NOTES ON FOURIER ANALYSIS (XXVI): LIPSCHITZ CONDITION OF PARTIAL SUMS OF FOURIER SERIES*)

By

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§ 1. Let $f(x)$ be a continuous function of period 2π. If $f(x)$ belongs to $\mathrm{Lip}\,\alpha\ (0<\alpha\leqq 1)$, that is,

$$\max_x |f(x+\delta)-f(x)| = O(|\delta|^\alpha),$$

then denoting by $s_n(x)$ the n-th partial sum of the Fourier series of $f(x)$, we have

$$(1)\qquad \max_x |s_n(x+\delta)-s_n(x)| = O(|\delta|^\alpha \log 1/|\delta|)$$

uniformly in n[1].

It arises the problem of the suppression of the logarithm in (1). We shall prove that this is closely related to the order of approximation of $f(x)$ by the partial sum $s_n(x)$.

THEOREM 1. *Let $f(x)$ be a continuous function of period 2π, and let $s_n(x)$ denoted the n-th partial sum of its Fourier series.*

(i) *If $0<\alpha\leqq 1$ and*

$$(2)\qquad \max_x |s_n(x+\delta)-s_n(x)| = O(|\delta|^\alpha)$$

uniformly in n, then $f(x)$ belongs to $\mathrm{Lip}\,\alpha$, *and*

$$(3)\qquad r_n(x) \equiv f(x)-s_n(x) = O(n^{-\alpha})^{2)}$$

uniformly in x.

(ii) *If $0<\alpha<1$, and* (3) *holds uniformly in x, then $f(x)$ belongs to* $\mathrm{Lip}\,\alpha$ *and* (2) *holds uniformly in n.*

(iii) *If $f(x)$ belongs to* $\mathrm{Lip}\,1$, *and* (3) *holds uniformly in x for $\alpha=1$, then* (2) *holds uniformly in n for $\alpha=1$.*

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1) Cf. A. Zygmund, Trigonometrical series, 1935, 4.7.9 p.106.

2) The expression "$A\equiv B$" means that "denote B by A".

For the proof we use the following lemmas:

LEMMA 1[3]. *Let* $\sigma_n(x)$ *denote the* $(C, 1)$ *mean of the Fourier series of* $f(x)$. *A necessary and Sufficient condition that* $f(x)$ *belongs to* $\mathrm{Lip}\,\alpha\ (0<\alpha\leqq 1)$, *is that the equality*

$$(4)\qquad \max_x |\sigma_n(x+\delta)-\sigma_n(x)| = O(|\delta|^{\alpha})$$

holds uniformly in n.

LEMMA 2[4]. *If* $f(x)$ *belongs to* $\mathrm{Lip}\,\alpha\ (0<\alpha\leqq 1)$, *then*

$$f(x)-\frac{1}{2}\left\{s_n\left(x+\frac{\pi}{2n}\right)+s_n\left(x-\frac{\pi}{2n}\right)\right\}=O(n^{-\alpha})$$

uniformly in x.

LEMMA 3[5]. *Denoting by* $\bar{f}(x)$ *and* $\bar{s}_n(x)$ *the conjugates of* $f(x)$ *and* $s_n(x)$ *respectively, if* $|f(x)-s_n(x)|=O(n^{-\alpha})$ $(\alpha>0)$ *uniformly in* x, *then* $|r_n(x)|\equiv|\bar{f}(x)-\bar{s}_n(x)|=O(n^{-\alpha})$ *uniformly in* x.

LEMMA 4[6]. *If* $f(x)$ *belongs to* $\mathrm{Lip}\,1$, *then* $|\bar{f}(x)-\bar{\sigma}_n(x)|=O(n^{-1})$ *uniformly in* x, *where* $\bar{\sigma}_n(x)$ *is conjugate to* $\sigma_n(x)$.

LEMMA 5[7]. *If* $0<\alpha<1$, *and if* (3) *holds uniformly in* x, *then* $f(x)$ *belonsgs to* $\mathrm{Lip}\,\alpha$.

PROOF.
$$|f(x+\delta)-f(x)|\leqq|f(x+\delta)-s_{[1/|\delta|]}(x+\delta)|+|f(x)-s_{[1/|\delta|]}(x)| + |s_{[1/|\delta|]}(x+\delta)-s_{[1/|\delta|]}(x)| = P+Q+R,$$

say. By the assumption, $P=O(|\delta|^{\alpha})$, and $Q=O(|\delta|^{\alpha})$ uniformly in x. And

$$R=-2\sum_{\nu=1}^{[1/|\delta|]}A_\nu(x)\sin^2\frac{\nu\delta}{2}+\sum_{\nu=1}^{[1/|\delta|]}\bar{A}_\nu(x)\sin\nu\delta=R'+R'',$$

say, where $f(x)\sim\sum_{\nu=0}^{n}A_\nu(x)$, $\bar{f}(x)\sim\sum_{\nu=0}^{\infty}\bar{A}_\nu(x)$.

Applying Abel's transformation, we have

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3) Zygmund, loc. cit.
4) W.Rogosinski, Uber die Abschnitte trigonometrischer Reihen, Math. Ann., 95 (1925) p. 110-131.
5) R. Salem-A.Zygmund, The approximation by partial sums of Fourier series, Trans. Amer. Math. Soc., 59 (1946) p. 14-22.
6) A.Zygmund, On the degree of approximations of functions by Fejér means, Bull. Amer. Math. Soc., 51 (1945) p. 274-278.
7) de la Vallée Poussin, Leçons sur l'approximation, Paris, 1919.

$$|R'|\leqq\left|\sum_{\nu=0}^{[1/|\delta|]} r_\nu(x)\left(\sin^2\frac{(\nu+1)\delta}{2}-\sin^2\frac{\nu\delta}{2}\right)\right|+\left|r_{[1/|\delta|]}(x)\sin^2\frac{([1/|\delta|]+1)\delta}{2}\right|$$

$$=O\left(\sum_{\nu=1}^{[1/|\delta|]}\nu^{-\alpha}|\delta|\right)+O(|\delta|^\alpha)=O(|\delta|^\alpha),$$

and

$$|R''|\leqq\left|\sum_{\nu=0}^{[1/|\delta|]}\bar r_\nu(x)\{\sin(\nu+1)\delta-\sin\nu\delta\}\right|+\left|\bar r_{[1/|\delta|]}(x)\sin([1/|\delta|]+1)\delta\right|$$

$$=O\left(\sum_{\nu=1}^{[1/|\delta|]}\nu^{-\alpha}|\delta|\right)+O(|\delta|^\alpha)=O(|\delta|^\alpha), \qquad \text{(by Lemma 3).}$$

Hence summing up the above results we conclude that $f(x)$ belongs to Lip α.

Proof of Theorem 1. (i) If (2) holds uniformly in n, then from Lemma 1, we see immediately that $f(x)$ belongs to Lip α. By Lemma 2 and our assumption we have

$$\begin{aligned}|f(x)-s_n(x)|&\leqq\left|f(x)-\frac{1}{2}\left\{s_n\left(x+\frac{\pi}{2n}\right)+s_n\left(x-\frac{\pi}{2n}\right)\right\}\right|\\&+\frac{1}{2}\left|s_n\left(x+\frac{\pi}{2n}\right)-s_n(x)\right|+\frac{1}{2}\left|s_n\left(x-\frac{\pi}{2n}\right)-s_n(x)\right|\\&=O(n^{-\alpha})+O(n^{-\alpha})+O(n^{-\alpha})=O(n^{-\alpha}).\end{aligned}$$

(ii) and (iii). From Lemma 5 it is enough to prove them under the assumption that $f(x)$ belongs to Lip α $(0<\alpha\leqq 1)$ and (3) holds uniformly in x. We shall divide the proof into two steps:

1°) $n\delta\geqq 1$ and 2°) $n\delta<1$ (suppose $\delta>0$).

1°) Let $n\delta\geqq 1$, then

$$\begin{aligned}|s_n(x+\delta)-s_n(x)|&\leqq|s_n(x+\delta)-f(x+\delta)|+|f(x)-s_n(x)|+|f(x+\delta)-f(x)|\\&=O(n^{-\alpha})+O(n^{-\alpha})+O(\delta^\alpha)=O(\delta^\alpha).\end{aligned}$$

2°) Let $n\delta<1$, that is $n<1/\delta$.

$$s_n(x+\delta)-s_n(x)=-2\sum_{\nu=1}^{n}A_\nu(x)\sin^2\frac{\nu\delta}{2}+\sum_{\nu=1}^{n}\bar A(x)\sin\nu\delta=-2I+J,$$

say. By Abel's transformation

$$|I|\leqq\left|\sum_{\nu=0}^{n}r_\nu(x)\{\sin^2(\nu+1)\delta/2-\sin^2\nu\delta/2\}\right|+\left|r_n(x)\sin^2(n+1)\delta/2\right|$$

$$=O\left(\sum_{\nu=1}^{n}\nu^{-\delta}\nu\delta^2\right)+O(n^{-\alpha}n^2\delta^2)=O(\delta^\alpha).$$

Putting $\bar{\tau}_n(x) \equiv \bar{f}(x) - \bar{\sigma}_n(x) = \frac{1}{n+1}\sum_{\nu=1}^{n} \bar{r}_\nu(x)$, we have, combining the known results for $0 < \alpha < 1$ and Lemma 4,

$$\tau_n(x) = O(n^{-\alpha}) \tag{5}$$

uniformly in x. Applying Abel's transformation twice[8] in J

$$J = -\sum_{\nu=0}^{n} (\nu+1)\bar{\tau}_\nu(x)\{\sin \nu\delta - 2\sin(\nu+1)\delta + \sin(\nu+2)\delta\}$$
$$+ (n+1)\bar{\tau}_n(x)\{\sin(n+2)\delta - \sin(n+1)\delta\} - \bar{r}_n(x)\sin(n+1)\delta.$$

Using (5) and Lemma 3,

$$|J| = \sum_{\nu=1}^{n} O(\nu\, \nu^{-\alpha}\delta^2) + O(n\, n^{-\alpha}\delta) + O(n^{-\alpha}\, n\delta)$$
$$= O(\delta^2 n^{2-\alpha}) + O(\delta n^{1-\alpha}) = O(\delta^\alpha).$$

Summing up the estimations of $|I|$ and $|J|$, we conclude that

$$s_n(x+\delta) - s_n(x) = O(\delta^\alpha)$$

uniformly in x and n.

Thus (ii) and (iii) of Theorem 1 were proved at the same time. q.e.d.

REMARK. In Theorem 1 (iii), the assumption that $f(x)$ belongs to Lip 1 is indispensable. For example the function

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$$

satisfies the equality (3) for $\alpha = 1$, and does not belong to Lip1.

§ 2. For the integrated Lipschitz condition, we have the theorem analoguous to Theorem 1:

THEOREM 2. *Let $f(x)$ be a function of period 2π and integrable L_p $(p \geqq 1)$, and we adopt the same notations as in* § 1.

(i) *If $0 < \alpha \leqq 1$ and*

$$\left(\int_0^{2\pi} \left|s_n(x+\delta) - s_n(x)\right|^p dx\right)^{1/p} = O(|\delta|^\alpha) \tag{6}$$

uniformly in n, then $f(x)$ belongs to Lip(α, p):

8) In the case $0 < \alpha < 1$, it is enough for our purpose to apply Abel's transformation only once.

$$(7) \qquad \left(\int_0^{2\pi} \left|f(x+\delta) - f(x)\right|^p dx\right)^{1/p} = O(|\delta|^\alpha),$$

and

$$(8) \qquad \left(\int_0^{2\pi} \left|f(x) - s_n(x)\right|^p dx\right)^{1/p} = O(n^{-\alpha}).$$

(ii) *If* $0 < \alpha < 1$, *and* (8) *holds, then* $f(x)$ *belongs to* Lip (α, p) *and* (6) *holds uniformly in n.*

(iii) *If* $f(x)$ *belongs to* Lip $(1, p)$, *then* (8) *implies the equality* (6) *uniformly in n for* $\alpha = 1$.

The proof may be done under the same device of the proof of Theorem 1, and we shall omit it stating the main lemmas to be used.

Lemma 6[9]. *If* $f(x)$ *belongs to* Lip (α, p) $(0 < \alpha \leqq 1,\ p \geqq 1)$, *then*

$$\left(\int_0^{2\pi} \left|f(x) - \frac{1}{2}\left\{s_n\left(x + \frac{\pi}{2n}\right) + s_n\left(x - \frac{\pi}{2n}\right)\right\}\right|^p dx\right)^{1/p} = O(n^{-\alpha}).$$

Lemma 7[10]. *If* $\alpha > 0$, $p \geqq 1$ *and*

$$\left(\int_0^{2\pi} |f(x) - s_n(x)|^p dx\right)^{1/p} = O(n^{-\alpha}),$$

then

$$\left(\int_0^{2\pi} |\bar{f}(x) - \bar{s}_n(x)|^p dx\right)^{1/p} = O(n^{-\alpha}).$$

Lemma 8[11]. *If* $f(x)$ *belongs to* Lip $(1, p)$, *then*

$$\left(\int_0^{2\pi} |\bar{f}(x) - \bar{\sigma}_n(x)|^p dx\right)^{1/p} = O(1/n).$$

§3. We shall add some remarks concering the Lipschitz conditions of the Cesàro means of Fourier series.

As we saw in §1, the uniform Lipschitz condition of partial sums has the close relation with the order of approximation of the function by partial sums. On the contrary, that of the Cesàro means of order $\beta\,(0 < \beta \leqq 1)$: $\sigma_n^{(\beta)}(x)$ has no effect even on the approximation of $f(x)$ of Lip β, by $\sigma_n^{(\beta)}(x)$. In fact we may prove the following

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9) The proof may be done similarly as Lemma 2, using the Minkowski inequality.
10) R.Salem-A.Zygmund, loc. cit.
11) A.Zygmund, loc. cit. [6]

THEOREM 3. *Let $f(x)$ be a continuous function of period 2π, and $\sigma_n^{(\beta)}(x)$ denote the (C,β) mean of Fourier series of $f(x)$, $(0<\beta\leqq 1)$, then a necessary and sufficient condition that $f(x)$ belongs to* Lip α $(0<\alpha\leqq 1)$, *is that*

$$\max_x |\sigma_n^{(\beta)}(x+\delta)-\sigma_n^{(\beta)}(x)| = O(|\delta|^\alpha)$$

uniformly in n.

PROOF. Let $f(x)$ belong to Lip α. By the usual notation, as we know,

$$\sigma_n^{(\beta)}(x) = \frac{1}{\pi}\int_0^\pi \{f(x+t)+f(x-t)\} K_n^{(\beta)}(t)\,dt,$$

where $K_n^{(\beta)}(t)$ is the (C,β) mean of Dirichlet's kernel, and it satisfies the inequalities

$$|K_n^{(\beta)}(t)| \leqq 2n,\ |K_n^{(\beta)}(t)| \leqq Cn^{-\beta}t^{-\beta-1} \text{ for } 1/n \leqq t \leqq \pi,$$

C being an absolute constant (supposing $0<\beta<1$). Hence we find that

$$\begin{aligned}\left|\sigma_n^{(\beta)}(x+\delta)-\sigma_n^{(\beta)}(t)\right| &= \frac{1}{\pi}\left|\int_0^\pi \Big\{f(x+\delta+t)+f(x+\delta-t)\right.\\ &\qquad\left. -f(x+t)-f(x-t)\Big\} K_n^{(\beta)}(t)\,dt\right|\\ &= O\left(\int_0^{1/n}|\delta|\ 2n\,dt\right)+O\left(\int_{1/n}^\pi |\delta|^\alpha n^{-\beta}t^{-\beta-1}\,dt\right)\\ &= O(|\delta|^\alpha),\end{aligned}$$

which proves the necessity.

The sufficiency is almost evident.

Analogously we may prove the

THEOREM 4. *A necessary and sufficient condition that $f(x)$ belongs to* Lip (α, p) $(\alpha>0,\ p\geqq 1)$, *is that, for $\beta>0$,*

$$\left(\int_0^{2\pi} |\sigma_n^{(\beta)}(x+\delta)-\sigma_n^{(\beta)}(x)|^p\,dx\right)^{1/p} = O(|\delta|^\alpha)$$

uniformly in n.

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