REMARKS ON THE METRIZATION PROBLEM*)

By
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§1. Let E be a topological space. We call E a general metric space when to every two points a and b in E there corresponds a non-negative real number ab, called distance function. A general metric space is called a metric space when it satisfies the following conditions:

I. $ab = 0$ *if and only if* $a = b$.

II. (Symmetry) $ab = ba$.

III. (Triangle property) $ac \leqq ab + bc$.

Further, when a general metric space satisfies I and II or I and III, it is said to be a *semi-metric* or a *quasi-metric* space, respectively. It is well known that II is implied by the following condition III′ (Lindenbaum):

III′. $ac \leqq ab + cb$.

We shall here consider the following conditions IV – VI besides I, II and III:

IV. $ab < \varepsilon$ *and* $cb < \varepsilon$ *imply* $ac < 2\varepsilon$.

V. (Uniformly regular). *For every* $\varepsilon > 0$ *there exists* $\varphi(\varepsilon) > 0$ *such that* $ab < \varphi(\varepsilon)$ *and* $cb < \varphi(\varepsilon)$ *imply* $ac < \varepsilon$.

VI. *For every* $\varepsilon > 0$ *and every* $a \in E$ *there exists* $\varphi(a, \varepsilon) > 0$ *such that* $ab < \varphi(a, \varepsilon)$, $cb < \varphi(a, \varepsilon)$ *and* $cd < \varphi(a, \varepsilon)$ *imply* $ad < \varepsilon$.

E. Chittenden proved the following theorem:

Theorem A. *A topological space E is homeomorphic to a metric space provided that a distance function ab is defined in E and satisfies the conditions I, II and V.*

In the proof he used the new distance function defined by

$$d(a, b) = \underset{x_1, \dots x_n \in E}{\text{g.l.b.}} (ax_1 + \cdots + x_{n-1}x_n + x_n b).$$

Moreover, E. Chittenden and A. H. Frink proved the following:

Theorem B. *A semi-metric space is metrizable provided that the distance function satisfies V.*

Also, A. H. Frink proved the following two theorems[1]:

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1) Cf. A. H. Frink, "Distance functions and the metrization problem". Bull, Amer. Math. Soc., 43(1937).

Theorem C. *A quasi-metric space satisfying the condition V is metrizable.*

Theorem D. *A quasi-metric space satisfying the condition VI is metrizable.*

The object of this paper is to prove the following two theorems:

Theorem 1. *A general metric space satisfying the conditions I and V is metrizable.*

Theorem 2. *A general mestic space satisfying the conditions I and VI is metrizable.*

These theorems are generalizations of Theorems C and D. For the proof we use Theorem A and the well known lemma due to Alexandorff and Urysohn. The latter reads as follows:

Lemma. *The following conditions are necessary and sufficient that a neighbourhood space E is metrizable:*

There exists in the space E a sequence of families of sets $\{G_1\}$, $\{G_2\}$, $\cdots$, $\{G_n\}$, $\cdots$, each family $\{G_n\}$ covering whole space E, such that:

A. If two sets G_n and G_n' of the n-th family $(n>1)$ have a common point, then there is a set of the $(n-1)$-th family, containing both G_n and G_n'.

B. If a and b are distinct points, there exists an n such that no set G_n of the n-th family contains both a and b.

C. Let $S_n(x)$ be the sum of all sets G_n of the n-th family, containing the point x. Then the sets $\{S_n(x)\}$orm a complete system of neighbourhood of the point x.

§ 2. Proof of Theorem 1.

Let $b=a$ in V. Then $ca<\varphi(\varepsilon)$ implies $ca<\varepsilon$. Hence, $aa_n\to 0$ is equivalent to $a_na\to 0$. If we define $d(a,b)\equiv\max(ab,ba)$, then $d(a,b)$ is symmetric and equivalent to ab (and so ba) topologically. Putting now $\varphi^2(\varepsilon)\equiv\varphi(\varphi(\varepsilon))$, $d(a,b)<\varphi^2(\varepsilon)$ and $d(c,b)<\varphi^2(\varepsilon)$ imply $ab<\varphi^2(\varepsilon)$ and $cb<\varphi^2(\varepsilon)$. On the other hand $ac<\varphi(\varepsilon)$ implies $ca<\varepsilon$. Then we have

$$d(a,c)=\max[ac,ca]<\max[\varphi(\varepsilon),\varepsilon]=\varepsilon,$$

by $\varphi(\varepsilon)<\varepsilon$. If we put $\varphi^2(\varepsilon)\equiv\psi(\varepsilon)$, then $d(a,b)<\psi(\varepsilon)$ and $b(c,b)<\psi(\varepsilon)$ imply $d(a,c)<\varepsilon$. Since we can take $\psi(\varepsilon)\leqq\varepsilon/2$, if we define $r_1\equiv 1, r_2\equiv\varphi(r_1),\cdots, r_{n+1}\equiv\varphi(r_n),\cdots$, then $r_n\to 0$.

Let us now introduce a new metric $\rho(a,b)$ such that,

$$\rho(a,b)\equiv 1 \qquad (ab\geqq r_1)$$

$$\rho(a,b)\equiv 1/2^n \quad (r_n>ab\geqq r_{n-1}) \quad (n=1,2,\cdots).$$

As easily may be seen, $\rho(a,b)$ satisfies the condition IV and is equivalent to $d(a,b)$ and then to ab topologically. The symmetricity is also preserved.

Therefore, by Theorem A, E is metrizable by using

$$\delta(a,b) \equiv \underset{x_1, \cdots x_n \in E}{\text{g.l.b.}} \quad (\rho(ax_1) + \cdots + \rho(x_{n-1}x_n) + \rho(x_n b))$$

as a new distance function. q.e.d.

We can see that the condition V in Theorem I may be replaced by the following:

V′. *For every* $\varepsilon > 0$ *there exists* $\rho(\varepsilon) > 0$ *such that* $ba < \varphi(\varepsilon)$ *and* $bc < \varphi(\varepsilon)$ *imply* $ac < \varepsilon$.

§3. Proof of Theorem 2. Let E be a general metric space which satisfies the conditious I and VI. Let $U_n(a) \equiv \underset{x}{E}(ax < 1/n)$, which gives the same topology as the original one. $\{U_n(a)\}$ is decreasing and by the conditions I, $\bigwedge_i U_i(a) = a$. Let us now put

$$n_1(x) \equiv 1,$$
$$n_2(x) \equiv \min[m; 1/m < \varphi(x, n_1(x))],$$
$$n_{r+1}(x) \equiv \min[m; 1/m < \varphi(x, n_r(x))],$$

and

$$U_{n_r(x)}(x) = V_r(x).$$

$\{V_r(x)\}$ is obviously the equivalent system of neighbourhoods to $\{U_i(x)\}$. We shall prove that, $\{V_r(x); x \in E\} \equiv \mathfrak{M}_r$ satisfies the conditions A, B and C of Lemma.

Let $n \wedge 1$, $V_r(a)\, V_r(b) \neq 0$ and take $x \in V_r(a)\, V_r(b)$. $n_r(a) \geqq n_r(b)$ implies $ax < n_r(b)$ and then $bx < n_r(b) \leqq n_r(a)$. Hence

$$by < n_r(b) \leqq n_r(a) \text{ for all } y \in V_r(b).$$

From the condition VI, $ay < n_{r-1}(a)$ and then $y \in V_r(a)$, hence $V_r(b) \subset V_{r-1}(a)$. By $n_r(a) \leqq n_r(b)$, we have $V_r(a) \subset V_{r-1}(b)$. Therefore A is satisfied by $\{\mathfrak{M}_r\}$.

Let $a \neq b$. Form $\bigwedge_i U_i(x) = x$ there is an n such as $b \overline{\in} U_n(a)$. If we take $m = \min[m; 1/m < \varphi(a,n) < 1/n]$, then $V_m(x)$ does not contain both a and b. For, if $a, b \in V_m(x)$, then $V_m(x) = U_{n_m(x)}(x) \subset U_m(x)$ and $U_m(a) \subset U_n(a)$; therefore $a \in V_m(a) V_m(x)$. Since $xa < \varphi(a,n)$ and $xb < \varphi(a,n)$ from the codition VI, we have $ab < 1/n$; i.e., $b \in U_n(a)$, which is a contradiction. Therefore B is satisfied by $\{\mathfrak{M}_r\}$.

$S_r(a) \equiv \vee V_r(x)$ for $a \in V_r(x)$. So that $S_r(a) \supset V_r(a) = U_{n_r(a)}(a)$. Hence it is sufficient to prove that there is an m such that $S_m(a) \subset V_r(a)$ for every $V_r(a)$. Take $V_r(a) = U_{n_r}(a)$, and put $k \equiv \min[k; 1/k < 1/n_r(a)]$. Taking l such as $\varphi(a,k) > 1/l$, we have $S_l(a) \subset V_r(a)$ for this l. For, $b \in S_l(a)$ if and only if there is $V_l(x)$ such as $a, b \in V_l(x)$. So that $b \in$

$S_l(a)$ if and only if there is an x such as $xa < 1/l$ and $xb < 1/l$. For such x, $xa < 1/l < \varphi(a,k) < \varphi(a, n_r(a))$. And then $\min\{1/m : 1m < \varphi(a, n_r(a))\} > \varphi(a,k)$: that is, $n_{r+1}(a) = 1/m > \varphi(a,k))$. The same is true for xb. Hence $ab < n_r(a)$, and then $b \in V_r(a)$. q. e. d.

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