ON SYSTEMS OF STRICT IMPLICATION

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(Received May 1, 1950)

Introduction. Lewis attempted to construct a logistic system[1] containing an implication-relation $\prec$ such that $p \prec q$ (p strictly implies q) is synonymous with "q is deducible from p". In his calculus, the propositions,

19.74 $\quad \sim \Diamond p \cdot \prec \cdot p \prec q,$

that is, if p is impossilbe then p strictly implies any proposition, and

19.75 $\quad \sim \Diamond \sim p \cdot \prec \cdot q \prec p,$

that is, if p is necessary then any proposition q strictly implies p, and further some paradoxical propositions have been proved[2]. Emch[3] and Vredenduin[4] showed that such paradoxical propositions should *not always* hold in usual logic, and they attempted in two different points of view, to construct new systems, the implication-relations of which seem to accord with the usual deducibility. Vredenduin accepts only $p \prec q \cdot \prec \cdot \sim \Diamond (p \sim q)$ as a postulate, but not $\sim \Diamond (p \sim q) \cdot \prec \cdot p \prec q$, then he assumes $\prec$ as an *undefined* term. On the other hand, Emch assumes a unary operation $\bigcirc$ as an *undefined* term, by which he defines his implications $\backsim$, and develops his system in the analogous way to Lewis.

It is the purpose of this paper to present, in I some investigation of Vredenduin's suggestions, and certain properties in his system according to Mckinsey's results[5], in II the equivalence between Vredenduin's system and Emch's one, and in III certin extensions of their systems from a viewpoint of modality.

I. *Vredenduin's calculus of propositions* is as follows: Undefined ideas; elementary propositions p, q, r, *etc.*, negation $\sim p$, possibility $\Diamond p$, product pq or $p \cdot q$, implication $p \prec q$, and equivalence $p = q$.

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1) LEWIS AND LANGFORD, Symbolic Logic.

2) *Op. cit.*, p. 248.

3) A. F. EMCH, Implication and deducibility, Journ. of Symbolic Logic, vol. 1 (1936), pp. 26–35; Addendum to this paper, *op. cit.* p. 58.

C. I. LEWIS, Emch's calculus and strict implication, *op. cit.*, pp. 77–86.

A. F. EMCH, Deducibility with respect to necessary and impossible propositions, *op. cit.*, vol. 2 (1937), pp. 78–81.

4) P. G. J. VREDENDUIN, A system of strict implication, *op cit.*, vol. 14 (1939), pp. 73–76.

5) J. C. C. MCKINSEY, On the number of complete extensions of the Lewis's system, *op. cit.*, vol. 9 (1944), pp. 42–45.

J. C. C. MCKINSEY, proof that there are infinitely many modalities in S2, *op. cit.*, vol. 5 (1940), pp. 110–112.

Postulates;

V1 $pq \prec qp$
V2 $pq \prec p$
V3 $p \prec pp$
V4 $(pq) r \prec p (pr)$
V5 $p \prec \sim\sim p$
V6 $p \prec q \cdot q \prec r : \prec \cdot p \prec r$
V7 $p \cdot p \prec q : \prec \cdot q$
V8 $\Diamond (pq) \prec \Diamond p$
V9 *Substitution* (*a*)
V10 *Substitution* (*b*)
V11 *Adjunction*
V12 *Inference*
V13 $\sim p \prec q \cdot \prec \cdot \sim q \prec p$
V14 $pq \prec r \cdot \prec \cdot p \sim r \prec \sim q$
V15 $p \prec q \cdot \prec \cdot qr \prec pr$
V16 $p \prec q \cdot r \prec s : \prec \cdot pr \prec qs$
V17 $p \prec \Diamond p$
V18 $p \prec q \cdot \Diamond p : \prec \cdot \Diamond q$
V19 $p \prec q \cdot \prec \cdot \sim \Diamond (p \sim q)$

Definitions;

V01 $p \vee q \cdot = \cdot \sim (\sim p \sim q)$
V02 $p = q \cdot = \cdot p \prec q \cdot q \prec p$
V03 $p \supset q \cdot = \cdot \sim (p \sim q)$
V04 $p \equiv q \cdot = \cdot p \supset q \cdot q \supset p$
V05 $p \circ q \cdot = \cdot \Diamond (pq)$

1. It is obvious that this system is included in *Lewis's system* S2. Vredenduin states in his paper that *the asserted propositions* 17.51, 17.52 *and* 19.47 *of S2 can not be deduced from his assumptions*, but if it were, 16.33, 16.34, 17.5, 19.46, 19.48, 19.49, 19.5, 19.51 and 19.52 could not be deduced because any of them can deduce some of 17.51, 17.52 and 19.47 in his calculus. (See the later proofs of 17.51 etc.) We are sure that these propositions have no paradoxical structure. In the following, we will show that *they are all deducible in his calculus.* The head numbers of propositions shall be identical with those in *Symbolic Logic.*

12.1 —— 16.32 and 16.4 —— 16.86 are proved in similar way in *Symbolic Logic.*

LEMMA 1. $p \prec q \cdot \prec : p \vee r \cdot \prec \cdot q \vee r$ [V15, 12.44, V01, 12.3]

LEMMA 2. $p \prec q \cdot r \prec s : \prec : p \vee r \cdot \prec \cdot q \vee s$ [V16, 12.44, V01, 12.3]

LEMMA 3. $p \sim p \cdot \prec \cdot q, \quad p \cdot \prec \cdot q \vee \sim q$

[V14] $pq \prec p : \prec : p \sim p \cdot \prec \cdot \sim q$

[V 2] QED.

LEMMA 4. $p \cdot = \cdot p (q \vee \sim q), \quad p \cdot = \cdot p \vee (q \sim q)$

[V15] $p \cdot \prec \cdot q \vee \sim q : \prec : pp \cdot \prec \cdot (q \vee \sim q) p$

[LEM. 3, 12.7, 12.15] $p \cdot \prec \cdot p (q \vee \sim q)$ (1)

[V 2] $p (q \vee \sim q) \cdot \prec \cdot p$ (2)

[(1), (2), V02] QED.

LEMMA 5. $p \cdot \prec \cdot q \vee r : = : \sim r \cdot \prec \cdot q \vee \sim p$ [12.6, 12.44, 12.3, V01, V2]

16.33 $p \prec q \cdot = \cdot p \prec pq$

[V15] $p \prec q \cdot \prec \cdot pp \prec qp$

[12.7, 12.15] $p \prec q \cdot \prec \cdot p \prec pq$ (1)

[LEM. 1] $p \prec pq \cdot \prec : p \vee \sim p \cdot \prec \cdot pq \vee \sim p$

[16.73] $p \prec pq \cdot \prec : p \vee \sim p \cdot \prec \cdot (p \vee \sim p)\ (q \vee \sim p)$

[LEM. 4] $p \prec pq \cdot \prec : p \vee \sim p \cdot \prec \cdot q \vee \sim p$

[LEM. 5] $p \prec pq \cdot \prec : p \cdot \prec \cdot q \vee (p \sim p)$

[LEM. 4] $p \prec pq \cdot \prec : p \prec q$ (2)

[(1), (2)] QED.

16.34 $p \prec q : = : p \vee q \cdot \prec \cdot q$ [16.33, 12.44]

16.35 $p \cdot = : p \vee q \cdot p$

[13.2] $p \prec p \vee q$

[16.33] $p : \prec : p \cdot p \vee q$ (1)

[V 2] $p \cdot p \vee q : \prec : p$ (2)

[(1), (2)] QED.

17.01 —— 17.4 is easily proved except 17.01′ and 17.12′,

†17.01′ $\sim (p \prec \sim q) \cdot \prec \cdot p \circ q$

†17.12′ $\sim (p \circ \sim q) \cdot \prec \cdot p \prec q$

*17.5 is considered later. 17.51 and 17.52 are proved against the statement of Vredenduin as follows:

17.51 $p \prec \sim r \cdot q \prec r : \prec \cdot \sim (p \circ q)$

[V16] $p \prec \sim r \cdot q \prec r : \prec : pq \cdot \prec \cdot r \sim r$ (1)

[V19] $pq \cdot \prec \cdot r \sim r : \prec : \sim \Diamond (pq \sim (r \sim r))$ (2)

[V01] $\sim \Diamond : (pq \sim (r \sim r)) \cdot = \cdot \sim \Diamond (pq (r \vee \sim r))$

[LEM. 4] $\cdot = \cdot \sim \Diamond (pq)$

[V05] $\cdot = \cdot \sim (p \circ q)$ (3)

[(1), (2), (3), V6] QED.

17.52 $p \prec q \cdot p \prec \sim q : \prec \cdot \sim (p \circ p)$ [17.5, q/p p/q]

17.53 —— 17.71 are all proved except 17.592, 17.7 and 17.71, where 17.7, 17.71 need not to be considered, because of 19.692.

*17.592 $p \circ p \cdot \prec : p \circ q \cdot \vee \cdot p \circ \sim q$

18.1 —— 18.92 are proved except the following propositions:

†18.1′ $\sim (p \prec \sim p) \cdot \prec \cdot \Diamond p$

†18.12′ $\sim \Diamond p \cdot \prec \cdot p \prec \sim p$

†18.13′ $\sim (\sim p \prec p) \cdot \prec \cdot \Diamond \sim p$

†18.14′ $\sim \Diamond \sim p \cdot \prec \cdot \sim p \prec p$

†18.2′ $\sim (p \sim q \cdot \circ \cdot p \sim q) \cdot \prec \cdot p \prec q$

†18.3′ $\sim (p \prec \sim q) \cdot \prec \ p \circ q$

†18.31′ $\sim (p \circ q) \cdot \prec \cdot p \prec \sim q$

†18.35′ $\sim (pq \cdot \prec \sim r) \prec \Diamond (pqr)$ etc.

†18.36′ $\sim (qrs \cdots \prec \cdot \sim p) \cdot \prec \cdot \Diamond (pqrs \cdots)$ etc.

†18.61 $\sim \Diamond \sim p \cdot pq \prec r : \prec \cdot q \prec r$

†18.7′ $\sim \Diamond \sim (p \supset q) \cdot \prec \cdot p \prec q$

19.02 —— 19.451 are easily deduced.

19.51 $p \prec r \cdot \prec \cdot pq \prec r$

[16.33] $p \prec r \cdot = \cdot p \prec pr$ (1)

[V15] $p \prec pr \cdot \prec \cdot pq \prec prq$ (2)

[16.33] $pq \prec r \cdot = \cdot pq \prec pqr$ (3)
[(1), (2), (3)] QED.
19.52 $q \prec r \cdot \prec \cdot pq \prec r$ [19.51, 12.15]
19.5 $p \prec r \cdot \vee \cdot q \prec r : \prec : pq \prec r$ [LEM. 2, 19.51, 19.52, 13.31]
19.48 $p \prec q : \prec : p \cdot \prec \cdot q \vee r$ [19.51, 12.44]
19.49 $p \prec r : \prec : p \cdot \prec \cdot q \vee r$ [19.52, 12.44]
19.46 $p \prec q \cdot \vee \cdot p \prec r : \prec : p \cdot \prec \cdot q \vee r$ [19.48, 19.49 LEM. 2]
19.47 $p \prec q \cdot \vee \cdot p \prec r \cdot \vee \cdot \sim q \prec r : \prec : p \cdot \prec \cdot q \vee r$
[19.48] $\sim q \prec r : \prec : \sim q \cdot \prec \cdot r \vee \sim p$ (1)
[LEM. 5] $\sim q \cdot \prec \cdot r \vee \sim p : = : p \cdot \prec \cdot r \vee q$ (2)
[(1), (2), 13.11] $\sim q \prec r : \prec : p \cdot \prec \cdot q \vee r$ (3)
[(3), 19.46, LEM. 2, 13.31] QED.
19.57 $p \cdot q \sim q : = : q \sim q$
[LEM. 3] $q \sim q \cdot \prec : p \cdot q \sim q$ (1)
[12.17] $p \cdot q \sim q : \prec \cdot q \sim q$ (2)
[(1), (2)] QED.
19.58 (= second part of LEM. 4)
19.6 is identical with V15, and 19.61 is a special case of V16.
19.62 $p \prec qr : \prec : p \prec q \cdot p \prec r$
[V15] $p \prec qr : \prec : p \sim r \cdot \prec \cdot qr \sim r$
[19.57] $p \prec qr : \prec : p \sim r \cdot \prec \cdot r \sim r$ (1)
[12.6] $p \sim r \cdot \prec \cdot r \sim r : = : p(r \vee \sim r) \cdot \prec \cdot r$
[LEM. 4] $: = : p \prec r$ (2)
[(1), (2)] $p \prec qr \cdot \prec \cdot p \prec r$ (3)
[(3), 12.15] $p \prec qr \cdot \prec \cdot p \prec q$ (4)
[(3), (4), V16] QED.
19.63 $p \prec qr : = : p \prec q \cdot p \prec r$ [19.61, 19.62]
19.64 is identical with LEM. 1.
19.65 is a special case of LEM. 2.
19.66 —— 19.682 are easily deduced from the above formulas.
∗19.69 $p \cdot \circ \cdot q \vee r : \prec : p \circ q \cdot \vee \cdot p \circ r$ is considered later.
19.692 —— 19.92 are easily proved except the following parts:
∗19.692′ (= 19.69), ∗19.7′ (= 17.592)
∗19.71′ $\Diamond p \cdot \prec : \Diamond (pq) \cdot \vee \cdot \Diamond (p \sim q)$
†19.72′-1 $\sim \Diamond p : \prec : p \prec \sim q \cdot p \prec q$
†19.72′-2 $\sim (p \circ q) \sim (p \circ \sim q) : p \prec \sim q \cdot p \prec q$
∗19.72′-3 $\sim (p \circ q) \cdot \sim (p \circ \sim q) : \prec : \sim \Diamond p$
†19.73′ $\sim \Diamond \sim p \prec : q \prec p \cdot \sim q \prec p$
†19.74, †19.75
†19.76 $\sim (p \prec q) \cdot \prec \cdot \Diamond p$
19.77 $\sim (q \prec p) \cdot \prec \cdot \Diamond \sim p$
∗19.8′ $\sim \Diamond p \sim \Diamond q \cdot \prec \cdot \sim \Diamond (p \vee q)$
∗19.81′ $\sim \Diamond \sim p \sim \Diamond \sim q \cdot \prec \cdot \sim \Diamond \sim (pq)$

$*19.82'$ $\Diamond(p \vee q) \cdot \prec : \Diamond p \cdot \vee \cdot \Diamond q$
$\dagger 19.83$ $\sim\Diamond p \sim\Diamond q \cdot \prec \cdot p = q$
$\dagger 19.84$ $\sim\Diamond \sim p \sim\Diamond\sim q \cdot \prec \cdot p = q$

2. $*$-Propositions. (i) It is obvious that 17.592, 19.71′ and 19.72′–3 are deducible from one another, and 19.8′, 19.81′ and 19.82′ are deducible from one another.

(ii) 19.82′ is deducible from 19.71′:

[19.71′] $\Diamond(p \vee q) \cdot \prec : \Diamond(p \vee q \cdot q) \cdot \vee \cdot \Diamond(p \vee q \cdot \sim q)$ (1)
[16.35, 16.72] $\Diamond(p \vee q \cdot q) \cdot \vee \cdot \Diamond(p \vee q \cdot \sim q)$
$:=: \Diamond q \vee \cdot \Diamond(p \sim q \cdot \vee \cdot q \sim q)$
[LEM. 4] $:=: \Diamond q \cdot \vee \cdot \Diamond(p \sim q)$ (2)
[V8] $\Diamond(p \sim q) \prec \Diamond p$
[LEM. 1] $\Diamond(p \sim q) \cdot \vee \cdot \Diamond q : \prec : \Diamond p \cdot \vee \cdot \Diamond q$ (3)
[(1), (2), (3), V6] $\Diamond(p \vee q) \cdot \prec : \Diamond p \cdot \vee \cdot \Diamond q$ QED.

(iii) 19.69 is deducible from 19.82′:

[19.82′] $\Diamond(pq \cdot \vee \cdot pr) : \prec : \Diamond(pq) \cdot \vee \cdot \Diamond(pr)$ (1)
[16.72] $pq \cdot \vee \cdot pr : = : p(q \vee r)$ (2)
[(1), (2), V05] $p \cdot \circ \cdot q \vee r : \prec : p \circ q \cdot \prec \cdot p \circ r$ QED.

(iv) 17.592 is deducible from 19.69. [LEM. 4, 12.7]

(v) 17.5 is deducible from 19.81′:

[16.35] $pr : = : (pr \vee q)pr$
[16.72] $: = : pr \vee pqr$
[12.1] $\sim\Diamond(pr) \cdot \prec \cdot \sim\Diamond(pr \cdot \vee \cdot pqr)$ (1)
$\sim\Diamond(q \sim r) \cdot \prec \cdot \sim\Diamond(q \sim r \cdot \vee \cdot pq \sim r)$ (2)
[(1), (2), V16] $\sim\Diamond(pr) \sim\Diamond(q \sim r) \cdot \prec \cdot \sim\Diamond(pr \cdot \vee \cdot pqr)$
$\sim\Diamond(q \sim r \cdot \prec \cdot pq \sim r)$ (3)
[19.81] $\sim\Diamond(pr \cdot \vee qpr) \sim\Diamond(q \sim r \cdot \vee \cdot pq \sim r)$
$: = : \sim\Diamond(pr \cdot \vee \cdot pqr \cdot \vee \cdot q \sim r \cdot \vee \cdot pq \sim r)$
[16.72] $: = : \sim\Diamond(pr \cdot \vee \cdot q \sim r \cdot \vee \cdot pq(r \vee \sim r))$
[LEM. 4] $: = : \sim\Diamond(pr \cdot \vee \cdot q \sim r \vee \cdot pq)$
[19.82] $: = : \sim\Diamond(pr \cdot \vee q \sim r) \sim\Diamond(pq) : = : \sim\Diamond(pr)$
$\sim\Diamond(q \sim r) \sim\Diamond(pq)$ (4)
[(3), (4)] $\sim\Diamond(pr) \sim\Diamond(q \sim r) \cdot \prec \cdot \sim\Diamond(pr) \sim\Diamond(q \sim r) \sim\Diamond(pq)$
[16.33] $\sim\Diamond(pr) \sim\Diamond(q \sim r) \cdot \prec \cdot \sim\Diamond(pq)$
[V05] $\sim(p \circ r) \sim(q \circ \sim r) \cdot \prec \cdot \sim(p \circ q)$ QED.

(vi) 19.7′ is deducible from 17.5 [17.5, p/q q/r, 12.44]

Hence, every $*$-proposition is deducible from one another in Vredenduin's system, but we can show that they are not deducible from his system. In order to deduce them we translate V18 into a new stronger postulate V18′;

V18′ $\sim\Diamond(p \sim q) \cdot \Diamond p : \prec \cdot \Diamond q$

If we designate this system by V_2, then in V_2, we can deduce V18 and all the $*$-propositions which have no paradoxical structure, and also Vredenduin's aim is attained as will be shown later.

[V18′] $\sim\Diamond(p\sim q)\Diamond p\cdot\prec\cdot\Diamond q$

[V14] $\sim\Diamond(p\sim q)\sim\Diamond q\cdot\prec\cdot\sim\Diamond q$

[$p\vee q/p$] $\sim\Diamond(p\vee q\cdot\sim q)\sim\Diamond q\cdot\prec\cdot\sim\Diamond(p\vee q)$ (1)

[16.72, 12.15] $p\vee q\cdot\sim q:=:p\sim q\cdot\vee\cdot q\sim q$

[LEM. 4] $:=:p\sim q$ (2)

[19.16, V15] $\sim\Diamond p\sim\Diamond q\cdot\prec\cdot\sim\Diamond(p\sim q)\sim\Diamond q$ (3)

[(1), (2), (3), V6] $\sim\Diamond p\sim\Diamond q\cdot\prec\cdot\sim\Diamond(p\vee q)$ (= 19.81′)

Hence, $*$-propositions have been all asserted in V_2.

3. †-propositions and certain properties of the system V_2. We next consider about *†-propositions*, which seem to have more or less evidently paradoxical structures. As Vredenduin shows the independency of 19.74 and 19.75, of his system, we can show it in his way that none of †-propositions can be deduced in this system V_2. As he states, all the assumptions of the system V_2 are altered to asserted propositions of the system S2 if the $\Diamond$-symbols are omitted. Effectively, postulate V18′ are then altered to $\sim(p\sim q)\cdot p:\prec\cdot q$. In the system S2, $\sim(p\sim q)p:=:\sim p\vee q\cdot p:=:\sim pp\cdot\vee\cdot qp:=:pq$, and $pq\prec q$ are asserted, hence that is asserted. By Vredenduin's method we can show that 18.1′, 19.61, 19.75, 19.84 are all independent of the system V_2.[6] On the other hand, we can deduce any of 18.1′, 19.61, 19.75, 19.84 from †11.01′ $\sim\Diamond(p\sim q).\ \prec\cdot p\prec q$ or from any of the remainders of †-propositions in the system V_2, then every paradoxical proposition is independent.

Further, the following fact is to be noticed: Halldén Sören shows[7] that certain analogues of the paradoxes are deducible in S1, in which the *consistency postulate* regarded as the cause of the paradoxes is independent, namely

†(1) $\sim\Diamond p\cdot\supset\cdot p\prec q$

†(2) $\sim\Diamond\sim p\cdot\supset\cdot q\prec p$.

It can easily be shown in such a way as above, that (1) and (2) are not deducible in the system V_2. ((1); $p=3$, $q=2$, (2), $p=2$, $q=1$, in

6) 18.1′ ...... $\sim(p\prec\sim p)\cdot\prec\cdot p$ (1)
18.61 ...... $\sim(\sim p)\cdot pq\prec r:\prec\cdot q\prec r$ (2)
19.84 ...... $pq\cdot\prec\cdot p=q$ (3)

Lewis and Langford give the following normal S2-matrix (S. L., p. 493, Group I)

pq	1	2	3	4	$\sim p$	$\Diamond p$	$p\prec q$	1	2	3	4
1	1	2	3	4	4	1	1	2	4	4	4
2	2	2	4	4	3	1	2	2	2	4	4
3	3	4	3	4	2	1	3	2	4	2	4
4	4	4	4	4	1	3	4	2	2	2	2

Every proposition asserted in S2 has one of the designated values 1 and 2. Now choose $p=2$, then (1) has the value 4, choose $p=2$, $q=1$ and $r=2$, then (2) has 4; choose $p=1$ and $q=2$, then (3) has 4. Hence (1), (2) and (3) are independent of S2.

7) HALLDÉN SÖREN, A note concerning the paradoxes of strict implication, Journ. of Symbolic Logic, vol. 13 (1948), pp. 138–139.

Group 1.)

THEOREM 1. Every asserted proposition of S2 in *Symbolic Logic* is asserted in the system V_2 except †-propositions which are not deducible in V_2.

The following theorem are evident by the comparison with the assumptions of the systems S2 and V_2.

THEOREM 2. Every asserted proposition of V_2 is also asserted in S2.

THEOREM 3. The system V_2 has infinitely many complete extensions[8].

This is easily shown by the Theorem 2 and by the fact that the system S2 has infinitely many complete extensions.[8]

THEOREM 4. The number of irreducible molalities[9] in the system V_2 is infinite.

The proof is trivial by the Theorem 2 and by the fact that there are an infinite number of irreducible molalities in the system S2[10]

II. Relations among Lewis's system, Emch's and Vredenduin's.

Designate the following *system* by S2°, which are obtained when the primitive symbol $\Diamond$ is altered to symbol $\bigcirc$. Let us add to the system V_2 a definition

V06 $\qquad \bigcirc p \cdot = \cdot \sim (p \prec \sim p).$

Designate Emch's system by E_2, which are obtained when we translate his symbol ∞ (logical implcation) into $\prec$, $\prec$ (strict implication) into $<$, logical equivalence into $=$, and strict equivalence into I in Emch's system[11],

Emch's system E_2:

Undefined ideas; Elementary propositions p, q, r, *etc.*, negation $\sim p$, product pq or $p \cdot q$, possibility $\Diamond p$, consistency $\bigcirc p$, and equivalence $p = q$.

Postulates;		Definitions:	
L1	$pq \prec qp$	L01	$p \vee q \cdot = \cdot \sim (\sim p \sim q)$
L2	$p \prec pp$	L02	$p \prec q \cdot = \cdot \sim \bigcirc (p \sim q)$
L3	$(pq) r \prec p (qr)$	L03	$p = q \cdot = \cdot p \prec q \cdot q \prec p$
L4	$p \prec \sim (\sim p)$	(L04)	$p < q \cdot = \cdot \sim \Diamond (p \sim q)$
L5	$p \prec q \cdot q \prec r : \prec \cdot p \prec r$	(L05)	$p \,\mathrm{I}\, q \cdot = \cdot p < q \cdot q < p$
L6	$p \cdot p \prec q : \prec q$	L06	$p \supset q \cdot = \cdot \sim (p \sim q)$
L7	$\Diamond p \cdot \Diamond q \cdot \prec \cdot \bigcirc p \bigcirc q$	L07	$p \equiv q \cdot = \cdot p \supset q \cdot q \supset p$
L8	$\Diamond (pq) \cdot \prec \cdot \Diamond p$		

8) McKINSEY, *op. cit.*, the first paper.

9) W. T. PARRY, Modalities in the survey system of strict implication, Journ. of Symbolic Logic, vol. 4 (1939), p. 144.

10) McKINSEY, *op. cit.*, the second paper.

11) Emch had an attempt such that the logical implication coincides with the usual deducibility, then there arises no essential absurdity though we alter the symbol.

L11 $p \prec \Diamond p$
L12 $\bigcirc (pq) \cdot \prec \cdot \bigcirc p$
L13 $\Diamond (pq); \prec : \Diamond (p \sim r) \cdot \vee \cdot \Diamond (rq)$
L14 *Substitution* (a)
L15 *Substituion* (b)
L16 *Adjunction*
L17 *Inference*

$qp \prec p$ is deducible in E_2, and the other assumptions of S2° are included in one of E_2, then the following theorem is shown:

THEOREM 5. The asserted propositions in S2° are all asserted in the system E_2[12].

Next, we show that the systems E_2 and V_2 are equivalent. Designate the corresponding proposition of S2° to one of S2 by the same number having the sign ° at the shoulder.

1. V1, V3 — V12, V17, V01 — V04 are included in the assumptions of the system E_2 and V2 is easily proved in E_2. V13, V14, V15, and V16 are proved in S2 and invariant by the above translation, then they are proved also in the system E_2 by Theorem 5.

V18′: [L8] $\Diamond (qp) \cdot \prec \cdot \Diamond q$
[12.15°] $\Diamond (pq) \cdot \prec \cdot \Diamond q$
[19.64°] $\Diamond (pq) \cdot \prec \cdot \Diamond (p \sim q) : \prec : \Diamond q \cdot \vee \cdot \Diamond (p \sim q)$ (1)
[L13] $\Diamond (pp) \cdot \prec : \Diamond (p \sim q) \cdot \vee \cdot \Diamond (pq)$ (2)
[(1), (2), L5, 12.7°, 13.11°] $\Diamond p \cdot \prec : \Diamond q \cdot \vee \cdot \Diamond (p \sim q)$
[12.44°, L01] $\sim \Diamond (p \sim q) \sim \Diamond q \prec \sim \Diamond p$
[12.6°] $\sim \Diamond (p \sim q) \Diamond p \cdot \prec \Diamond q$ QED.

V19 : [L7, 12.7°] $\Diamond p \prec \bigcirc p$
[12.44°] $\sim \bigcirc p \prec \sim \Diamond p$ (1)
[(1), $p \sim q/p$, L02] $p \prec q \cdot \prec \sim \Diamond (p \sim q)$ QED.

V06 : [L02] $\sim \bigcirc (p \sim \sim p) \cdot = \cdot (p \prec \sim p)$
[12.3°, 12.7°] $\bigcirc p \cdot = \cdot \sim (p \prec \sim p)$ QED.

2. L01. L03, L06, L07, L1 — L6, L8, L11, L14 — L17 are included in the assumptions of the system V_2.

L02 : [V06, 12.3, 12.44] $\sim \bigcirc p \cdot = \cdot p \prec \sim p$ (1)
[(1) $p \sim q/p$] $\sim \bigcirc (p \sim q) \cdot = \cdot p \sim q \cdot \prec \cdot \sim (p \sim q)$ (2)
[V01] $p \sim q \cdot \prec \cdot \sim (p \sim q) : = : p \sim q \cdot \prec \cdot \sim p \vee p$
[13.11] $: = : p \sim q \cdot \prec \cdot q \vee \sim p$
[LEM. 5] $: = : p \cdot \prec \cdot q \vee q \vee \sim p$
[13.31] $: = : p \cdot \prec \cdot q \vee \sim p$
[13.11] $: = : p \cdot \prec \cdot \sim p \vee q$
[LEM. 5] $: = : \sim q \cdot \prec \cdot \sim p \vee \sim p$
[13.31] $: = : \sim q \cdot \prec \cdot \sim p$

12) This was shown by Lewis, *op. cit.*, 3)

[12.44] $: = : p \prec q$ (3)
[(2),(3)] $\sim \bigcirc (p \sim q) \cdot = \cdot p \prec q$ QED.
L7 ; [V19, 12.44, 12.3, 12.7] $\Diamond p \prec \sim (p \prec \sim p)$ (1)
$\Diamond q \prec \sim (q \prec \sim q)$ (2)
[(1),(2),V16] $\Diamond p \Diamond q \cdot \prec \cdot \sim (p \prec \sim p) \sim (q \prec \sim q)$
[V06] $\Diamond p \Diamond q \cdot \prec \cdot \bigcirc p \bigcirc q$ QED.
L12 : [V15] $p \prec \sim p \cdot \prec : ppq \cdot \prec \cdot \sim ppq$ (1)
[12.7, 19.57] $ppq \cdot \prec \cdot \sim ppq : = : pq \cdot \prec \cdot p \sim p$
[12.6] $: = : p (p \vee \sim p) \cdot \prec \cdot \sim q$
[LEM. 4] $: = : p \prec \sim q$ (2)
[(1), (2)] $p \prec \sim p : \prec \cdot p \prec \sim q$
[V06, 12.44, L02] $\bigcirc (pq) \prec \bigcirc p$ QED.

L13 is easily proved by 17.5.

Hence the following theorem has been established;

THEOREM 6. The asserted propositions of the system V_2 are all asserted in the system E_2, and *vice versa.*

Thus we have concluded by the theorems 1, 2, 5 and 6 that (*i*) *the system S2 includes the systems V_2 and E_2*[13], (*ii*) *the systems V_2 and E_2 are equivalent, and* (*iii*) *the systems V_2 and E_2 include the system S2°, hence, both in V_2 and in E_2, we can deduce not only the propositions which have been asserted in S2, but also deducible one in S2, if they are invariant by the translation $\Diamond$ into $\bigcirc$*[14].

III. Certain extensions of system V_2. Becker[15] and the others[16] made attempts to construct extensions of the system S2 or S3 in view of modality. Whether can we hold the analogous extensions concerning to the system V_2 without loss of Vredenduin's purpose or not, and how many modalities have they? As Tang[17] and Parry[18] show in S2, we can deduce the following lemmas in the system V_2.

LEMMA 6. $p \prec q \cdot = \cdot p = pq$
[19.62, V2, V6] $p \cdot \prec pq \cdot \prec \cdot p \prec p \cdot$ (1)
[19.51] $p \prec p \cdot \prec \cdot pq \prec p$ (2)
[(1), (2), V6] $p \prec pq \cdot \prec \cdot pq \prec q$

13) We must, of course, add the Definition $\bigcirc p . = . \sim (p \prec \sim p)$ to the system S2.

14) If, in V_2, we prove only the propositions used in the proof of $V_2 \to E_2$, certain proofs in I, of course, are unnecessary.

15) O. BECKER, Zur Logik der Modalitäten, Jahrbuch für Philosophie und phänomenologische Forschung, vol. 11(1930), pp 497–548.

16) PARRY, op. cit. and C. W. CHURCHMAN, On finite and infinite modal systems, Journ. of Symbolic Logic, vol. 3 (1938), pp. 77–82

17) T. C. TANG, The theorem $p \prec q . = . pq = p$ and Huntington's relation between Lewis's strict implication and Boolean algebra. Bull. Amer. Math. Soc., vol. 42 (1936), pp. 743–746.
E. V. HUNTINGTON, Postulates for assertion, conjunction, negation, and equality., Proceed. of Arts and Sciences, vol. 72 (1938), pp. 1–44, Theorem 97 (p. 24)

18) Parry, *op. cit.*

[16.33] $p \prec pq \cdot \prec : p \prec pq \cdot pq \prec p$ (3)
[V2] $p \prec pq \cdot pq \prec p : \prec \cdot p \prec pq$ (4)
[(3), (4), V02] $p = pq \cdot = \cdot p \prec pq$
[16.33] $p = pq \cdot = \cdot p \prec q$ QED.

LEMMA 7. If $p \prec q$ has been asserted, then $\Diamond p \prec \Diamond q$, $\sim \Diamond q \prec \sim \Diamond p$, and $\sim \Diamond \sim \Diamond p \prec \sim \Diamond \sim \Diamond q$ may be asserted.

[Hyp., LEM. 6] $p = pq$ (1)
[V8] $\Diamond (qp) \cdot \prec \cdot \Diamond q$ (2)
[(1), (2)] $\Diamond p \cdot \prec \cdot \Diamond q$
[12.44] $\sim \Diamond q \cdot \prec \cdot \sim \Diamond p$ (3)
[(3)] $\sim \Diamond \sim \Diamond p \cdot \prec \cdot \sim \Diamond \sim \Diamond q$ QED.

LEMMA 8. $\sim \Diamond \sim p \cdot \prec \cdot \sim \Diamond \sim \Diamond p$. [18.41, LEM. 7]

1. Designate by V_4 the system deduced from the set (V_2 and the following postulate C10).

C10 $\sim \Diamond \sim p \cdot \prec \cdot \sim \Diamond \sim \sim \Diamond \sim p$

Then the following lemmas can be deduced in V_4.

LEMMA 9. $\Diamond_n p = \Diamond p$, where by "$\Diamond_n p$" we mean the formula which is formed by putting n "$\Diamond$" symbols in front of p.

[C10, 12.3] $\sim \Diamond \sim p \cdot \prec \cdot \sim \Diamond_2 \sim p$
[12.44] $\Diamond_2 \sim p \cdot \prec \cdot \Diamond \sim p$
[12.3] $\Diamond_2 p \cdot \prec \cdot \Diamond p$ QED.

LEMMA 10. $\sim \Diamond p \cdot \prec \cdot \sim \Diamond \sim \Diamond \sim \Diamond p$

[LEM. 8, $\sim \Diamond p/p$] $\sim \Diamond \sim \sim \Diamond p \prec \sim \Diamond \sim \Diamond \sim \Diamond p$
[LEM. 9] $\sim \Diamond p \prec \sim \Diamond \sim \Diamond \sim \Diamond p$ QED.

LEMMA 11. $\sim \Diamond \sim \Diamond p \cdot = \cdot \sim \Diamond \sim \Diamond \sim \Diamond \sim \Diamond p$

[LEM. 10, LEM. 7] $\sim \Diamond \sim \Diamond \sim \Diamond \sim \Diamond p \cdot \prec \cdot \sim \Diamond \sim \Diamond p$ (1)
[LEM. 10, $\sim \Diamond p/p$] $\sim \Diamond \sim \Diamond p \cdot \prec \cdot \sim \Diamond \sim \Diamond \sim \Diamond \sim \Diamond p$ (2)
[(1), (2)] QED.

LEMMA 12. $\sim \Diamond \sim \Diamond \sim \Diamond \sim p \cdot \prec \cdot \Diamond \sim \Diamond \sim p$ [18.42], $\sim \Diamond \sim p \cdot \prec \cdot \sim \Diamond \sim \Diamond \sim \Diamond \sim p$ [LEM. 10] $\sim \Diamond \sim \Diamond \sim \Diamond \sim p \prec \sim \Diamond \sim \Diamond p$ [LEM. 7, 18.42], $\Diamond \sim \Diamond \sim p \prec \Diamond \sim \Diamond \sim \Diamond p$ [LEM. 8,7], $\sim \Diamond \sim \Diamond p \prec \Diamond \sim \Diamond \sim \Diamond p$ [V 17], $\Diamond \sim \Diamond \sim \Diamond p \prec \Diamond p$ [LEM. 10] and the propositions deduced from these propositions by 12.44.

This system V_4 is evidently included in S4 by Theorem 2, and it was proved by Parry[19] that these fourteen modalities cannot be further reduced in S4.

THEOREM 7. The propositions asserted in V_4 are all asserted in S4 and the number of irreducible modalities in V_4 is fourteen.

2. Designate by $V_{4\cdot5}$ the system deduced from the set (V_4 and a new postulate C 4.5).

C 4.5 $\sim \Diamond \sim \Diamond \sim \Diamond p \prec \sim \Diamond p$

LEMMA 13. $\sim \Diamond \sim \Diamond \sim \Diamond p \cdot = \cdot \sim \Diamond p$, $\Diamond \sim \Diamond \sim \Diamond p \cdot = \cdot \Diamond p$,

19) Parry, *op. cit.*

$$\Diamond\sim\Diamond\sim\Diamond\sim p \cdot = \cdot \Diamond\sim p, \quad \sim\Diamond\sim\Diamond\sim\Diamond\sim p \cdot = \cdot \sim\Diamond\sim p.$$

Hence the fourteen modalities are reduced to ten in $V_{4\cdot5}$.

THEOREM 8. The number of irreducible modalities in $V_{4\cdot5}$ is at most ten.

3. Designate by $V_{4\cdot51}$ the system deduced from the set (V_4 and a new postulate C 16).

C 16 $$\Diamond\sim\Diamond\sim p \cdot = \cdot \sim\Diamond\sim\Diamond p$$

Then, $$\sim\Diamond\sim p \prec \begin{matrix}\sim\Diamond\sim\Diamond p \\ p\end{matrix} \prec \Diamond p, \quad \sim\Diamond p \prec \begin{matrix}\Diamond\sim\Diamond p \\ \sim p\end{matrix} \prec \sim\Diamond p.$$

In the corresponding system of Parry, including this system $V_{4\cdot51}$, the above eight modalities can not be further reduced.

THEOREM 9. The number of irreducible modalities in $V_{4\cdot51}$ is eight.

4. Designate by V_5 the system deduced from the set (V_2 and a new postulate C 11).

C 11 $$\Diamond p \cdot \prec \cdot \sim\Diamond\sim\Diamond p$$

As in S5, it is easily proved that, C11 is deducible from C 10 and C 12 $p \cdot \prec \cdot \sim\Diamond\sim\Diamond p$ in V_5, and *vice versa.* C4. 5 is dedubible from C11 in V_5, hence V_4 and $V_{4\cdot5}$ are included in V_5.

LEMMA 14. $\Diamond p \cdot = \cdot \sim\Diamond\sim\Diamond p$

[18. 41, $\sim\Diamond p/p$] $\sim\Diamond\sim\Diamond p \cdot \prec \cdot \sim\sim\Diamond p$

[12. 3] $\sim\Diamond\sim\Diamond p \cdot \prec \cdot \Diamond p$ (1)

[(1), C11] QED.

Then, by Lemma 14 and the considerations in $V_{4\cdot5}$, every modality in V_5 is reduced to one of the following six $\sim\Diamond\sim p \cdot \prec \cdot p \cdot \prec \cdot \Diamond p$; $\sim\Diamond p \cdot \prec \cdot \sim p \cdot \prec \cdot \Diamond\sim p$.

On the other hand, this system V_5 is included in S5, and these six modalities can not be further reduced in S5, hence,

THEOREM 10. The propositions asserted in V_5 are all asserted in S5, and the number of irreducible modalities in V_5 is six.

It is noticed that all these extensions of the system V_2 have no paradoxical †-propositions as Vredenduin's system. If we omit the symbol $\Diamond$ in C11 and C 16, we have $p \cdot \prec \cdot \sim\sim p$ and $\sim\sim p \cdot = \cdot \sim\sim p$ respectively, and they are deducible in S2.

THEOREM 11. Any paradox of †-propositions can not be deduced in each extension V_4, $V_{4\cdot5}$, $V_{4\cdot51}$, V_5 of the system V_2.

The author expresses here his hearty thanks to Professor M. Ito to whom he has been indebted for his many valuable remarks and suggestions.

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