ON THE GROUP OF AUTOMORPHISMS OF A TOPOLOGICAL GROUP*)

By

KATSUMI NOMIZU AND MORIKUNI GOTÔ

Let G be a locally compact topological group and let $A(G)$ be the group of all (continuous) automorphisms of G. In § 1 of the present note, we introduce a topology into $A(G)$ such that $A(G)$ becomes a topological group[1]. Our topology is a very natural one in view of the properties stated there. Our main concern, in the present note, will be then to investigate the condition for $A(G)$ to be compact with respect to this topology. For a connected Lie group, the problem is completely solved (§ 2). But the situations are more complicated in the case of a general locally compact connected group and we can not yet obtain a satisfactory result. Some remarks and examples will be added in § 3.

§ 1. Let G be a locally compact group and $A(G)$ the group of all (continuous) automorphisms of G. Cosider the totality Σ of pairs $\alpha = (K, V)$, where K is, in each pair, a compact subset of G and V is a neighbourhood of the identity e of G. If we define a partial order in Σ such that, for $\alpha = (K, V)$ and $\alpha' = (K', V')$, $\alpha > \alpha'$ means $K \supset K'$ and $V \subset V'$, Σ becomes a directed set. For any $\alpha \in \Sigma$, $\alpha = (K, V)$, put

$$U_\alpha = \{\sigma \in A(G) ;\ x^{\sigma-1} \in V \text{ and } x^{\sigma^{-1}-1} \in V \text{ for all } x \in K\}.$$

Clearly $U_\alpha = U_\alpha^{-1}$ and $U_\alpha \subset U_{\alpha'}$ if $\alpha > \alpha'$ in Σ. It is easy to verify that, by taking the set of all U_α, $\alpha \in \Sigma$ as a complete system of neighbourhoods of the identity, $A(G)$ becomes a topological group.

This topology of $A(G)$ has the following properties :

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1) For a group of homeomorphisms of a topological space, see H. Nagao: On the Topologies of Homeomorphism Groups of Topological Spaces, Osaka Math. Journ. vol. 1. No 1 1949. Also, J. Dieudonné: On Topological Groups of Homeomorphisms, Amer, Journ, of Math, vol, LXX, No. 3 (1948),

(1) x^{σ} is continnous in $(x, \sigma) \in G \times A(G)$. Morever this topology is the weakest one among the topologies of $A(G)$ having this property.

(2) The natural homomorphism from G into $A(G)$ letting correspond to an element $x \in G$ the inner automorphism of G induced by x is continuous. The subgroup $I(G)$ of all inner automorphisms is not necessarily closed in $A(G)$.

(3) If G is moreover connected, we can define same topology of $A(G)$ by considering the system ot $U(V) = U_{\alpha}$, $\bar{\alpha} = (W, V)$, where W is an arbitrary but fixed neighbouhood of e with compact closure.

(4) If G is a connected Lie group, $A(G)$ is a Lie group as a linear group of automorphisms of the Lie algebra of G. The topology of this Lie group $A(G)$ coincides with the one defined above.

(5) Let G be a locally compact abelian group and G^* the character group of G. The well known dual isomorphism[2] of $A(G)$ and $A(G^*)$ is topological if $A(G)$ and $A(G^*)$ are topologized as above.

§2. In this § we shall prove the following.

THEOREM 1. *Let G be a connected Lie group. Then necessary and sufficient condition that $A(G)$ be compact is that G is compact and has a center of dimenssion less than 2.*

PROOF. *Necessity*: Let $A(G)$ be compact. Then the adjojnt group $I(G)$ of G is fully reduecible as a linear subgroup of the compact $A(G)$. From this we see that

$$G = Z^0 \cdot S, \qquad [Z^0, S] = e,$$

where Z^0 is the component of the identity of the center Z of G and S is the maximal semi-simple normal subgroup of G. G/Z is semi-simple as it is locally isomorphic with G/Z^0. Moreover G/Z is maximaly almost periodic, for it is continuously isomorphic with $I(G)$ which is maximally almost periodic. Thus G/Z is compact. Aa S is semi-simple and locally isomorphic with compact G/Z, S is also compact by a theorem of Weyl. As $Z^0 = V \times T$ we have $G = (V \times T)S = V \times (TS)$, where V is a vector group, T is a toroidal group and S is a semi-simple compact normal subgroup of G. But as $A(G)$ is compact, G can not contain V. Hence $G = TS$. We must show that the dimension of T is $\leqq 1$. If σ is an automorphism of T leaving invariant every element of $T \cap S$, we obtain an automorphism of G by setting $g^{\sigma} = t^{\sigma} s$

2) E.g., M. Abe Uber Automorphismen der lokal-kompakten abelschen Gruppen, Proc, Imp. Acad, Tokyo, vol, **16** (1940) .

for $g=ts$, $t\in T$, $s\in S$. Hence we can conclude our proof by the following lemma which is easy to prove.

LEMMA. Let T be a toroidal group of dimension $\geqq 2$, and let F be any finite subgroup of T. Then there exist infinitely many automorphisms of T which leave invariant every element of F.

Sufficiency: First, if G is compact and semi-simple, $A(G)$ is clearly compact. Let $G=TS$, where T is a one-dimensional central toroidal group and S is s semi-simple normal subgroup of compact G. Then $A(G)$ is a subgroup of $A(T)\times A(S)$, which is compact. $I(G)$ is isomorphic with $I(S)$, which is known to be the component of the identity of $A(S)$. So $A(T)\times A(S)/I(S)$ is finite.

Theorem 1 is thus proved.

When G is not a Lie group, we can prove, by a similar arguement as in the first half of Theorem 1, the following

THEOREM 2. *Let G be a locally compact connected group. If $A(G)$ is compact, then G is also compact.*

The proof is omitted.

§3. We want to add some supplementary remarks and examples concerning the problem in the case of a compact connected abelian group. In the following G will always denote such a group.

(1) As is remarked in §1, $A(G)$ is isomorphic with $A(G^*)$, G^* being the discrete character group of G. G^* contains no element of finite order and the rank of G^* is equal to the dimension of G. Hence, if G is finite-dimensional. G^* is a subgroup of the finite direct product of the discrete additive group R of rational numbers and $A(G)=A(G^*)$ is discrete.

(2) Let dim $G=$ rank $G^*=1$. Then G^* is a subgroup of R and $A(G)=A(G^*)$ is finite (of order 2) if and only if G^* is isomorphic with the group of integers or G^* satisfies the following condition:

In the irreducible fractional expressions of the elements of G^*, every prime number p appears in the denominators with bounded exponents.

(3) By (2) we see that, for any integer $n>0$, there exists an n-dimensional G with finite $A(G)$. Such an example can be constructed as follows:

Divide the set of all prime numbers into n disjoint classes P_k ($k=1,2,\cdots,n$), each containing infinitely many prime numbers. Let G_k^* ($k=1, 2, \cdot, n$) be the subgroup of R consisting of rational numbers ot the form $q/p_{i1}\cdots p_{is}$, where q is an integer and $p_{i1},\cdots,p_{is}$ are any different prime numbers belonging to P_k.

The character group G of the direct product group of G_k^* $(k=1,2,\cdots,n)$ is the one required.

In an analogous way, we can show the existence of an infinite-dimensional G with compact $\mathcal{A}(G)$.

(4) If G contains a toroidal group of any dimension as a proper direct factor, then $\mathcal{A}(G)$ is not compact.

Mathematical Institute, Nagoya University, Nagoya.