ON SOME DIVERGENCE PROBLEMS*)

By

TAMOTSU TSUCHIKURA

This paper contains three §§ on different problems. In the first § we shall concern the problem of J. D. Hill and S. Kakutani, in the second the extension of a theorem of H. Steinhaus, and in the third a property of subseries of divergent series with function-terms.

§ 1. On a problem of J. D. Hill and S. Kakutani.

Let $\{r_n(x)\}$ be the Rademacher system, J. D. Hill ([2]) and S. Kakutani ([4]) proposed the problem, whether, for the sequence $0 \leqq p_1 \leqq p_2 \leqq \cdots \to \infty$ such that $p_n/(p_1 + \cdots + p_n) \to 0\ (n \to \infty)$, the Riesz mean

$$\varphi_n(x) = \frac{p_1 r_1(x) + \cdots + p_n r_n(x)}{p_1 + \cdots + p_n}$$

tends to zero almost everywhere as $n \to \infty$.

I was told that this problem had been solved negatively by S. Kakutani, but I am not aware of his precise result. On the other hand, using the independency of the system, G. Maruyama recently gave a negative example using the Kolmogoroff lemma on the law of the iterated logarithm ([5]).

We shall give here another negative example ([9]) by an elementary lemma of S. Mazur and W. Orlicz ([6]):

LEMMA. *Let $f(x)$ be a measurable function of period 1. Then for any sequence of positive numbers $n_1 < n_2 < \cdots \to \infty$, we have, for almost all x,*

$$\limsup_{i \to \infty} |f(n_i x)| = \text{ess. max.} |f(x)|.$$

First of all we note the evident relation $r_n(x) = r_1(2^{n-1}x)$, and let $r_1(x) \equiv r(x)$ for brevity. In the sequel we shall use these notations of the Rademacher functions without any notice.

Our purpose is to construct a sequence $\{p_n\}$ such that $0 < p_1 < p_2 < \cdots \to \infty$, $p_n/(p_1 + \cdots + p_n) \to 0\ (n \to \infty)$ and that if we put

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*) Received Aug. 20, 1949.

$$(1) \qquad \varphi_n(x) = \frac{p_1 r(x) + p_2 r(2x) + \cdots p_n r(2^{n-1}x)}{p_1 + p_2 + \cdots + p_n},$$

then we have

$$(2) \qquad \limsup |\varphi_n(x)| = 1$$

almost everywhere. Since the set (we understand throughout this § that the set is included in (0,1)) of convergence of $\{\varphi_n(x)\}$ is of measure 0 or 1, and since if it converges almost everywhere, its limit is necessarily zero (see [2]), the equality (2) implies its almost everywhere divergence.

Let us define a sequence of integers

$$(3) \qquad k_1^{(n)}, k_2^{(n)}, \cdots, k_{\nu_n}^{(n)} \quad (n = 1, 2, \cdots)$$

by induction. Let $k_1^{(1)} = k_{\nu_1}^{(1)} = 1$ and suppose that the integers

$$k_1^{(s)}, k_2^{(s)}, \cdots, k_{\nu_s}^{(s)} \quad (s = 1, 2, \cdots, n-1)$$

were defined. Put $k_1^{(n)} = n + k_{\nu_{n-1}}^{(n-1)}$ and $k_m^{(n)} = n + k_{m-1}^{(n)} + 1 \quad (m > 1)$.

If we consider the function which is the sum of n number of the Rademacher functions,

$$f_n(x) = r(x) + r(2x) + \cdots + r(2^n x),$$

then by the Mazur-Orlicz lemma, we have

$$(4) \qquad \limsup_{m\to\infty} |f_n(2^{k_m^{(n)}} x)| = \limsup_{m\to\infty} |r(2^{k_m^{(n)}} x) + \cdots + r(2^{n+k_m^{(n)}} x)| = n$$

almost everywhere, for $k_1^{(n)} < k_2^{(n)} < \cdots \to \infty$ and ess. max. $|f_n(x)| = n$. Accordingly there exist an integer ν_n and a set E_n, $|E_n| \geqq 1 - n^{-2}$ such that at least one of the terms

$$(5) \qquad |f_n(2^{k_1^{(n)}} x)|, \; |f_n(2^{k_2^{(n)}} x)|, \; \cdots, \; |f_n(2^{k_{\nu_n}^{(n)}} x)|$$

is greater than $n(1 - n^{-1})$ for $x \in E_n$. The sequence (3) is thus completely defined. Putting now $m = 1, 2, \cdots, \nu_n$, and $n = 1, 2, \cdots$, we arrange the fnnctions $f_n(2^{k_m^{(n)}} x)$ successively, which we denote by

$$(9) \qquad [1], [2], \cdots, [i], \cdots.$$

Then we can easily see that every $r_n(x)$ appears in (9) once and only once in order of its index.

Let $E = \liminf E_n$ and N_i be the number of the Rademacher functions included in $[i]$. Then $|E| = 1$, and we see from the definition of the sequence

(3) that

$$\limsup_{i\to\infty} |[i]/N_i| = \lim_{i\to\infty}(1-N_i^{-1}) = 1 \quad (x \in E). \tag{7}$$

Let

$$q_1 = 1 \text{ and } q_i = (q_1 N_1 + \cdots + q_{i-1} N_{i-1})/N_i^{1/2} \quad (i > 1). \tag{8}$$

Then we have

$$q_i/(q_1 N_1 + \cdots + q_{i-1} N_{i-1}) = N_i^{-1/2} \to 0 \quad \text{as } i \to \infty, \tag{9}$$

and from (8) and (9) we deduce that

$$F_i(x) = \left| \frac{q_1[1] + q_2[2] + \cdots + q_i[i]}{q_1 N_1 + q_2 N_2 + \cdots + q_i N_i} \right| \geqq \left| \frac{q_i[i]}{q_i N_i^{1/2} + q_i N_i} \right|$$

$$- \left| \frac{q_1 N_1 + \cdots + q_{i-1} N_{i-1}}{q_i N_i^{1/2} + q_i N_i} \right|$$

$$\geqq (1 + N_i^{-1/2})^{-1} |[i] N_i^{-1}| - o(1).$$

Hence from this and (7) we have

$$\limsup_{i\to\infty} |F_i(x)| = \limsup_{i\to\infty} |[i] N_i^{-1}| = 1 \tag{10}$$

for $x \in E$, that is, almost everywhere.

Let us now define a sequence $\{p_n\}$. If $r_n(x)$ is included in $[i]$ we put $p_n = q_i$. From (8) we see immediately that $p_n/(p_1 + \cdots + p_n) \to 0$ as $n \to \infty$, and from (10) that $\limsup_{n\to\infty} |\varphi_n(x)| = 1$ almost everywhere.

Thus the negative example is constructed.

The sequence $\{p_n\}$ defined above is not strictly increasing, but it is easy to find a seqneuce $\{p_n\}$ with this additional property. The detail may be omitted.

Remark. Let $f(x)$ be a function of period 1, of $\mathrm{Lip}\,\alpha\,(0 < \alpha \leqq 1)$ (more generally, of $\mathrm{Lip}(\alpha, 2)$) and such that

$$\int_0^1 f(x)\,dx = 0, \qquad \int_0^1 f^2(x)\,dx = 1.$$

Let $\{p_k\}$ be a monotone sequence of positive numbers tending to ∞, and such that $p_k/(p_1 + \cdots + p_k) \to 0$ as $k \to \infty$, and let $\{n_k\}$ be a lacunary sequence of integers: $n_{k+1}/n_k > q > 1$ $(k = 1, 2, \cdots)$.

Then by M. Kac ([3]), the sequence

$$\psi_k(x) = \frac{p_1 f(n_1 x) + \cdots + p_k f(n_k x)}{p_1 + \cdots + p_k} \qquad (k = 1, 2, \cdots)$$

tends to zero in L^2-mean as $k \to \infty$.

If, moreover, n_{k+1} is a multiple of n_k $(k = 1, 2, \cdots)$, the set of convergence of $\{\psi_k(x)\}$ is of measure 0 or 1. In this case, by the similar device as above, we can also construct two sequences $\{p_k\}$ and $\{n_k\}$, $n_{k+1}/n_k > q > 1$ such that $\{\psi_k(x)\}$ diverges almost everywhere.

§ 2. A property of divergent series.

THEOREM. *Consider the series of bounded measurable functions*

$$(1) \qquad \sum_{k=1}^{\infty} a_k(x)$$

If, in a set E, $0 < |E| \leqq \infty$, we have

$$(2) \qquad \limsup_{n\to\infty} |s_n(x)| = \infty.$$

where $s_n(x)$ denote the n-th partial sum of the series (1), *then there exists a monotone sequence of positive numbers $\{\lambda_k\}$ such that $\lambda_k \to 0$ as $k \to \infty$ and that*

$$(3) \qquad \limsup_{n\to\infty} |\sigma_n(x)| = \infty$$

almost everywhere in E, where $\sigma_n(x)$ denotes the n-th partial sum of the series

$$(4) \qquad \sum_{k=1}^{\infty} \lambda_k a_k(x).$$

H. Steinhaus proved this theorem in the case $a_k(x) \geqq 0$ $(k = 1, 2, \cdots)$ ([7]).

PROOF. Let us define a sequence of positive integers $\{n_i\}$ by induction. By (2) we can find a positive integer n_1 and a set $E_1 \subset E \cdot (-1, 1)$ such that at least one of the terms

$$|a_1(x)|, |a_1(x) + a_2(x)|, \cdots, |a_1(x) + \cdots + a_{n_1}(x)|$$

is greater than 1 in E_1, and, $|E_1| > |E \cdot (-1, 1)| - 1$.

Suppose that the integers n_ν and the sets E_ν $(\nu = 1, 2, \cdots, i-1)$ were all defined. Then by (2) we can determine a positive integer n_i and a set $E_i \subset E \cdot (-i, i)$ such that at least one of the terms

$$|a_{n_{i-1}+1}(x)|, |a_{n_{i-1}+1}(x) + a_{n_{i-1}+2}(x)|, \cdots, |a_{n_{i-1}+1}(x) + \cdots + a_{n_i}(x)|$$

is greater than i^2 in E_i, aud $|E_i| > |E \cdot (-i, i)| - i^{-2}$.

The sequences $\{n_i\}$ and $\{E_i\}$ are thus defined. Let us now put

$$\lambda_k = 1 \ (1 \leqq k \leqq n_1), \qquad = i^{-1} \ (n_{i-1} < k \leqq n_i), \ (i > 1),$$

and $E^* = \liminf E_i$. Evidently $\lambda_k \downarrow 0$ as $k \to \infty$ and the set E^* coincides with E except a null set. In fact, we have

$$E - E^* = \prod_{p=1}^{\infty} \sum_{i=p}^{\infty} (E - E_i) \subset \prod_{p=1}^{\infty} \sum_{i=p}^{\infty} \Big[\{E \cdot (-i, i) - E_i\} + (-\infty, -i) + (i, \infty) \Big]$$

$$= \prod_{p=1}^{\infty} \Big[\sum_{i=p}^{\infty} \{E \cdot (-i, i) - E_i\} + (-\infty, -p) + (p, \infty) \Big]$$

$$= \prod_{p=1}^{\infty} \sum_{i=p}^{\infty} \{E \cdot (-i, i) - E_i\},$$

which is clearly of measure zero.

For any $x \varepsilon E^*$ we have the equality (3), since for large i_0 we have $x \varepsilon \prod_{i=i_0}^{\infty} E_i$ and from the definition of $\{n_i\}$, for any N there exists an integer $l > n_{N-1}$ such that

$$\left| \sum_{k=n_{N-1}+1}^{l} \lambda_k a_k(x) \right| > N. \qquad \text{q. e. d.}$$

REMARKS (i). If the series (1) is oscillating finitely, we can not necessarily find a sequence $\{\lambda_k\}$, $\lambda_k \downarrow 0$ $(k \to \infty)$ such that the series (4) diverges almost everywhere in E. For example, the series

$$\frac{1}{2} + \cos x + \cos 2x + \cdots$$

is divergent for all x, but for any $\{\lambda_k\}$, $\lambda_k \downarrow 0 (k \to \infty)$ the series

$$\frac{\lambda_1}{2} + \lambda_2 \cos x + \lambda_3 \cos 2x + \cdots$$

is convergent for $x \not\equiv 0$ (mod. 2π).

(ii) In our theorem we can choose the sequence $\{\lambda_k\}$ as to be convex. The proof of this fact is not so difficult, and may be omitted.

(iii) From remark (ii) and known theorems (see [10], p. 104), if we can find a Fourier-Stieltjes series (Fourier-Lebesgue series of bounded function), which is divergent oscillating infinitely almost everywhere, then we may determine a Fourier series of Lebesgue integrable (continuous) functions with the same property.

§ 3. Subseries of divergent series.

Let us consider the series

$$a_1 + a_2 + \cdots$$

with real or complex terms which is not absolutely convergent. R. P. Agnew ([1]) proved that there exists a sequence of indices

$$1 \leqq n_1 < n_2 < \cdots$$

such that

$$n_{k+1} - n_k \to \infty \qquad \text{as } k \to \infty \tag{1}$$

and the series

$$a_{n_1} + a_{n_2} + \cdots$$

is divergent.

We shall consider now the corresponding result for the series of functions, and prove the following theorem.

THEOREM 1. *Let* $a_n(x)$ $(n = 1, 2, \cdots)$ *be bounded measurable functions and suppose that the series*

$$a_1(x) + a_2(x) + \cdots \tag{2}$$

diverges in a set E, $0 < |E| \leqq \infty$, *in the following manner: the set* E *is decomposed into at most enumerable sum of sets* E_i, $|E_i| > 0$ $(i = 1, 2, \cdots)$ *and subseries of the series* (2)

$$\sum_{k=1}^{\infty} a_{m_k^{(i)}}(x) \qquad (i = 1, 2, \cdots) \tag{3}$$

diverge to $\pm\infty$ *or oscillate infinitely, that is,*

$$\limsup_{N\to\infty} \left| \sum_{k=1}^{N} a_{m_k^{(i)}}(x) \right| = \infty \tag{4}$$

for $x \in E_i$ $(i = 1, 2, \cdots)$.

Then there exists a sequence of indices $\{n_k\}$ *such that the expression* (1) *holds and moreover the series*

$$a_{n_1}(x) + a_{n_2}(x) + \cdots \tag{5}$$

diverges to $\pm\infty$ *or oscillates infinitely almost everywhere in* E.

For the proof of this theorem we first prove the lemma which is a special case of the theorem.

LEMMA. *Let* $a_n(x)$ $(n = 1, 2, \cdots)$ *be bounded measurable functions, and suppose that the series* (2) *diverges to* $\pm\infty$ *or oscillates infinitely, that is,*

$$\limsup_{N\to\infty}\left|\sum_{n=1}^{N} a_n(x)\right| = \infty$$

for $x \in E$, $0 < |E| < \infty$.

Then the conclusion of Theorem 1 *is true.*

PROOF OF LEMMA. We shall define the sequence $\{n_k\}$ by induction. Let $n_1 = 1$ and suppose that the indices $n_1, n_2, \cdots, n_k$ are determined. Denote

$$M_1 = E\cdot\underset{x}{\mathrm{E}}\left[\begin{array}{l}\text{the series } \sum_{j=1}^{\infty} a_{n_k+kj}(x) \text{ diverges to } \pm\infty \\ \text{or oscillates infinitely}\end{array}\right],$$

$$M_2 = E\cdot\underset{x}{\mathrm{E}}\left[\text{the same as above for the series } \sum_{j=1}^{\infty} a_{n_k+1+kj}(x)\right], \tag{6}$$

$$\cdots\cdots\cdots,$$

$$M_k = E\cdot\underset{x}{\mathrm{E}}\left[\text{the same as above for the series } \sum_{j=1}^{\infty} a_{n_k+(k-1)+kj}(x)\right].$$

We have then $E = M_1 + M_2 + \cdots + M_k$. In fact, if x is not contained in the set-sum of the right-hand side, then all the partial sum of the series in the brackets of (6) are equelly bounded, and then the partial sums of the series (2) are also bounded, this contradicts the assumption (3), whence

$$E \subset M_1 + M_2 + \cdots + M_k;$$

the opposite inclusion is obviously true.

There exists a natural number λ_1 such that if we put

$$N_1 = M_1\cdot\underset{x}{\mathrm{E}}\left(\begin{array}{l}\text{at least one of the partial sums of the series} \\ \sum_{j=1}^{\lambda_1} a_{n_k+kj}(x) \text{ is } \geqq k \text{ in absolute value}\end{array}\right), \tag{7}$$

then

$$|N_1| \geqq |M_1| - 2^{-(k+1)}.$$

Since $a_n(x)$ are bounded functions, there exists a natural number λ_2 such that by denoting

$$N_2 = M_2\cdot\underset{x}{\mathrm{E}}\left[\text{ the same as in (7) for } \sum_{j=\lambda_1+1}^{\lambda_2} a_{n_k+1+kj}(x)\right],$$

we have

$$|N_2| \geqq |M_2| - 2^{-(k+2)}.$$

Proceeding in this manner we may define $N_1, N_2, \cdots$ successively and finally

N_k such that

$$N_k = M_k' \cdot \underset{x}{\mathrm{E}} \left[\text{the same as in (7) for } \sum_{i=\lambda_{k-1}+1}^{\lambda_k} a_{n_k+(k-1)+kj}(x) \right]$$

and

$$|N_k| \geqq |M_k| - 2^{-2k}.$$

Obviously we have

$$\begin{aligned} |E_k| &\equiv |N_1 + N_2 + \cdots + N_k| \\ &\geqq |M_1 + M_2 + \cdots + M_k| - (2^{-(k+1)} + 2^{-(k+2)} + \cdots + 2^{-2k}) > |E| - 2^{-k}. \end{aligned}$$

Let the indices of the terms of the series appeared in the definition of $N_1, N_2, \cdots, N_k$ be

$$n_{k+1}, n_{k+2}, \cdots, n_{k+\nu_k}$$

successively, then

$$n_{k+p+1} - n_{k+p} \geqq k \quad (p = 0, 1, 2, \cdots, \nu_k - 1). \tag{8}$$

The sequence $\{n_k\}$, thus determined, has the property (1) in virtue of (8), Let $E^* = \liminf E_k$, then $|E^*| = |E|$ as we can easily see, and the point $x \in E^*$ is contained in every E_k for sufficiently large k, whence, as we see immediately from the definition of N_j, the series (5) diverges to $\pm\infty$ or oscillates infinitely. This proves the lemma.

We are now in a position to prove Theorem 1. Without loss of generality we may assume that $0 < |E_i| < \infty$ $(i = 1, 2, \cdots)$, for if $|E_j| = \infty$, the set E_j is decomposed into an enumerable sum of sets of finite measure.

By lemma there exist subsequences $\{n_k^{(i)}\}_k \subset \{m_k^{(i)}\}_k$ $(i = 1, 2, \cdots)$ such that $n_{k+1}^{(i)} - n_{k+1}^{(i)} \to \infty$ $(k \to \infty)$ and the series

$$\sum_{k=1}^{\infty} a_{n_k^{(i)}}(x) \tag{9}$$

diverges to $\pm\infty$ or oscillates infinitely almost everywhere in E_i $(i = 1, 2, \cdots)$.

In order to determine the required sequence $\{n_k\}$ we shall use the "diagonally selecting method".

First of all, let

$$a_{n_1^{(1)}}(x) + a_{n_2^{(1)}}(x) + \cdots + a_{n_{k_1}^{(1)}}(x)$$

be the partial sum of the series (9) $(i = 1)$ such that at least one of its partial sum is greater than 1 in absolute value, and every difference of the consecutive indices is greater than 1.

Secondly, let

$$a_{n_{k_2}^{(2)}}(x) + a_{n_{k_2+1}^{(2)}}(x) + \cdots + a_{n_{h_2}^{(2)}}(x)$$

be the partial sum of the series (9) ($i=2$) such that

$$n_{k_2}^{(2)} \geqq n_{h_1}^{(1)} + 2,$$

and at least one of its partial sums is greater than 2 in absolute value and the difference of the consecutive indices is greater than 2.

Thirdly, we find the partial sum of (9) ($i=1$)

$$a_{n_{k_3}^{(1)}}(x) + a_{n_{k_3+1}^{(1)}}(x) + \cdots + a_{n_{h_3}^{(1)}}(x)$$

such that

$$n_{k_3}^{(1)} \geqq n_{k_2}^{(2)} + 3$$

and the other properties in the second step hold replacing 2 by 3.

Taking the seires (9) for $i = 3, 2, 1, 4, 3, 2, 1, 5, 4, 3, \cdots$ successively, we may obtain by the similar device the sequence of indices of their terms

$$n_1^{(1)}, \cdots, n_{h_1}^{(1)};\ n_{k_2}^{(2)}, \cdots, n_{h_2}^{(2)};\ \cdots;\ n_{k_\nu}^{(\mu)}, \cdots, n_{h_\nu}^{(\mu)};\ \cdots \qquad (\mu = \mu\ (\nu))$$

which we denote by $\{n_k\}$, then the sequence $\{n_k\}$ is the required one, q.e.d.

Corollary. *Let*

$$\sum_{n=1}^{\infty} a_n^{(i)} \qquad (i = 1, 2, \cdots) \tag{10}$$

be series of constant terms (real or complex), and suppose that each of them is not absolutely convergent. Then there exists a sequence of suffices $\{n_k\}$ *such that* $n_k - n_{k-1} \to \infty$ *as* $k \to \infty$ *and the series*

$$\sum_{k=1}^{\infty} a_{n_k}^{(i)}$$

diverges for every $i = 1, 2, \cdots$.

Without loss of generality we may assume that all the terms of (10) are real. If (10) is not absolutely convergent either the series of its positive terms or that of negative terms diverges to $\pm\infty$ ($i = 1, 2, \cdots$). Let $E_i = (i, i+1)$ ($i = 1, 2, \cdots$) and let $a_n(x) = a_n^{(i)}$ if $x \in E_i$ Then from Theorem 1 we may get the required sequence.

Our theorem may be generalized as follows:

THEOREM 2. *Under the same assumption as in Theorem 1, if* $0 < \mu_n$, $\mu_n/\mu_{n+1} \to 1$ *as* $n \to \infty$ *and* $\sum \mu_n = \infty$, *then there exists a sequence* $\{n_k\}$ *such that*

$$\lim_{k\to\infty} \sum_{i=1}^{k} \mu_{n_i} / \sum_{j=1}^{n_k} \mu_j = 0$$

and the series (4) *diverges almost everywherein E.*

This is an immediate consequence of Theorem 1 and Mr. Sunouchi's Theorem ([8]).

Comparing Theorem 1 and the Agnew theorem, stated in the beginning of this §, it may arise the question whether the restriction of divergence-manner such as (4) is superfluous or not for the validity of Theorem 1, but it remains open.

Tôhoku University, Sendai.

References

[1] Agnew, R. P., Subseries of series which are not absolutely convergent, Bull. Amer. Math. Soc., 53;2 (1947), p. 118-120.

[2] Hill, J. D., Summability of sequences of 0's and 1's, Ann. of Math., 46;4 (1945), p. 556-562.

[3] Kac, M., Convergence of certain gap series, Ann. of Math., 44;3 (1943), p. 411-415.

[4] Kakutani, S., Monthly of Real Analysis, 1(1947) (in Japanese).

[5] Maruyama, G., ibid., 2-9 (1949) (in Japanese).

[6] Mazur, S. and Orlicz, W., Sur quelques propriétés de fonctions périodiques et presque périodiques, Stud. Math., 9 (1940), p. 1-16.

[7] Steinhaus, H., Sur une question concernant la convergence de séries des fonctions, Fund. Mnth., 11 (1928). p. 186-192.

[8] Sunouchi, G., On the distribution of the signs of the terms of unconditionally convergent series, Monthly of Real Analysis, 3-3 (1949), (in Japanese).

[9] Tsuchikura, T., Monthly of Real Analysis, 2-9 (1949) (in Japanese).

[10] Zygmund, A., Trigonometrical series, 1935.