ON THE UNIFORM SPACE*)

By

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The notion of a uniform space has been introduced by A.Weil (cf. [1][1]). We shall prove that some topological properties of metric space can be discussed in fully normal space (cf. [2]), using structure theory of uniform space (§ 1)[2]. From this, we see that a part of conjecture of J.W.Tukey (cf. [2]) is proper. Next, we shall discuss on a metrization condition (*) (cf, § 2) for uniform space (§ 2), and prove that condition is closely related to the completeness of uniform space (§ 3). Furthermore, we consider a local property in uniform space and answer the Kakutani's problem negatively.

§ 1. THEOREM 1. *In the fully normal space E, following five conditions are equivalent to each other*:

(1) *E is compact.*
(2) *E is countably compact.*
(3) *Every real valued continuous function on E is bounded.*
(4) *E is precompact for any uniform structure compatible with its topology.*
(5) *Uniform structure compatible with its topology is unique.*

PROOF. (1) → (2) and (2) → (3) are evident. (3) → (4) is proved by R. Doss (cf [4]). We will now prove (1) → (5). After A. Weil, if E is a compact uniform space defined by structure $\{V_\alpha\}$, and $\mathfrak{G}$ is an open covering of E, then for every $p \in E$ there exists α such that $V_\alpha(p)$ contains some set G in $\mathfrak{G}$. Hence every structure of E is equivalent to the structure defined by all open coverings. Thus the uniform structure of E is unique.

We will next prove (5) → (4). E has a unique structure which has to coincide with the uniform structure of Weil, that is, defined by all bounded continuous functions on E. As easily may be seen, E is precompact for this structure. Thus (4) has been proved. It remains to prove (4) → (1). Let E be a fully normal T_1-space, and $\{\mathfrak{M}_\alpha\}$ be the family of all open coverings of

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1) Numbers in brackets refer to the bibliography at the end of the paper.

2) See N. Bourbaki, [3].

E. We denote by $S(H, \mathfrak{M})$ the union of all the sets of $\mathfrak{M}$ meeting H. Further let $\mathfrak{M}^*$ be the family of sets $S(M, \mathfrak{M})$, where $M \in \mathfrak{M}$, and put $V_\alpha(p) = S(p, \mathfrak{M}_\alpha)$. Then, from the definition of fully normal space, $\{V_\alpha\}$ definies the uniform st ucture in E compatible with its topology. For arb.trary open covering of E, there exists α such as $\mathfrak{M}_\alpha^* < \mathfrak{M}$[3], from the definition of full normality. Since E is prececompact for $\{V_\alpha\}$, there exist finite pointes $p_{\alpha i}$ $(i = 1, 2, \cdots, n)$ of E such that $E = \bigcup_{i=1}^{n} V_\alpha(p_{\alpha i})$. While, there exist $M_i \in \mathfrak{M}$, $(i = 1, 2, \cdots, n)$ such as $V_\alpha(P_{\alpha i}) \subset M_i$.

Hence $E = \bigcup_{i=1}^{n} M_i$, that is, E is compact. Thus the theorem is completely proved.

Since, after A.H. Stone [5], full normality and paracompactness (cf, [6]) are equivalent in T_1-space, we set the following corollary:

COROLLARY 1.1. *Hausdorff space is compact if and only if it is countably compact and paracompact.*

REMARK. Even if E is a locally compact and countably compact uniform space, and moreover its uniform structure is unique, it need not necessarily be compact. This is shown by the following example.

EXAMPLE 1. Let E be the set of all ordinal numbers $< \Omega$, Ω being the first uncountable number. The topology of E is the usual one of an ordered set, an open base being given by the family of all open intervals. It is well known that E is a locally compact, countably compact and completely normal space. It is also easy to see that every two disjoint closed sets are normally separable and at least one of separator may be compact. Hence uniform structure of E compatible with this topology is unique, by R. Doss's theorem (cf. [7]). But E is not compact. Thus E is the required one. By Theorem 1 E is not fully normal. Consequently,

COROLLARY 1.2. *Locally compact, countably compact and completely normal space is not always fully normal.*

That is, locally compact, countably compact and normal space is not always paracompact.

§2. Let us now turn to the metrization problem of uniform space. A.Weil has proved that uniform space E with uniform structure $\{V_\alpha\}$ is uniformly homeomorph with metric space if and only if there exists a

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3) A covering $\mathfrak{M}$ which is a refinement of a covering $\mathfrak{N}$, is written as $\mathfrak{M} < \mathfrak{N}$.

countable system of entrouges $\{V_n\}$ equivalent to $\{V_\alpha\}$. For uniformly locally countably compact space, we shall give another necessary and sufficient condition, that is,

THEOREM 2. *Uniformly locally countably compact uniform space E is uniformly homeomorph with metric space if and only if there exists a countable system of entrouges* $\{V_n\} \subset \{V_\alpha\}$ *such that*

$$(*) \qquad \bigcap_{n=1}^{\infty} V_n = \Delta^{4)}.$$

PROOF. Since necessity is evident, we need to prove sufficiency only. Let $V_\alpha(p)$ be countably compact for every p in E, and $\{V_{\beta_n}\}$ and $\{V_{\gamma_n}\}$ satisfy the relations

$$V^3_{\beta_n} \subset V_\alpha \cap V_n, \; V_{\gamma_n} \subset \bigcap_{i=1}^{n} V_{\beta_i}, \qquad n = 1, 2, \cdots.$$

In the first place, we prove $\{V_{\gamma_n}(p)\}$ is fundamental neighbourhood system for each p in E. Otherwise there exist $p \varepsilon E$ and an open set $G(p)$ containing p, such that $V_{\gamma_n}(p) \not\subset G(p)$, $n = 1, 2, \cdots$. From this we find a countable sequence $\{p_n\} \subset E$ such that $p_n \varepsilon V_{\gamma_n}(p) - G(p)$, $n = 1, 2, \cdots$. Thus $\{p_n\} \subset V_\alpha(p)$. If $\{p_n\}$ contains infinitely many different points then $\{p_n\}$ has an accumulation point p_0 in $V_\alpha(p)$. In the contrary case there is a point p_0 coinciding with infinitely many p_n.

Since we have

$$p_m \varepsilon V_{\gamma_n}(p) \subset V_{\beta_n}(p) \qquad \text{for } m > n, \; n = 1, 2, \cdots.$$

This is a contradiction.

We can also similarly prove, that $\{V_{\gamma_n}\}$ is equivalent to $\{V_\alpha\}$. From the proof of this theorem, we have

COROLLARY 2.1. *Locally countably compact regular space, in which each point is* G_δ, *satisfies the first countability axiom.*

In perfectly normal space, every closed set is a G_δ set. Hence locally countably compact perfectly normal space satisfies the first countability axiom. From this,

COROLLARY 2.2. *Compact completely normal space is not always perfectly normal.*

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4) Such notion of countability was introduced by J. von Neumann [8].

For, let E' be Ω adjoined by E in the example of § 1, the neighbourhood system of Ω being defined as in § 1. then E' is compact and completely normal, so that is fully normal, but not perfectly normal.

COROLLARY 2.3. *Countably compact and completely regular space is metrizable if and only if, there exists a countable set of real valued continuous functions $\{f_n\}$ defined on E such that, for any two distinct points p, q of E there exists an f_n in $\{f_n\}$ such as $f_n(p) \neq f_n(q)$.*

PROOF. Since necessity is evident, we prove sufficiency. Let us put $V_{\varphi_1, \cdots, \varphi_n, \varepsilon} = \{(p, q) | \varphi_i(p) - \varphi_i(q) | < \varepsilon, \ i = 1, 2, \cdots, n\}$ for any continuous functions φ_i $(i = 1, 2, \cdots, n)$, any positive number n, and any positive number $\varepsilon > 0$. Then $\{V_{\varphi_1, \cdots, \varphi_n, \varepsilon}\}$ $\varphi_1, \cdots, \varphi_n, \varepsilon > 0$ defines a uniform topology in E compatible with its original topology. Since $\bigcap_{n\ m} V_{f_n, 1/m} = \Delta$, Theorem 2 completes the proof.

§ 4. In this section we consider a condition of completeness of uniform space. J.Dieudonné[5] has proved that uniformizable space, which is metrizable, has a uniform structure compatible with its topology, and that such space is complete. The essential part of his proof lies in that, if for every $p^* \in E^* - E$, E being uniform space and E^* being completer of E, there exists a real valued function f on E such that f is continuous in E and $f(p)$ tends to $+\infty$ when p tends to p^* varying points in E, then E has a structure of complete space.

Applying this method, we can replace the metrizability condition by the condition (*) of Theorem 2.

THEOREM 3. *If E is a uniform space satisfying the condition* (*), *then E has a uniform structure compatible with its original topology which makes E compacte.*

PROOF. We shall first prove that there exists a uniformly continuous distance function "d" satisfying the condition of semi-metric. Let us take $\{V_{\alpha_n}\}$ such as

$$V_{\alpha_1} = V_1, \quad V_{\alpha_n} = \overset{-1}{V}_{\alpha_n}, \quad \overset{2}{V}_{\alpha_{n+1}} \subset V_{\alpha_n} \cap V_n, \qquad n = 1, 2, \cdots,$$

and define d_n such that

$$d_n(p, q) = 0 \qquad \text{if } (p, q) \in V_{\alpha_n},$$
$$d_n(p, q) = 1 \qquad \text{if } (p, q) \bar{\in} V_{\alpha_n},$$

5) See J. Dieudonné [9], [10].

and put

$$d(p, q) = \sum_{n=1}^{\infty} d_n(p, q)/2^n.$$

Then clearly $d(p, q) = 0$ if and only if $p = q$ (by the condition (*)), and $d(p, q) = d(q, p)$ (by $V_{\alpha_n} = \overset{-1}{V}_{\alpha_n}$). That is, distance function $d(p, q)$ is semi-metric. Since $d(p, q)$ is uniformly continuous, from a theorem A.Weil, $\bar{d}(p, q)$, extension to E^* of $d(p, q)$, is also uniformly continuous. If $p^* \varepsilon E^* - E$, then there exists at most one point p_0 in E such that $\bar{d}(p^*, p_0) = 0$. From this and using the method of Dieudonné (cf, [9]), the proof is compl ted.

Remark. We can show, using a Theorem of W.A.Wilson (cf [11]), that the above distance function $d(p, q)$ satisfies also the triangular axiom of metric.

Consequently, if $\{V_n\}$ is equivalent to $\{V_\alpha\}$, then E is uniformly homeomorphic with a metric space.

From this theorem we can deduce the following corollaries.

Corollary 3.1. *In a uniform space E satisfying the condition* (*), *the five conditions of theorem* 1 *are equivalent to each other.*

Proof. It is enough to prove (4) $\to$ (1) only. But this is clear, since precompact and complete uniform space is compact.

Corollary 3.2. *If E is a uniformizable space which is enumerable sum of compact sets, then E has a uniform structure, by which E is complete.*

Proof. From the postulate, there exists a (enumerable) sequence of compact sets $\{K_n\}$, such that $E = \bigcup_n K_n$. Let E^* be the completer of E for a uniform structure of E compatible with its topology. Then each K_n is also compact in E^*. For any $p^* \varepsilon E^* - E$, there exists a sequence of continuous functions $\{f_u(p)\}$, defined in E^* such that $f_n(p^*) = 0$, $f_n(p) = 1$ for $p \varepsilon K_n$ and $0 \leqq f_n(p) \leqq 1$ for every $p \varepsilon E^*$. If we put $f(p) = \Sigma_{n=1}^{\infty} f_n(p)/2^n$, then clearly $f(p)$ is continuous in E^* and $f(p) = 0$ implies $p \varepsilon E^* - E$. We put $\varphi(p) = 1/f(p)$ for $p \varepsilon E$. Then $\varphi(p)$ is continuous in E and tends to $+\infty$ when p tends to p^* varying points in E. Thus our proof is completed (cf. the first part of this section).

§ 5. We will conclude the paper by discussing local properties of uniform space. Let P be a hereditary property. In the topological group G, if G has the property P locally, then G has property P uniformly locally. S.Kakutani proposed the problem that, if E, a uniform space, has the property P

locally, then has E it uniformly locally, or does there exist a uniform structure compatible with its topology such that E has the property P uniformly locally. We can answer this problem negatively.

A.Weil has proved that if a uniform space E is uniformly locally compact, then E is complete. While, in Example 1 of §1, we have known that countably compact and locally compact completely normal space need not be compact, and then it is not complete. Thus, from the above theorem of Weil, it is not uniformly locally compact for any uniform structure compatible with its topology. Thus the problem is soloved negatively.

We will remark that, if E is a fully normal space, then this problem was positively solved by Shiroda (cf. [12]).

Now regular T_0-space having an enumerable covering compact sets, is normal. This is proved similarly as Tychonoff's theorem (cf. [13]); that is, regular T_0-space satisfying the second countability axiom is normal.

THEOREM 4. *Connected, uniformly loally compact uniform space is normal.*

Finally we will give an extension of a Kakutani's theorem in metric space.

THEOREM 5. *Connected uniform space E satisfying the second countability axiom uniformly locally, is metrizable.*

PROOF. There exists α such that $V_\alpha(p)$ satisfy the second countability axiom for all $p \in E$. For fixed $p_0 \in E$, there exists $\{p_i\} \subset V_\alpha(p_0)$ such that $\{p_i\}$ is dense in $V_\alpha(p_0)$. And for each i, there exists $\{p_{ij}\}_j \subset V_\alpha(p_i)$ such that $\{p_{ij}\}_j$ is dense in $V_\alpha(p_i)$. In general, for each $i, j, \cdots, m$, there exists $\{p_{ij\cdots mn}\}_n \subset V_\alpha(p_{ij\cdots m})$ such that $\{p_{ij\cdots mn}\}_n$ is dense in $V_\alpha(p_{ij\cdots m})$. We put

$$S = \cup\,\{\cup_i V_\alpha(p_i),\ \cup_{ij} V_\alpha(p_{ij}), \cdots,\ \cup_{ji\cdots n} V_\alpha(p_{ij\cdots n}), \cdots\}.$$

Then S is open and satisfies the second countability axiom. If we can prove that S is closed, then S coincides with the whole space E by connectedness. If S is not closed, then we can find a point $p \in \bar{S} - S$. Since $S \cap V_\beta(p) \neq \phi$ where $V_\beta = \overset{-1}{V_\beta} \subset V_\alpha$, there exists $p_{ij\cdots n} \in V_\beta(p)$, whence $q \in V_\beta(p_{ij\cdots n}) \subset V_\alpha(p_{ij\cdots n}) \subset S$. This is a contradiction.

If connected uniform space satisfies the second countability axiom merely locally, then E is not metrizable in general. This can be seen by the following example (cf. Alexandorff-Hopf [14]).

EXAMPLE 2. Let E be the Euclidean half plane: $\{(x,y): y \geqq 0\}$. For $p = (x,y)$, $y > 0$, let its neighbourhood $V(p)$ be open spheres with center p lying in E. For $p = (x,y)$, $y = 0$, let its neighbourhood $V(p)$ be the open sphere touching at p with x-axis. Then E is regular locally compact T_0-space and satisfies the second countability axiom locally. Thus E is completely regular, but not normal. And clearly E is connected. From Theorem 4 and 5, E is not uniformly locally compact, and satisfies the second countability axiom uniformly locally. Hence E is not metrizable.

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