\xi',\eta'}(f)\|_p d\xi' d\eta', \quad 1 < p < \infty .$$

Since $\|\tilde{b}_\alpha\|_{L^1(S^{m-1}\times S^{n-1})} \le 4$, to prove Lemma 1, it remains to prove that $M_{\xi',\eta'}$ is bounded on $L^p(\boldsymbol{R}^m \times \boldsymbol{R}^n \times \boldsymbol{R})$, $1 < p < \infty$. Let $\mathbf{1} = (1, 0, \dots, 0) \in S^{m-1}$, $\tilde{\mathbf{1}} = (1, 0, \dots, 0) \in S^{n-1}$. For each fixed (ξ', η'), choose a rotation $\rho = \rho_1 \otimes \rho_2$ such that $\rho_1\xi = \mathbf{1}$ and $\rho_2\eta = \tilde{\mathbf{1}}$. Let $\rho^{-1} = \rho_1^{-1} \otimes \rho_2^{-1}$ be the inverse of ρ. We define the function f_ρ by $f_\rho(x, y, z) = f(\rho_1 x, \rho_2 y, z)$. So

$$f(x - u\xi', y - v\eta', z - \gamma(u, v)) = f_{\rho^{-1}}(\rho_1 x - u\mathbf{1}, \rho_2 y - v\tilde{\mathbf{1}}, z - \gamma(u, v)) .$$

This, together with the condition (1.2) in Theorem 1 and a change of variables, show that

$$\|M_{\xi',\eta'}(f)\|_{L^p(\boldsymbol{R}^m\times\boldsymbol{R}^n\times\boldsymbol{R})} \le C\|f\|_{L^p(\boldsymbol{R}^m\times\boldsymbol{R}^n\times\boldsymbol{R})}, \quad 1 < p < \infty ,$$

where C is independent of (ξ', η'). Lemma 1 is thus proved. □

LEMMA 2. *Let $\Omega = \sum_\alpha C_\alpha b_\alpha = \sum_\alpha C_\alpha \tilde{b}_\alpha$ be as in* (3.4). *Suppose that* $1 < q \le 2$, *and $|Q_\alpha| < 1$ for each α. Then for each α*:

(i) $|\hat{\sigma}^{s,t}_{\alpha;j,k}(\xi, \eta, \zeta)| \le C|2^j s\xi||2^k t\eta|$;

(ii) *if $|Q_\alpha| < e^{2q/(1-q)}$, then*

$$|\hat{\sigma}^{s,t}_{\alpha;j,k}(\xi, \eta, \zeta)| \le C \min\{1, |2^j s\xi|^{2/\log|Q_\alpha|}|2^k t\eta|, |2^j s\xi||2^k t\eta|^{2/\log|Q_\alpha|}, |2^j s\xi|^{2/\log|Q_\alpha|}|2^k t\eta|^{2/\log|Q_\alpha|}\}; \tag{3.9}$$

(iii) *if $|Q_\alpha| \ge e^{2q/(1-q)}$, then*

$$|\hat{\sigma}^{s,t}_{\alpha;j,k}(\xi, \eta, \zeta)| \le C \min\{1, |2^j s\xi|^{-\varepsilon}|2^k t\eta|, |2^j s\xi||2^k t\eta|^{-\varepsilon}, |2^j s\xi|^{-\varepsilon}|2^k t\eta|^{-\varepsilon}\} , \tag{3.10}$$

where ε, C are positive constants independent of $s, t \in \boldsymbol{R}^+$, $j, k \in \boldsymbol{Z}$, $(\xi, \eta, \zeta) \in \boldsymbol{R}^m \times \boldsymbol{R}^n \times \boldsymbol{R}$, and the block $b_\alpha(\cdot, \cdot)$.

PROOF. First we consider the case $m > 2$ and $n > 2$. By (3.2) and (3.3), (i) is obvious. By the definition of $\hat{\sigma}^{s,t}_{\alpha;j,k}$ and (3.3), it is easy to see that

$$|\hat{\sigma}^{s,t}_{\alpha;j,k}(\xi, \eta, \zeta)| \le C . \tag{3.11}$$

Now we prove the other cases of (ii) and (iii). Let $\mathbf{1} = (1, 0, \dots, 0) \in S^{m-1}$ and $\tilde{\mathbf{1}} = (1, 0, \dots, 0) \in S^{n-1}$. For any fixed $\xi \ne 0$ and $\eta \ne 0$, by the method of rotation, without loss of generality, we may write

$$
\begin{aligned}
&|\hat{\sigma}^{s,t}_{\alpha;j,k}(\xi,\eta,\zeta)| \\
&\quad\le \frac{1}{2^{j+k}st}\int_{2^j s}^{2^{j+1}s}\int_{2^k t}^{2^{k+1}t}\left|\iint_{S^{m-1}\times S^{n-1}}\tilde{b}_\alpha(x',y')e^{-iu|\xi|\mathbf{1}\cdot x'}[e^{-iv|\eta|\tilde{\mathbf{1}}\cdot y'}-1]dx'dy'\right|dudv \\
&\quad\le \frac{C}{2^{j+k}st}\int_{2^k t}^{2^{k+1}t}v|\eta|\int_{2^j s}^{2^{j+1}s}\int_{S^{n-1}}\left|\int_{S^{m-1}}\tilde{b}_\alpha(x',y')e^{-iu|\xi|\mathbf{1}\cdot x'}dx'\right|dy'dudv \\
&\quad\le C|2^k t\eta|\int_{S^{n-1}}\int_{2^j s|\xi|}^{2^{j+1}s|\xi|}(2^j s|\xi|)^{-1}\left|\int_{S^{m-1}}\tilde{b}_\alpha(x',y')e^{-ix_1'u}dx'\right|dudy' \\
&\quad\le C|2^k t\eta|\int_{S^{n-1}}\int_{2^j s|\xi|}^{2^{j+1}s|\xi|}(2^j s|\xi|)^{-1}\left|\int_{\boldsymbol{R}}\phi_{y'}(x_1')e^{-ix_1'u}dx_1'\right|dudy' \\
&\quad\le C|2^k t\eta|\int_{S^{n-1}}\int_{2^j s|\xi|}^{2^{j+1}s|\xi|}(2^j s|\xi|)^{-1}|\hat{\phi}_{y'}(u)|dudy',
\end{aligned}
$$

where

$$\phi_{y'}(x_1') = (1-x_1'^2)^{(m-3)/2}\chi_{\{|x_1'|<1\}}(x_1')\int_{S^{m-2}}\tilde{b}_\alpha(x_1',(1-x_1'^2)^{1/2}\bar{x}',y')d\bar{x}'$$

is a one-dimensional function. Then, for any $\omega\in(1,q)$, it follows from Hölder's inequality that

$$|\hat{\sigma}^{s,t}_{\alpha;j,k}(\xi,\eta,\zeta)| \le C|2^k t\eta|\int_{S^{n-1}}(2^j s|\xi|)^{-1/\omega'}\|\hat{\phi}_{y'}\|_{L^{\omega'}(\boldsymbol{R})}dy',$$

where $\omega'=\omega/(\omega-1)$. By the Hausdorff-Young inequality, we have

$$|\hat{\sigma}^{s,t}_{\alpha;j,k}(\xi,\eta,\zeta)| \le C|2^k t\eta||2^j s\xi|^{-1/\omega'}\int_{S^{n-1}}\|\phi_{y'}\|_{L^{\omega}(\boldsymbol{R})}dy'. \tag{3.12}$$

On the other hand, by the definition of $\phi_{y'}$ and Hölder's inequality, we have

$$
\begin{aligned}
\int_{S^{n-1}}\|\phi_{y'}\|_{L^\omega(\boldsymbol{R})}dy' &\le C\|\tilde{b}_\alpha\|_{L^\omega(S^{m-1}\times S^{n-1})} \\
&\le C\|\tilde{b}_\alpha\|_{L^q(S^{m-1}\times S^{n-1})}|Q_\alpha|^{1/\omega-1/q} \le C|Q_\alpha|^{-1/\omega'}.
\end{aligned}
$$

This together with (3.12) shows that for any $\omega\in(1,q)$,

$$|\hat{\sigma}^{s,t}_{\alpha;j,k}(\xi,\eta,\zeta)| \le C|2^j s\xi|^{-1/\omega'}|2^k t\eta||Q_\alpha|^{-1/\omega'}. \tag{3.13}$$

If $|Q_\alpha|<e^{2q/(1-q)}$, taking $\omega=\log|Q_\alpha|/(2+\log|Q_\alpha|)$, then we obtain

$$|\hat{\sigma}^{s,t}_{\alpha;j,k}(\xi,\eta,\zeta)| \le C|2^j s\xi|^{2/\log|Q_\alpha|}|2^k t\eta|. \tag{3.14}$$

If $|Q_\alpha|\ge e^{2q/(1-q)}$, taking $\omega=q^{1/3}$ and letting $\varepsilon=1/\omega'$, then

$$|\hat{\sigma}^{s,t}_{\alpha;j,k}(\xi,\eta,\zeta)| \le C|2^j s\xi|^{-\varepsilon}|2^k t\eta|. \tag{3.15}$$

Switching the variables ξ and η in the proof of (3.13), we can get that for any $\omega \in (1, 2]$ such that $\omega < q$,

$$|\hat{\sigma}^{s,t}_{\alpha;j,k}(\xi,\eta,\zeta)| \le C|2^j s\xi||2^k t\eta|^{-1/\omega'}|Q_\alpha|^{-1/\omega'}\,. \tag{3.16}$$

By the same arguments as those used in proving (3.14) and (3.15), it follows from (3.16) that

$$|\hat{\sigma}^{s,t}_{\alpha;j,k}(\xi,\eta,\zeta)| \le C|2^j s\xi||2^k t\eta|^{2/\log|Q_\alpha|}, \quad \text{if } |Q_\alpha| < e^{2q/(1-q)}\,; \tag{3.17}$$

$$|\hat{\sigma}^{s,t}_{\alpha;j,k}(\xi,\eta,\zeta)| \le C|2^j s\xi||2^k t\eta|^{-\varepsilon}, \quad \text{if } |Q_\alpha| \ge e^{2q/(1-q)}\,. \tag{3.18}$$

It remains to prove that

$$|\hat{\sigma}^{s,t}_{\alpha;j,k}(\xi,\eta,\zeta)| \le C|2^j s\xi|^{2/\log|Q_\alpha|}|2^k t\eta|^{2/\log|Q_\alpha|}, \quad \text{if } |Q_\alpha| < e^{2q/(1-q)}\,; \tag{3.19}$$

$$|\hat{\sigma}^{s,t}_{\alpha;j,k}(\xi,\eta,\zeta)| \le C|2^j s\xi|^{-\varepsilon}|2^k t\eta|^{-\varepsilon}, \quad \text{if } |Q_\alpha| \ge e^{2q/(1-q)}\,. \tag{3.20}$$

By the method of rotation, we have

$$\begin{aligned}
&|\hat{\sigma}^{s,t}_{\alpha;j,k}(\xi,\eta,\zeta)| \\
&\quad \le C|2^j s\xi|^{-1}|2^k t\eta|^{-1}\int_{2^j s|\xi|}^{2^{j+1}s|\xi|}\int_{2^k t|\eta|}^{2^{k+1}t|\eta|}\left|\iint_{\boldsymbol{R}\times\boldsymbol{R}}\Phi(x_1',y_1')e^{-i\{x_1'u+y_1'v\}}dx_1'dy_1'\right|dudv \\
&\quad \le C|2^j s\xi|^{-1}|2^k t\eta|^{-1}\int_{2^j s|\xi|}^{2^{j+1}s|\xi|}\int_{2^k t|\eta|}^{2^{k+1}t|\eta|}|\hat{\Phi}(u,v)|dudv\,,
\end{aligned}$$

where

$$\Phi(x_1',y_1') = (1-{x_1'}^2)^{(m-3)/2}(1-{y_1'}^2)^{(n-3)/2}\chi_{\{|x_1'|<1,|y_1'|<1\}}(x_1',y_1')\Theta(x_1',y_1')$$

and

$$\Theta(x_1',y_1') = \iint_{S^{m-2}\times S^{n-2}}\tilde{b}_\alpha(x_1',(1-{x_1'}^2)^{1/2}\bar{x}',y_1',(1-{y_1'}^2)^{1/2}\bar{y}')d\bar{x}'d\bar{y}'\,.$$

Using Hölder's inequality and the Hausdorff-Young inequality again, we obtain

$$|\hat{\sigma}^{s,t}_{\alpha;j,k}(\xi,\eta,\zeta)| \le C|2^j s\xi|^{-1}|2^k t\eta|^{-1}\left\{\int_{2^j s|\xi|}^{2^{j+1}s|\xi|}\int_{2^k t|\eta|}^{2^{k+1}t|\eta|}dudv\right\}^{1/\omega}\|\Phi\|_{L^\omega(\boldsymbol{R}\times\boldsymbol{R})}\,.$$

It is easy to verify that $\|\Phi\|_{L^\omega(\boldsymbol{R}\times\boldsymbol{R})} \le |Q_\alpha|^{-1/\omega'}$. Thus

$$|\hat{\sigma}^{s,t}_{\alpha;j,k}(\xi,\eta,\zeta)| \le C|Q_\alpha|^{-1/\omega'}|2^j s\xi|^{-1/\omega'}|2^k t\eta|^{-1/\omega'}\,.$$

By the same arguments as those used in proving (3.14) and (3.15), we get (3.19) and (3.20). Summarizing (3.11), (3.14), (3.15) and (3.17)–(3.20), we obtain (ii) and (iii). Lemma 2 is thus proved for the case $m > 2$ and $n > 2$.

Next we consider the cases $m = 2$ or $n = 2$. The arguments are essentially the same as those in the previous case. Only two things must be modified: (a) the estimate of $\|\phi_{y'}\|_{L^w(\boldsymbol{R})}$; (b) the estimate of $\|\Phi\|_{L^w(\boldsymbol{R}\times\boldsymbol{R})}$.

We consider the following three cases separately.

(i) $m = 2$ and $n > 2$. For $\phi_{y'}(x_1')$, we have

$$\phi_{y'}(x_1') = (1 - x_1'^2)^{-1/2}\chi_{(-1,1)}(x_1')[\tilde{b}_\alpha(x_1', (1 - x_1'^2)^{1/2}, y') + \tilde{b}_\alpha(x_1', -(1 - x_1'^2)^{1/2}, y')]\,,$$

and

$$\begin{aligned}
&\|\phi_{y'}\|_{L^w(\boldsymbol{R})} \\
&= \left(\int_{-1}^{1} |(1 - x_1'^2)^{-1/2}[\tilde{b}_\alpha(x_1', (1 - x_1'^2)^{1/2}, y') + \tilde{b}_\alpha(x_1', -(1 - x_1'^2)^{1/2}, y')]|^w dx_1'\right)^{1/w} \\
&= \left(\int_{-1}^{-1+|Q_\alpha|^2} (1-x_1'^2)^{-w/2}|\tilde{b}_\alpha(x_1', (1-x_1'^2)^{1/2}, y') + \tilde{b}_\alpha(x_1', -(1 - x_1'^2)^{1/2}, y')|^w dx_1'\right)^{1/w} \\
&\quad + \left(\int_{-1+|Q_\alpha|^2}^{1-|Q_\alpha|^2} (1-x_1'^2)^{-w/2}|\tilde{b}_\alpha(x_1', (1-x_1'^2)^{1/2}, y') + \tilde{b}_\alpha(x_1', -(1-x_1'^2)^{1/2}, y')|^w dx_1'\right)^{1/w} \\
&\quad + \left(\int_{1-|Q_\alpha|^2}^{1} (1-x_1'^2)^{-w/2}|\tilde{b}_\alpha(x_1', (1-x_1'^2)^{1/2}, y') + \tilde{b}_\alpha(x_1', -(1-x_1'^2)^{1/2}, y')|^w dx_1'\right)^{1/w} \\
&:= I^{1/w} + II^{1/w} + III^{1/w}\,.
\end{aligned}$$

For II, we have

$$\begin{aligned}
II &= \int_{-1+|Q_\alpha|^2}^{1-|Q_\alpha|^2} (1 - x_1'^2)^{-(w-1)/2}(1 - x_1'^2)^{-1/2} \\
&\quad \times |\tilde{b}_\alpha(x_1', (1 - x_1'^2)^{1/2}, y') + \tilde{b}_\alpha(x_1', -(1 - x_1'^2)^{1/2}, y')|^w\, dx_1' \\
&\le C|Q_\alpha|^{-(w-1)} \int_{-1+|Q_\alpha|^2}^{1-|Q_\alpha|^2} (1 - x_1'^2)^{-1/2} \\
&\quad \times [|\tilde{b}_\alpha(x_1', (1 - x_1'^2)^{1/2}, y')|^w + |\tilde{b}_\alpha(x_1', -(1 - x_1'^2)^{1/2}, y')|^w]\, dx_1'\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{S^{n-1}} II^{1/w} dy' &\le C|Q_\alpha|^{-1/w'}\Bigg(\int_{S^{n-1}}\int_{-1+|Q_\alpha|^2}^{1-|Q_\alpha|^2} (1 - x_1'^2)^{-1/2}[|\tilde{b}_\alpha(x_1', (1 - x_1'^2)^{1/2}, y')|^w \\
&\qquad + |\tilde{b}_\alpha(x_1', -(1 - x_1'^2)^{1/2}, y')|^w] dx_1' dy'\Bigg)^{1/w} \\
&\le C|Q_\alpha|^{-1/w'}\|\tilde{b}_\alpha\|_{L^w(S^1\times S^{n-1})} \le C|Q_\alpha|^{-2/w'}.
\end{aligned}$$

For III, we choose r such that $rw = q$ and $q < 2r - 1$. By Hölder's inequality and the fact that $(w - 1)r' < 1$, we have

$$\begin{aligned}
III &= \int_{1-|Q_\alpha|^2}^{1} (1-x_1'^2)^{-1/2r'-(w-1)/2-1/2r} \\
&\quad \times |\tilde{b}_\alpha(x_1', (1-x_1'^2)^{1/2}, y') + \tilde{b}_\alpha(x_1', -(1-x_1'^2)^{1/2}, y')|^w dx_1' \\
&\le \left(\int_{1-|Q_\alpha|^2}^{1} (1-x_1'^2)^{-1/2-(w-1)r'/2} dx_1'\right)^{1/r'}
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\int_{1-|Q_\alpha|^2}^{1}(1-x_1'^2)^{-1/2}|\tilde{b}_\alpha(x_1', (1-x_1'^2)^{1/2}, y') + \tilde{b}_\alpha(x_1', -(1-x_1'^2)^{1/2}, y')|^q dx_1'\right)^{1/r}\\
&\le C\left(\int_{1-|Q_\alpha|^2}^{1}(1-x_1')^{-1/2-(w-1)r'/2}dx_1'\right)^{1/r'}\\
&\quad\times \left(\int_{1-|Q_\alpha|^2}^{1}(1-x_1'^2)^{-1/2}|\tilde{b}_\alpha(x_1', (1-x_1'^2)^{1/2}, y') + \tilde{b}_\alpha(x_1', -(1-x_1'^2)^{1/2}, y')|^q dx_1'\right)^{1/r}\\
&\le C|Q_\alpha|^{1/r'-(w-1)}\left(\int_{-1}^{1}(1-x_1'^2)^{-1/2}[|\tilde{b}_\alpha(x_1', (1-x_1'^2)^{1/2}, y')|^q\right.\\
&\qquad\qquad\left. + |\tilde{b}_\alpha(x_1', -(1-x_1'^2)^{1/2}, y')|^q]dx_1'\right)^{1/r}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\int_{S^{n-1}} III^{1/w}dy' &\le C\left(\int_{S^{n-1}} III dy'\right)^{1/w} \le C|Q_\alpha|^{1/wr'-1/w'}\|\tilde{b}_\alpha\|_{L^q(S^1\times S^{n-1})}\\
&\le C|Q_\alpha|^{-2/w'}.
\end{aligned}
$$

Similarly,

$$
\int_{S^{n-1}} I^{1/w}dy' \le C|Q_\alpha|^{-2/w'}.
$$

Thus,

$$
\int_{S^{n-1}} \|\phi_{y'}\|_{L^w(\boldsymbol{R})}dy' \le C|Q_\alpha|^{-2/w'}.
$$

For $\Phi(x_1', y_1')$, we have

$$
\Phi(x_1', y_1') = (1-x_1'^2)^{-1/2}(1-y_1'^2)^{(n-3)/2}\chi_{\{|x_1'|<1, |y_1'|<1\}}(x_1', y_1')\Theta(x_1', y_1'),
$$

where

$$
\begin{aligned}
\Theta(x_1', y_1') = \int_{S^{n-2}}&[\tilde{b}_\alpha(x_1', (1-x_1'^2)^{1/2}, y_1', (1-y_1'^2)^{1/2}\bar{y}')\\
&+ \tilde{b}_\alpha(x_1', -(1-x_1'^2)^{1/2}, y_1', (1-y_1'^2)^{1/2}\bar{y}')]d\bar{y}'.
\end{aligned}
$$

We write

$$
\begin{aligned}
\|\Phi\|_{L^w(\boldsymbol{R}\times\boldsymbol{R})} &= \left(\int_{-1}^{1}\int_{-1}^{1}(1-x_1'^2)^{-w/2}(1-y_1'^2)^{-w/2}|\Theta(x_1',y_1')|^w dx_1' dy_1'\right)^{1/w} \\
&\le \left(\int_{-1}^{1}(1-y_1'^2)^{-1/2}\left[\int_{-1}^{-1+|Q_\alpha|^2}(1-x_1'^2)^{-w/2}|\Theta(x_1',y_1')|^w dx_1'\right.\right. \\
&\quad + \int_{-1+|Q_\alpha|^2}^{1-|Q_\alpha|^2}(1-x_1'^2)^{-w/2}|\Theta(x_1',y_1')|^w dx_1' \\
&\quad \left.\left. + \int_{1-|Q_\alpha|^2}^{1}(1-x_1'^2)^{-w/2}|\Theta(x_1',y_1')|^w dx_1'\right] dy_1'\right)^{1/w} \\
&:= \left(\int_{-1}^{1}(1-y_1'^2)^{-1/2}[J_1+J_2+J_3]dy_1'\right)^{1/w}.
\end{aligned}
\tag{3.21}
$$

By the arguments similar to those used in proving *I*, *II* and *III*, we can obtain that

$$
J_1, J_3 \le C|Q_\alpha|^{1/r'-(w-1)}\left(\int_{-1}^{1}(1-x_1'^2)^{-1/2}|\Theta(x_1',y_1')|^q dx_1'\right)^{1/r}
\tag{3.22}
$$

with $rw = q$, and

$$
J_2 \le |Q_\alpha|^{-(w-1)}\int_{-1}^{1}(1-x_1'^2)^{-1/2}|\Theta(x_1',y_1')|^w dx_1'.
\tag{3.23}
$$

By (3.21), (3.22) and (3.23), it is not hard to verify that $\|\Phi\|_{L^w(\boldsymbol{R}\times\boldsymbol{R})} \le C|Q_\alpha|^{-2/w'}$.

(ii) $m > 2$ and $n = 2$. To estimate $\|\phi_{y'}\|_{L^w(\boldsymbol{R})}$, we argue in exactly the same way as in the previous case and obtain

$$
\int_{S^1}\|\phi_{y'}\|_{L^w(\boldsymbol{R})}dy' \le C|Q_\alpha|^{-1/w'}.
$$

Similarly, we apply the same method as in case (i) to estimate

$$
\|\Phi\|_{L^w(\boldsymbol{R}\times\boldsymbol{R})} \le C|Q_\alpha|^{-2/w'}.
$$

(iii) $m = n = 2$. Again, the same arguments as those in case (i) to derive an upper bound on $\|\phi_{y'}\|_{L^w(\boldsymbol{R})}$ also work for this case and we obtain

$$
\int_{S^1}\|\phi_{y'}\|_{L^w(\boldsymbol{R})}dy' \le C|Q_\alpha|^{-2/w'}.
$$

For $\Phi(x_1', y_1')$, we have

$$
\Phi(x_1',y_1') = (1-x_1'^2)^{-1/2}(1-y_1'^2)^{-1/2}\chi_{\{|x_1'|<1,|y_1'|<1\}}(x_1',y_1')\Theta(x_1',y_1'),
$$

where

$$
\begin{aligned}
&\Theta(x_1',y_1') \\
&= \tilde b_\alpha(x_1',(1-x_1'^2)^{1/2},y_1',(1-y_1'^2)^{1/2}) + \tilde b_\alpha(x_1',-(1-x_1'^2)^{1/2},y_1',(1-y_1'^2)^{1/2}) \\
&\quad + \tilde b_\alpha(x_1',(1-x_1'^2)^{1/2},y_1',-(1-y_1'^2)^{1/2}) + \tilde b_\alpha(x_1',-(1-x_1'^2)^{1/2},y_1',-(1-y_1'^2)^{1/2}).
\end{aligned}
$$

We can write

$$
\begin{aligned}
&\|\Phi\|_{L^w(\boldsymbol{R}\times\boldsymbol{R})}\\
&\le \left(\int_{-1}^{-1+|Q_\alpha|}\int_{-1}^{-1+|Q_\alpha|}(1-x_1'^2)^{-w/2}(1-y_1'^2)^{-w/2}|\Theta(x_1',y_1')|^w dx_1'dy_1'\right)^{1/w}\\
&\quad+\left(\int_{-1}^{-1+|Q_\alpha|}\int_{-1+|Q_\alpha|}^{1-|Q_\alpha|}(1-x_1'^2)^{-w/2}(1-y_1'^2)^{-w/2}|\Theta(x_1',y_1')|^w dx_1'dy_1'\right)^{1/w}\\
&\quad+\left(\int_{-1}^{-1+|Q_\alpha|}\int_{1-|Q_\alpha|}^{1}(1-x_1'^2)^{-w/2}(1-y_1'^2)^{-w/2}|\Theta(x_1',y_1')|^w dx_1'dy_1'\right)^{1/w}\\
&\quad+\left(\int_{-1+|Q_\alpha|}^{1-|Q_\alpha|}\int_{-1}^{-1+|Q_\alpha|}(1-x_1'^2)^{-w/2}(1-y_1'^2)^{-w/2}|\Theta(x_1',y_1')|^w dx_1'dy_1'\right)^{1/w}\\
&\quad+\left(\int_{-1+|Q_\alpha|}^{1-|Q_\alpha|}\int_{-1+|Q_\alpha|}^{1-|Q_\alpha|}(1-x_1'^2)^{-w/2}(1-y_1'^2)^{-w/2}|\Theta(x_1',y_1')|^w dx_1'dy_1'\right)^{1/w}\\
&\quad+\left(\int_{-1+|Q_\alpha|}^{1-|Q_\alpha|}\int_{1-|Q_\alpha|}^{1}(1-x_1'^2)^{-w/2}(1-y_1'^2)^{-w/2}|\Theta(x_1',y_1')|^w dx_1'dy_1'\right)^{1/w}\\
&\quad+\left(\int_{1-|Q_\alpha|}^{1}\int_{-1}^{-1+|Q_\alpha|}(1-x_1'^2)^{-w/2}(1-y_1'^2)^{-w/2}|\Theta(x_1',y_1')|^w dx_1'dy_1'\right)^{1/w}\\
&\quad+\left(\int_{1-|Q_\alpha|}^{1}\int_{-1+|Q_\alpha|}^{1-|Q_\alpha|}(1-x_1'^2)^{-w/2}(1-y_1'^2)^{-w/2}|\Theta(x_1',y_1')|^w dx_1'dy_1'\right)^{1/w}\\
&\quad+\left(\int_{1-|Q_\alpha|}^{1}\int_{1-|Q_\alpha|}^{1}(1-x_1'^2)^{-w/2}(1-y_1'^2)^{-w/2}|\Theta(x_1',y_1')|^w dx_1'dy_1'\right)^{1/w}\\
&:= I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8+I_9.
\end{aligned}
$$

By Hölder's inequality and arguments similar to those in case (i), it is easy to show that $I_j \le C|Q_\alpha|^{-2/w'}$ for $j=1,2,\dots,9$. Consequently, $\|\Phi\|_{L^w(S^{m-1}\times S^{n-1})} \le C|Q_\alpha|^{-2/w'}$. We omit the details and finish the proof of Lemma 2.

4. Proofs of Theorems. We only need to prove Theorem 1, because Theorem 2 follows directly. In fact, let $\gamma(u,v)\equiv 0$. Then γ satisfies (1.2) in Theorem 1. For any function $f\in\mathcal{S}(\boldsymbol{R}^m\times\boldsymbol{R}^n)$, we let h be a function on $\mathcal{S}(\boldsymbol{R})$ such that $\|h\|_p\ne 0$. By definition and Theorem 1, it is easy to see that

$$\|h\|_{L^p(\boldsymbol{R})}\|\mu_\Omega(f)\|_{L^p(\boldsymbol{R}^m\times\boldsymbol{R}^n)} = \|\mu_{\Omega,\gamma}(f\otimes h)\|_{L^p(\boldsymbol{R}^m\times\boldsymbol{R}^n\times\boldsymbol{R})} \le C\|f\|_{L^p(\boldsymbol{R}^m\times\boldsymbol{R}^n)}\|h\|_{L^p(\boldsymbol{R})},$$

where $(f\otimes h)(x,y,z)=f(x,y)h(z)$. This implies Theorem 2.

Next we prove Theorem 1. Let δ be the Dirac δ-function. Take two radial Schwartz functions $\varphi\in\mathcal{S}(\boldsymbol{R}^m)$ and $\psi\in\mathcal{S}(\boldsymbol{R}^n)$ such that:

(a) $0\le\varphi,\psi\le 1$;

(b) $\operatorname{supp}(\varphi)\subseteq\{x\in\boldsymbol{R}^m;\,1/2\le|x|\le 2\}$ and $\operatorname{supp}(\psi)\subseteq\{y\in\boldsymbol{R}^n;\,1/2\le|y|\le 2\}$;

(c) $\sum_{d\in\boldsymbol{Z}}(\varphi(2^d x))^2 \equiv 1$ for all $x \in \boldsymbol{R}^m \setminus \{0\}$ and $\sum_{l\in\boldsymbol{Z}}(\psi(2^l y))^2 \equiv 1$ for all $y \in \boldsymbol{R}^n \setminus \{0\}$. Let $\varphi_d(x) = \varphi(2^d x)$ and $\psi_l(y) = \psi(2^l y)$. Define the multiplier operators Φ_d and Ψ_l by

$$\widehat{\Phi_d f}(\xi) = \varphi_d(\xi)\hat{f}(\xi) \quad \text{and} \quad \widehat{\Psi_l g}(\eta) = \psi_l(\eta)\hat{g}(\eta),$$

and $\Phi_d \otimes \Psi_l \otimes \delta$ by

$$((\Psi_d \otimes \Psi_l \otimes \delta)f)^{\wedge}(\xi, \eta, \zeta) = \varphi_d(\xi)\psi_l(\eta)\hat{f}(\xi, \eta, \zeta).$$

Then in the sense of $L^2(\boldsymbol{R}^m \times \boldsymbol{R}^n \times \boldsymbol{R})$,

$$f(x, y, z) = \sum_{d\in\boldsymbol{Z}}\sum_{l\in\boldsymbol{Z}}(\Phi_d \otimes \Psi_l \otimes \delta)^2 f(x, y, z). \tag{4.1}$$

By Minkowski's inequality, it follows from (3.7) that

$$\begin{aligned}
\mu_{\Omega,\gamma}(f)(x, y, z) &= \left(\int_0^\infty\int_0^\infty\left|\sum_\alpha C_\alpha \sum_{j=-\infty}^{-1}\sum_{k=-\infty}^{-1} 2^{j+k}\sigma_{\alpha;j,k}^{s,t} * f(x, y, z)\right|^2 \frac{dsdt}{st}\right)^{1/2} \\
&\le \sum_\alpha |C_\alpha| \sum_{j=-\infty}^{-1}\sum_{k=-\infty}^{-1} 2^{j+k}\left(\int_0^\infty\int_0^\infty |\sigma_{\alpha;j,k}^{s,t} * f(x, y, z)|^2 \frac{dsdt}{st}\right)^{1/2} \\
&\le \sum_\alpha |C_\alpha| \sum_{j=-\infty}^{-1}\sum_{k=-\infty}^{-1} 2^{j+k}\left(\int_0^\infty\int_0^\infty |\sigma_{\alpha;0,0}^{s,t} * f(x, y, z)|^2 \frac{dsdt}{st}\right)^{1/2} \\
&= \sum_\alpha |C_\alpha|\left(\int_0^\infty\int_0^\infty |\sigma_{\alpha;0,0}^{s,t} * f(x, y, z)|^2 \frac{dsdt}{st}\right)^{1/2} \\
&= \sum_\alpha |C_\alpha|\left(\int_1^2\int_1^2 \sum_{j\in\boldsymbol{Z}}\sum_{k\in\boldsymbol{Z}} |\sigma_{\alpha;j,k}^{s,t} * f(x, y, z)|^2 \frac{dsdt}{st}\right)^{1/2}.
\end{aligned}$$

Set

$$\widetilde{\mu}_{\alpha,\gamma}(f)(x, y, z) = \left(\int_1^2\int_1^2 \sum_{j\in\boldsymbol{Z}}\sum_{k\in\boldsymbol{Z}} |\sigma_{\alpha;j,k}^{s,t} * f(x, y, z)|^2 dsdt\right)^{1/2}.$$

Then

$$\|\mu_{\Omega,\gamma}(f)\|_p \le C\sum_\alpha |C_\alpha|\|\widetilde{\mu}_{\alpha,\gamma}(f)\|_p, \quad 1 < p < \infty. \tag{4.2}$$

By (4.1), we can write

$$\begin{aligned}
\widetilde{\mu}_{\alpha,\gamma}(f)(x, y, z) = \Bigg(\int_1^2\int_1^2 \sum_{j,k\in\boldsymbol{Z}}\Bigg|\sum_{d,l\in\boldsymbol{Z}} &(\Phi_{j+d} \otimes \Psi_{k+l} \otimes \delta) \\
&\times (\sigma_{\alpha;j,k}^{s,t} * (\Phi_{j+d} \otimes \Psi_{k+l} \otimes \delta)f)(x, y, z)\Bigg|^2 dsdt\Bigg)^{1/2}.
\end{aligned} \tag{4.3}$$

In the following, we give the proof of Theorem 1. Note that $B_{q_1}^{0,\nu}(S^{m-1} \times S^{n-1}) \subset B_{q_2}^{0,\nu}(S^{m-1} \times S^{n-1})$ for $1 < q_2 < q_1$, and $\nu > -1$. It suffices to consider the case $1 < q \le 2$.

By Lemma 2(ii) and (iii), without loss of generality, we may assume that for each b_α, the support Q_α of b_α is uniformly small such that $|Q_\alpha| < e^{2q/(1-q)}$.

PROOF OF THEOREM 1(i). First, we consider the mapping $\mathcal{G}$ defined by

$$\mathcal{G} : \{g^{s,t}_{j,k;d,l}(x, y, z)\}_{j,k;d,l\in\boldsymbol{Z}} \longrightarrow \left\{ \sum_{d,l\in\boldsymbol{Z}} ((\Phi_{j+d} \otimes \Psi_{k+l} \otimes \delta)(g^{s,t}_{j,k;d,l}))(x, y, z) \right\}_{j,k\in\boldsymbol{Z}} .$$

By Plancherel's theorem, we have

$$\begin{aligned}
&\left\| \left(\sum_{j,k\in\boldsymbol{Z}} \int_1^2 \int_1^2 \left| \sum_{d,l\in\boldsymbol{Z}} (\Phi_{j+d} \otimes \Psi_{k+l} \otimes \delta) g^{s,t}_{j,k;d,l} \right|^2 ds dt \right)^{1/2} \right\|_2^2 \\
&\quad = \sum_{j,k\in\boldsymbol{Z}} \int_1^2 \int_1^2 \int_{\boldsymbol{R}^m\times\boldsymbol{R}^n\times\boldsymbol{R}} \sum_{d,l\in\boldsymbol{Z}} (\Phi_{j+d} \otimes \Psi_{k+l} \otimes \delta) g^{s,t}_{j,k;d,l}(x, y, z) \\
&\qquad \times \sum_{d',l'\in\boldsymbol{Z}} \overline{(\Phi_{j+d'} \otimes \Psi_{k+l'} \otimes \delta) g^{s,t}_{j,k;d',l'}(x, y, z)} dx dy dz ds dt \\
&\quad = \sum_{j,k\in\boldsymbol{Z}} \sum_{d,l\in\boldsymbol{Z}} \sum_{d',l'\in\boldsymbol{Z}} \int_1^2 \int_1^2 \int_{\boldsymbol{R}^m\times\boldsymbol{R}^n\times\boldsymbol{R}} \varphi_{j+d}(\xi)\psi_{k+l}(\eta)\widehat{g^{s,t}_{j,k;d,l}}(\xi, \eta, \zeta) \\
&\qquad \times \overline{\varphi_{j+d'}(\xi)\psi_{k+l'}(\eta)\widehat{g^{s,t}_{j,k;d',l'}}(\xi, \eta, \zeta)} d\xi d\eta d\zeta ds dt \\
&\quad = \sum_{j,k\in\boldsymbol{Z}} \sum_{d,l\in\boldsymbol{Z}} \sum_{|d-d'|\le 2, |l-l'|\le 2} \int_1^2 \int_1^2 \int_{\boldsymbol{R}^m\times\boldsymbol{R}^n\times\boldsymbol{R}} \varphi_{j+d}(\xi)\psi_{k+l}(\eta)\widehat{g^{s,t}_{j,k;d,l}}(\xi, \eta, \zeta) \\
&\qquad \times \overline{\varphi_{j+d'}(\xi)\psi_{k+l'}(\eta)\widehat{g^{s,t}_{j,k;d',l'}}(\xi, \eta, \zeta)} d\xi d\eta d\zeta ds dt .
\end{aligned}$$

A trivial computation shows that

$$\begin{aligned}
&\Bigg| \sum_{d,l\in\boldsymbol{Z}} \sum_{|d-d'|\le 2, |l-l'|\le 2} \int_1^2 \int_1^2 \int_{\boldsymbol{R}^m\times\boldsymbol{R}^n\times\boldsymbol{R}} \varphi_{j+d}(\xi)\psi_{k+l}(\eta)\widehat{g^{s,t}_{j,k;d,l}}(\xi, \eta, \zeta) \\
&\qquad \times \overline{\varphi_{j+d'}(\xi)\psi_{k+l'}(\eta)\widehat{g^{s,t}_{j,k;d',l'}}(\xi, \eta, \zeta)} d\xi d\eta d\zeta ds dt \Bigg| \\
&\quad \le \sum_{d,l\in\boldsymbol{Z}} \sum_{|d-d'|\le 2, |l-l'|\le 2} \int_1^2 \int_1^2 \int_{\boldsymbol{R}^m\times\boldsymbol{R}^n\times\boldsymbol{R}} |\widehat{g^{s,t}_{j,k;d,l}}(\xi, \eta, \zeta)| \\
&\qquad \times |\widehat{g^{s,t}_{j,k;d',l'}}(\xi, \eta, \zeta)| d\xi d\eta d\zeta ds dt \\
&\quad \le C \sum_{d,l\in\boldsymbol{Z}} \int_1^2 \int_1^2 \int_{\boldsymbol{R}^m\times\boldsymbol{R}^n\times\boldsymbol{R}} |\widehat{g^{s,t}_{j,k;d,l}}(\xi, \eta, \zeta)|^2 d\xi d\eta d\zeta ds dt .
\end{aligned}$$

Therefore,

$$\left\|\left(\sum_{j,k\in\mathbf{Z}}\int_1^2\int_1^2\left|\sum_{d,l\in\mathbf{Z}}(\Phi_{j+d}\otimes\Psi_{k+l}\otimes\delta)g^{s,t}_{j,k;d,l}\right|^2 dsdt\right)^{1/2}\right\|_2^2 \tag{4.4}$$
$$\leq C\sum_{d,l\in\mathbf{Z}}\left\|\left(\sum_{j,k\in\mathbf{Z}}\int_1^2\int_1^2|g^{s,t}_{j,k;d,l}|^2 dsdt\right)^{1/2}\right\|_2^2,$$

which implies that $\mathcal{G}$ is a bounded operator from $l^2(L^2(\mathbf{R}^m\times\mathbf{R}^n\times\mathbf{R})(L^2([1,2]\times[1,2])(l^2)))$ to $L^2(\mathbf{R}^m\times\mathbf{R}^n\times\mathbf{R})(L^2([1,2]\times[1,2])(l^2))$. It follows from (4.3) and (4.4) that

$$\|\widetilde{\mu}_{\alpha,\gamma}(f)\|_2^2\leq C\sum_{d,l\in\mathbf{Z}}\left\|\left(\sum_{j,k\in\mathbf{Z}}\int_1^2\int_1^2|\sigma^{s,t}_{\alpha;j,k}*((\Phi_{j+d}\otimes\Psi_{k+l}\otimes\delta)f)|^2 dsdt\right)^{1/2}\right\|_2^2.$$

For each fixed $d,l\in\mathbf{Z}$, let

$$I_{d,l}(f)(x,y,z)=\left(\sum_{j,k\in\mathbf{Z}}\int_1^2\int_1^2|\sigma^{s,t}_{\alpha;j,k}*((\Phi_{j+d}\otimes\Psi_{k+l}\otimes\delta)f)(x,y,z)|^2 dsdt\right)^{1/2}.$$

Then

$$\|\widetilde{\mu}_{\alpha,\gamma}(f)\|_2^2\leq C\sum_{d,l\in\mathbf{Z}}\|I_{d,l}(f)\|_2^2. \tag{4.5}$$

Applying Plancherel's theorem, we know that

$$\begin{aligned}
&\|I_{d,l}(f)\|_2^2\\
&\quad=\int_1^2\int_1^2\sum_{j,k\in\mathbf{Z}}\int_{\mathbf{R}^m\times\mathbf{R}^n\times\mathbf{R}}|\varphi_{j+d}(\xi)\psi_{k+l}(\eta)|^2|\hat{\sigma}^{s,t}_{\alpha;j,k}(\xi,\eta,\zeta)|^2|\hat{f}(\xi,\eta,\zeta)|^2 d\xi d\eta d\zeta dsdt\\
&\quad\leq C\int_1^2\int_1^2\sum_{j,k\in\mathbf{Z}}\int_{E_{j,k;d,l}}|\hat{\sigma}^{s,t}_{\alpha;j,k}(\xi,\eta,\zeta)|^2|\hat{f}(\xi,\eta,\zeta)|^2 d\xi d\eta d\zeta dsdt,
\end{aligned}$$

where $E_{j,k;d,l}=\{(\xi,\eta)\in\mathbf{R}^m\times\mathbf{R}^n;2^{-j-d-1}\leq|\xi|\leq 2^{-j-d+1},2^{-k-l-1}\leq|\eta|\leq 2^{-k-l+1}\}\times\mathbf{R}$. Also, using Lemma 2, it is easy to see that, for $(\xi,\eta,\zeta)\in E_{j,k;d,l}$ and $s,t\in[1,2]$,

$$|\hat{\sigma}^{s,t}_{\alpha;j,k}(\xi,\eta,\zeta)|\leq C|2^js\xi||2^kt\eta|\leq C2^{-d-l},\quad d\geq 0,\ l\geq 0; \tag{4.6}$$

$$|\hat{\sigma}^{s,t}_{\alpha;j,k}(\xi,\eta,\zeta)|\leq C|2^js\xi||2^kt\eta|^{2/\log|Q_\alpha|}\leq C2^{-d-2l/\log|Q_\alpha|},\quad d\geq 0,\ l<0; \tag{4.7}$$

$$|\hat{\sigma}^{s,t}_{\alpha;j,k}(\xi,\eta,\zeta)|\leq C|2^js\xi|^{2/\log|Q_\alpha|}|2^kt\eta|\leq C2^{-2d/\log|Q_\alpha|-l},\quad d<0,\ l\geq 0; \tag{4.8}$$

$$|\hat{\sigma}^{s,t}_{\alpha;j,k}(\xi,\eta,\zeta)|\leq C(|2^js\xi||2^kt\eta|)^{2/\log|Q_\alpha|}\leq C2^{-2(d+l)/\log|Q_\alpha|},\quad d<0,\ l<0. \tag{4.9}$$

Hence,

$$\|I_{d,l}(f)\|_2 \le C2^{-d-l}\|f\|_2, \quad d \ge 0\,,\ l \ge 0\,; \tag{4.10}$$

$$\|I_{d,l}(f)\|_2 \le C2^{-d-2l/\log|Q_\alpha|}\|f\|_2, \quad d \ge 0\,,\ l < 0\,; \tag{4.11}$$

$$\|I_{d,l}(f)\|_2 \le C2^{-2d/\log|Q_\alpha|-l}\|f\|_2, \quad d < 0\,,\ l \ge 0\,; \tag{4.12}$$

$$\|I_{d,l}(f)\|_2 \le C2^{-2d/\log|Q_\alpha|-2l/\log|Q_\alpha|}\|f\|_2, \quad d < 0\,,\ l < 0\,. \tag{4.13}$$

Combing (4.5) with (4.10)–(4.13), we obtain

$$\begin{aligned}\|\widetilde{\mu}_{\alpha,\gamma}(f)\|_2^2 \le C\|f\|_2^2\bigg\{&\sum_{d\ge 0,l\ge 0} 2^{-d-l} + \sum_{d\ge 0,l<0} 2^{-d-2l/\log|Q_\alpha|}\\ &+ \sum_{d<0,l\ge 0} 2^{-2d/\log|Q_\alpha|-l} + \sum_{d<0,l<0} 2^{-2(d+l)/\log|Q_\alpha|}\bigg\}^2\\ &\le C\left(1+\log\frac{1}{|Q_\alpha|}\right)^2\|f\|_2^2\,.\end{aligned}$$

Consequently,

$$\|\widetilde{\mu}_{\alpha,\gamma}(f)\|_2 \le C\left(1+\log\frac{1}{|Q_\alpha|}\right)\|f\|_2\,,$$

which together with (4.2) implies

$$\|\mu_{\Omega,\gamma}(f)\|_2 \le C\sum_\alpha |C_\alpha|\left(1+\log\frac{1}{|Q_\alpha|}\right)\|f\|_2 \le C\|f\|_2\,.$$

This proves (i) of Theorem 1. □

PROOF OF THEOREM 1(ii). First, we prove that, for $1 < p < 2$ and $1 < r < p$,

$$\begin{aligned}&\|\widetilde{\mu}_{\alpha,\gamma}(f)\|_p^r\\ &\le C\sum_{d,l\in\mathbf{Z}}\left\|\left(\sum_{j,k\in\mathbf{Z}}\int_1^2\int_1^2 |\sigma_{\alpha;j,k}^{s,t} * ((\Phi_{j+d}\otimes\Psi_{k+l}\otimes\delta)f)|^2 ds dt\right)^{1/2}\right\|_p^r.\end{aligned} \tag{4.14}$$

Indeed, by Minkowski's inequality, we have that for $1 < p_0 < \infty$,

$$\begin{aligned}&\left\|\left(\sum_{j,k\in\mathbf{Z}}\int_1^2\int_1^2\left|\sum_{d,l\in\mathbf{Z}}(\Phi_{j+d}\otimes\Psi_{k+l}\otimes\delta)g_{j,k;d,l}^{s,t}\right|^2 ds dt\right)^{1/2}\right\|_{p_0}\\ &\le \sum_{d,l\in\mathbf{Z}}\left\|\left(\sum_{j,k\in\mathbf{Z}}\int_1^2\int_1^2 |(\Phi_{j+d}\otimes\Psi_{k+l}\otimes\delta)g_{j,k;d,l}^{s,t}|^2 ds dt\right)^{1/2}\right\|_{p_0}.\end{aligned} \tag{4.15}$$

Note that for each fixed $d, l \in \mathbf{Z}$, and any functions $\{h_{j,k}^{s,t}\}$,

$$\left\|\sum_{j,k\in\mathbf{Z}}\int_1^2\int_1^2 |(\Phi_{j+d}\otimes\Psi_{k+l}\otimes\delta)h_{j,k}^{s,t}| ds dt\right\|_1 \le \sum_{j,k\in\mathbf{Z}}\left\|\int_1^2\int_1^2 |h_{j,k}^{s,t}| ds dt\right\|_1,$$

and

$$\Big\| \sup_{j,k\in\mathbf{Z}} \sup_{s,t\in[1,2]} |(\Phi_{j+d}\otimes\Psi_{k+l}\otimes\delta)h^{s,t}_{j,k}| \Big\|_{p_0} \le C \Big\| \sup_{j,k\in\mathbf{Z}} \sup_{s,t\in[1,2]} |h^{s,t}_{j,k}| \Big\|_{p_0}, \quad 1 < p_0 < \infty .$$

Then the mapping $\mathcal{H}$ defined by

$$\mathcal{H} : \{h^{s,t}_{j,k}(x,y,z)\}_{j,k\in\mathbf{Z};s,t\in[1,2]} \longrightarrow \{(\Phi_{j+d}\otimes\Psi_{k+l}\otimes\delta)h^{s,t}_{j,k}(x,y,z)\}_{j,k\in\mathbf{Z};s,t\in[1,2]}$$

is bounded from $L^{p_0}(\boldsymbol{R}^m\times\boldsymbol{R}^n\times\boldsymbol{R})(L^\infty([1,2]\times[1,2])(l^\infty)))$ to itself for any $1 < p_0 < \infty$, and bounded from $L^1(\boldsymbol{R}^m\times\boldsymbol{R}^n\times\boldsymbol{R})(L^1([1,2]\times[1,2])(l^1)$ to itself. Therefore, for given $p\in(1,2)$, we choose p_0 such that $1 < p_0 < \infty$ and $2/p = 1 + 1/p_0$, and apply the standard interpolation argument to conclude that $\mathcal{H}$ is bounded from $L^p(\boldsymbol{R}^m\times\boldsymbol{R}^n\times\boldsymbol{R})(L^2([1,2]\times[1,2])(l^2)$ to itself. This together with (4.15) states that

$$\begin{aligned}
&\Big\| \Big(\sum_{j,k\in\mathbf{Z}} \int_1^2\int_1^2 \Big| \sum_{d,l\in\mathbf{Z}} (\Phi_{j+d}\otimes\Psi_{k+l}\otimes\delta) g^{s,t}_{j,k;d,l} \Big|^2 dsdt \Big)^{1/2} \Big\|_p \\
&\quad \le \sum_{d,l\in\mathbf{Z}} \Big\| \Big(\sum_{j,k\in\mathbf{Z}} \int_1^2\int_1^2 |g^{s,t}_{j,k;d,l}|^2 dsdt \Big)^{1/2} \Big\|_p, \quad 1 < p < 2 .
\end{aligned} \tag{4.16}$$

Interpolating between (4.4) and (4.16), we get that for each fixed $1 < p < 2$ and any $1 < r < p$,

$$\begin{aligned}
&\Big\| \Big(\sum_{j,k\in\mathbf{Z}} \int_1^2\int_1^2 \Big| \sum_{d,l\in\mathbf{Z}} (\Phi_{j+d}\otimes\Psi_{k+l}\otimes\delta) g^{s,t}_{j,k;d,l} \Big|^2 dsdt \Big)^{1/2} \Big\|_p^r \\
&\quad \le \sum_{d,l\in\mathbf{Z}} \Big\| \Big(\sum_{j,k\in\mathbf{Z}} \int_1^2\int_1^2 |g^{s,t}_{j,k;d,l}|^2 dsdt \Big)^{1/2} \Big\|_p^r .
\end{aligned}$$

This implies (4.14).

Second, we claim that, for $2 < p < \infty$ and any $1 < r < p' = p/(p-1)$,

$$\begin{aligned}
&\|\widetilde{\mu}_{\alpha,\gamma}(f)\|_p^r \\
&\quad \le C \sum_{d,l\in\mathbf{Z}} \Big(\int_1^2\int_1^2 \Big\| \Big(\sum_{j,k\in\mathbf{Z}} |\sigma^{s,t}_{\alpha;j,k} * ((\Phi_{j+d}\otimes\Psi_{k+l}\otimes\delta)f)|^2 \Big)^{1/2} \Big\|_p^2 dsdt \Big)^{r/2} .
\end{aligned} \tag{4.17}$$

Indeed, by Minkowski's inequality and the Littlewood-Paley theory (see [25, Chapter 4]), we have

$$\begin{aligned}
&\Big\| \Big(\sum_{j,k\in\mathbf{Z}} \int_1^2\int_1^2 |(\Phi_{j+d}\otimes\Psi_{k+l}\otimes\delta) g^{s,t}_{j,k;d,l}|^2 dsdt \Big)^{1/2} \Big\|_p^2 \\
&\quad \le \int_1^2\int_1^2 \Big\| \Big(\sum_{j,k\in\mathbf{Z}} |(\Phi_{j+d}\otimes\Psi_{k+l}\otimes\delta) g^{s,t}_{j,k;d,l}|^2 \Big)^{1/2} \Big\|_p^2 dsdt \\
&\quad \le C \int_1^2\int_1^2 \Big\| \Big(\sum_{j,k\in\mathbf{Z}} |g^{s,t}_{j,k;d,l}|^2 \Big)^{1/2} \Big\|_p^2 dsdt , \qquad 2 < p < \infty .
\end{aligned}$$

From this and (4.15), we know that

$$\begin{aligned}(4.18)\qquad &\Big\|\Big(\sum_{j,k\in\mathbf{Z}}\int_1^2\int_1^2\Big|\sum_{d,l\in\mathbf{Z}}(\Phi_{j+d}\otimes\Psi_{k+l}\otimes\delta)g^{s,t}_{j,k;d,l}\Big|^2 dsdt\Big)^{1/2}\Big\|_p \\ &\le C\sum_{d,l\in\mathbf{Z}}\Big(\int_1^2\int_1^2\Big\|\Big(\sum_{j,k\in\mathbf{Z}}|g^{s,t}_{j,k;d,l}|^2\Big)^{1/2}\Big\|_p^2 dsdt\Big)^{1/2}, \quad 2<p<\infty\,.\end{aligned}$$

On the other hand, it follows from (4.4) that

$$\begin{aligned}(4.19)\qquad &\Big\|\Big(\sum_{j,k\in\mathbf{Z}}\int_1^2\int_1^2\Big|\sum_{d,l\in\mathbf{Z}}(\Phi_{j+d}\otimes\Psi_{k+l}\otimes\delta)g^{s,t}_{j,k;d,l}\Big|^2 dsdt\Big)^{1/2}\Big\|_2 \\ &\le C\Big(\sum_{d,l\in\mathbf{Z}}\int_1^2\int_1^2\Big\|\Big(\sum_{j,k\in\mathbf{Z}}|g^{s,t}_{j,k;d,l}|^2\Big)^{1/2}\Big\|_2^2 dsdt\Big)^{1/2}.\end{aligned}$$

By an interpolation argument, the inequalities (4.18) and (4.19) show that, for each $2<p<\infty$ and any $1<r<p'=p/(p-1)$,

$$\begin{aligned}&\Big\|\Big(\sum_{j,k\in\mathbf{Z}}\int_1^2\int_1^2\Big|\sum_{d,l\in\mathbf{Z}}(\Phi_{j+d}\otimes\Psi_{k+l}\otimes\delta)g^{s,t}_{j,k;d,l}\Big|^2 dsdt\Big)^{1/2}\Big\|_p^r \\ &\le C\sum_{d,l\in\mathbf{Z}}\Big(\int_1^2\int_1^2\Big\|\Big(\sum_{j,k\in\mathbf{Z}}|g^{s,t}_{j,k;d,l}|^2\Big)^{1/2}\Big\|_p^2 dsdt\Big)^{r/2},\end{aligned}$$

which implies (4.17).

Now we establish the $L^p(\boldsymbol{R}^m\times\boldsymbol{R}^n\times\boldsymbol{R})$-boundedness of $\|\mu_{\Omega,\gamma}(f)\|_p$ in the following two cases.

CASE 1. $2/(1+\upsilon)<p<2$. By (4.14) and the definition of $I_{d,l}$, we have

$$(4.20)\qquad \|\widetilde{\mu}_{\alpha,\gamma}(f)\|_p^r\le C\sum_{d,l\in\mathbf{Z}}\|I_{d,l}(f)\|_p^r, \quad 1<p<2 \text{ and } 1<r<p\,.$$

Then we first estimate $\|I_{d,l}(f)\|_p$. By the definition of $\sigma^{s,t}_{\alpha;j,k}$ and Lemma 1, it is easy to see that, for any functions $\{h_{j,k}\}_{j,k\in\mathbf{Z}}$,

$$\Big\|\int_1^2\int_1^2\sum_{j,k\in\mathbf{Z}}|\sigma^{s,t}_{\alpha;j,k}*h_{j,k}|dsdt\Big\|_1\le C\Big\|\sum_{j,k\in\mathbf{Z}}|h_{j,k}|\Big\|_1,$$

and

$$\Big\|\sup_{j,k\in\mathbf{Z}}\sup_{s,t\in[1,2]}|\sigma^{s,t}_{\alpha;j,k}*h_{j,k}|\Big\|_{p_0}\le C\Big\|\sup_{j,k\in\mathbf{Z}}|h_{j,k}|\Big\|_{p_0}, \quad 1<p_0<\infty\,.$$

Thus,

$$\Big\|\Big(\int_1^2\int_1^2\sum_{j,k\in\mathbf{Z}}|\sigma^{s,t}_{\alpha;j,k}*h_{j,k}|^2 dsdt\Big)^{1/2}\Big\|_p\le C\Big\|\Big(\sum_{j,k\in\mathbf{Z}}|h_{j,k}|^2\Big)^{1/2}\Big\|_p, \quad 1<p<2\,.$$

In particular, let $h_{j,k}(x,y,z)=(\Phi_{j+d}\otimes\Psi_{k+l}\otimes\delta)f(x,y,z)$, and invoke the Littlewood-Paley theory. We get

$$\|I_{d,l}(f)\|_p\le C\left\|\left(\sum_{j,k\in\boldsymbol{Z}}|(\Phi_{j+d}\otimes\Psi_{k+l}\otimes\delta)f|^2\right)^{1/2}\right\|_p\le C\|f\|_p\,,\quad 1<p<2\,. \tag{4.21}$$

By interpolation, the inequalities (4.10)–(4.13) and (4.21) show that for any $1<p<2$ and $0<\theta<1$,

$$\begin{aligned}\|I_{d,l}(f)\|_p&\le C2^{-\theta d-\theta l}\|f\|_p\,,\quad d\ge 0\,,\ l\ge 0\,;\\ \|I_{d,l}(f)\|_p&\le C2^{-\theta d-2\theta l/\log|Q_\alpha|}\|f\|_p\,,\quad d\ge 0\,,\ l<0\,;\\ \|I_{d,l}(f)\|_p&\le C2^{-2\theta d/\log|Q_\alpha|-\theta l}\|f\|_p\,,\quad d<0\,,\ l\ge 0\,;\\ \|I_{d,l}(f)\|_p&\le C2^{-2\theta d/\log|Q_\alpha|-2\theta l/\log|Q_\alpha|}\|f\|_p\,,\quad d<0\,,\ l<0\,.\end{aligned}$$

These inequalities together with (4.20) imply that for $2/(1+\upsilon)<p<2$,

$$\|\widetilde{\mu}_{\alpha,\gamma}(f)\|_p\le C\left(1+\log\frac{1}{|Q_\alpha|}\right)^{1+\upsilon}\|f\|_p\le C\left\{1+\left(\log\frac{1}{|Q_\alpha|}\right)^{1+\upsilon}\right\}\|f\|_p\,.$$

Therefore,

$$\|\mu_{\Omega,\gamma}(f)\|_p\le C\sum_\alpha|C_\alpha|\left\{1+\left(\log\frac{1}{|Q_\alpha|}\right)^{1+\upsilon}\right\}\|f\|_p\le C\|f\|_p\,,\quad \frac{2}{1+\upsilon}<p<2\,.$$

CASE 2. $2<p<2/(1-\upsilon)$. For fixed $s,t\in[1,2]$ and $d,l\in\boldsymbol{Z}$, let

$$J_{d,l}^{s,t}(f)(x,y,z)=\left(\sum_{j,k\in\boldsymbol{Z}}|\sigma_{\alpha;j,k}^{s,t}*((\Phi_{j+d}\otimes\Psi_{k+l}\otimes\delta)f)(x,y,z)|^2\right)^{1/2}.$$

Then, by (4.17), we have that for any $1<r<p'=p/(p-1)$,

$$\|\widetilde{\mu}_{\alpha,\gamma}(f)\|_p^r\le C\sum_{d,l\in\boldsymbol{Z}}\left(\int_1^2\int_1^2\|J_{d,l}^{s,t}(f)\|_p^2\,ds\,dt\right)^{r/2}. \tag{4.22}$$

Using Plancherel's theorem, we get

$$\|J_{d,l}^{s,t}(f)\|_2^2=\sum_{j,k\in\boldsymbol{Z}}\iint_{\boldsymbol{R}^m\times\boldsymbol{R}^n\times\boldsymbol{R}}|\hat{\sigma}_{\alpha;j,k}^{s,t}(\xi,\eta,\zeta)|^2|\varphi_{j+d}(\xi)\psi_{k+l}(\eta)|^2|\hat{f}(\xi,\eta,\zeta)|^2d\xi d\eta d\zeta\,.$$

Similarly to proving the inequalities (4.10)–(4.13), it follows from Lemma 2 that for $s,t\in[1,2]$,

$$\|J_{d,l}^{s,t}(f)\|_2\le C2^{-d-l}\|f\|_2\,,\quad d\ge 0\,,\ l\ge 0\,; \tag{4.23}$$

$$\|J_{d,l}^{s,t}(f)\|_2\le C2^{-d-2l/\log|Q_\alpha|}\|f\|_2\,,\quad d\ge 0\,,\ l<0\,; \tag{4.24}$$

$$\|J_{d,l}^{s,t}(f)\|_2\le C2^{-2d/\log|Q_\alpha|-l}\|f\|_2\,,\quad d<0\,,\ l\ge 0\,; \tag{4.25}$$

$$\|J_{d,l}^{s,t}(f)\|_2\le C2^{-2d/\log|Q_\alpha|-2l/\log|Q_\alpha|}\|f\|_2\,,\quad d<0\,,\ l<0\,. \tag{4.26}$$

On the other hand, by Lemma 1, it is easy to verify that for any $p_0 \in (2, \infty)$, $s, t \in [1, 2]$,

$$\Big\| \Big(\sum_{j,k \in \mathbf{Z}} |\sigma^{s,t}_{\alpha;j,k} * h_{j,k}|^2 \Big)^{1/2} \Big\|_{p_0} \le C \Big\| \Big(\sum_{j,k \in \mathbf{Z}} |h_{j,k}|^2 \Big)^{1/2} \Big\|_{p_0},$$

where C is independent of b_α and $s, t \in [1, 2]$. This together with the Littlewood-Paley theory implies that for $s, t \in [1, 2]$, $2 < p_0 < \infty$,

$$\|J^{s,t}_{d,l}(f)\|_{p_0} \le C \Big\| \Big(\sum_{j,k \in \mathbf{Z}} |(\Phi_{j+d} \otimes \Psi_{k+l} \otimes \delta) f|^2 \Big)^{1/2} \Big\|_{p_0} \le C \|f\|_{p_0}. \tag{4.27}$$

Invoking the interpolation theorem again, the inequalities (4.23)–(4.27) tell us that, for any $2 < p < \infty$ and $0 < \theta < 1$,

$$\begin{aligned}
\|J^{s,t}_{d,l}(f)\|_p &\le C 2^{-\theta d - \theta l} \|f\|_p, \quad d \ge 0,\ l \ge 0; \\
\|J^{s,t}_{d,l}(f)\|_p &\le C 2^{-\theta d - 2\theta l/\log|Q_\alpha|} \|f\|_p, \quad d \ge 0,\ l < 0; \\
\|J^{s,t}_{d,l}(f)\|_p &\le C 2^{-2\theta d/\log|Q_\alpha| - \theta l} \|f\|_p, \quad d < 0,\ l \ge 0; \\
\|J^{s,t}_{d,l}(f)\|_p &\le C 2^{-2\theta d/\log|Q_\alpha| - 2\theta l/\log|Q_\alpha|} \|f\|_p, \quad d < 0,\ l < 0.
\end{aligned}$$

Using these inequalities with (4.22), the same argument as in Case 1 leads to our desired estimate. This completes the proof of Theorem 1. □

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