ON N. MATSUYAMA'S CLOSURE OPERATORS ON GENERAL NEIGHBOURHOOD SPACES

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1. N. Matsuyama[1] has recently introduced the interesting notion of φ-closure, by which a neighbourhood space turns to the space with φ-topology.

Now we will recall the definitions due to him.

Let S be a neighbourhood space and denote its points by $x, y, \cdots\cdot$. Suppose that for each x in S there corresponds at least one neighbourhood U_x of x such that

(N. 1) for each $x \in S$, U_x contains x,

(N. 2) if U_x and V_x are neighbourhoods of x, then $W_x = U_x \cap V_x$ is also a neighbourhood of x,

(N. 3) for any neighbourhood U_x of x, any set containing U_x is also a neighbourhood of x.

Let φ be a set-function on 2^S(= family of all subsets in S) such that

(φ. 1) for any subset A in S, $A \subset \varphi(A)$,

(φ. 2) $A \subset B$ implies $\varphi(A) \subset \varphi(B)$.

Let Φ be the class of all such φ.

For any φ_1 and φ_2 in Φ, we write $\varphi_1 < \varphi_2$ if and only if $\varphi_1(A) \subset \varphi_2(A)$ for all subsets A of S. Then Φ is a partially ordered system. Let $\varphi \in \Phi$. If there exists at least one sequence[2] $\{x_\alpha\}$ of points in A, and for each neighbourhood[3] V_x of x containing $\varphi(A)^c$ there exists an $\alpha_0 = \alpha_0(V_x)$ such that $\alpha > \alpha_0$ implies $x \in V_x$, then x is said to be a limiting point of A with respect to φ-topology and denote it by $x \in A^\varphi$. And A^φ is said to be φ-closure of A.

N. Matsuyama has proved that the φ-closure has several properties analogous to the closure operator of C. Kuratowski[4].

In this paper we will investigate the relations between the class Φ and the class A^φ from the viewpoint of order-topology, which he has not yet tried.

2. Let A and B are elements of 2^S. When we define $A > B$ if and only if $A \supset B$, then 2^S is a lattice.

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1) N. MATSUYAMA, A note on general topological spaces, Tôhoku Math. Journ. Second series, Vol. 1, (1949), pp. 22-25.

2) By a sequence we mean a finite or infinite directed system.

3) If such neighbourhood does not exist, we define that x is a limiting point of A with respect to φ-topology.

4) C. KURATOWSKI, Topologie I, pp. 15-16.

According to N. Matsuyama, if we define 0 and I by $0(A) \equiv A$ and $\mathrm{I}(A) \equiv S$ for all elements A of 2^S respectively, then 0 and I are contained in Φ and $0 < \varphi < \mathrm{I}$ for any $\varphi \in \Phi$. Further Φ is a lattice when we define $\varphi_1 \vee \varphi_2$, $\varphi_1 \wedge \varphi_2$ as follows: $(\varphi_1 \vee \varphi_2)(A) \equiv \varphi_1(A) \vee \varphi_2(A)$, $(\varphi_1 \wedge \varphi_2)(A) \equiv \varphi_1(A) \wedge \varphi_2(A)$. It is easy to verify that for any sequence $\{\varphi_n\}$ in Φ. $\limsup\{\varphi_n\} = \bigwedge_n \{\bigvee_{k\geqq n} \varphi_k\}$, $\liminf\{\varphi_n\} = \bigvee_n\{\bigwedge_{k\geqq n}\varphi_k\}$ always exist in Φ. We define $\varphi_n \to \varphi(0)$ if and only if $\limsup\{\varphi_n\} = \liminf\{\varphi_n\} = \varphi$. Similarly we define as usual the order-topology in 2^S.

Now we will prove the following

LEMMA 1. *$\bigwedge_n A^{\varphi_n} = A^{\check{n}\varphi_n}$, $\bigvee_n A^{\varphi_n} = A^{\hat{n}\varphi_n}$ for any element A of 2^S and any sequence $\{\varphi_n\}$ in Φ.*

PROOF. Since the latter is proved in a way similar to the proof of the former, we will only prove the former, that is, $\bigwedge_n A^{\varphi_n} = A^{\check{n}\varphi_n}$. As $\bigvee_n \varphi_n > \varphi_k$ $(k = 1, 2, \cdots)$, we have[5] $A^{\varphi_k} > A^{\check{n}\varphi_n}$. Hence $\bigwedge_n A^{\varphi_n} > A^{\check{n}\varphi_n}$. therefore it is sufficient to show that $\bigwedge_n A^{\varphi_n} < A^{\check{n}\varphi_n}$.

Let x be any point of $\bigwedge_n A^{\varphi_n}$, then $x \in A^{\varphi_k}$ $(k = 1, 2, \cdots)$. Therefore there exists $\{x_\alpha\}$ in A such that for each neighbourhood U_x of x containing $\varphi_k(A)^c$, there exists an $\alpha_0 = \alpha_0(U_x)$ for which $\alpha > \alpha_0$ implies $x_\alpha \in U_x$. Suppose, if possible, $x \,\overline{\in}\, A^{\check{n}\varphi_n}$. Then there exist a neighbourhood V_x of x containing the set $(\bigvee_n \varphi_n(A))^c = \bigwedge_n \varphi_n(A)^c$ and a cofinal sequence $\{x_\beta\}$ in $\{x_\alpha\}$ such that $x_\rho \,\overline{\in}\, V_x$ for any $\beta > \beta_0(V_x)$, where β_0 is determined by V_x. Let $U'_x = V_x \cup \varphi_k(A)^c$, then U'_x is a neighbourhood of x by (N. 3). As $x \in A^{\varphi_k}$, there exists an $\alpha'_0(U'_x)$ such that $\alpha > \alpha'_0$ implies $x_\alpha \in U'_x$. Therefore $\beta > \alpha'_0$ implies $x_\beta \in \varphi_k(A)^c$.

On the other hand, since $\varphi_k(A)^c < A^c$ follows from $\varphi_k(A) > A$, we have $x_\beta \in A^c$ for any $\beta > \alpha'_0$. This result is in contradiction with the fact that $\{x_\alpha\}$ is in A.

By means of Lemma 1 we can prove the following

THEOREM 1. *$\varphi_n \to \varphi(0)$ implies $A^{\varphi_n} \to A^{\varphi}(0)$ for any $A \in 2^S$.*

PROOF. On account of Lemma 1, we see that $\varphi_n \uparrow \varphi(0)$ implies $A^{\varphi_n} \downarrow A^{\varphi}(0)$ and $\varphi_n \downarrow \varphi(0)$ implies $A^{\varphi_n} \uparrow A^{\varphi}(0)$.

In fact, from $\varphi = \bigvee_n \varphi_n$, we have $A^{\varphi} = A^{\check{n}\varphi_n}$. By Lemma 1, $A^{\varphi} = A^{\check{n}\varphi_n} = \bigwedge A^{\varphi_n}$. From the hypothesis that $\varphi_n \uparrow \varphi(0)$, we have $A^{\varphi_n} \downarrow A^{\varphi}(0)$. By a similar argument, we see that $\varphi_n \downarrow \varphi(0)$ implies $A^{\varphi_n} \uparrow A^{\varphi}(0)$, therefore we get the theorem.

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5) N. MATSUYAMA, loc. cit. p. 24.

3. In this section we will investigate whether the converse of Theorem 1 is valid or not. As easily may be seen by example, there exist φ and φ' such that $\varphi \neq \varphi'$ and $A^{\varphi} = A^{\varphi'}$ for all A in 2^S. For this reason, it is convenient to divide the class Φ into equivalence classes as follows.

DEFINITION. φ is said to be equivalent to φ' if and only if $A^{\varphi} = A^{\varphi'}$ for all A in 2^S, and denote it by $\varphi \sim \varphi'$. If we denote the equivalence class containing φ by $[\varphi]$[5], then $\varphi' \in [\varphi]$ if and only if $\varphi' \sim \varphi$. We define an ordering relation on the system of equivalence classes by letting $[\varphi] > [\psi]$ if and only if $A^{\varphi} < A^{\psi}$ for all A in 2^S.

Then it is evident that the system of equivalence classes $[\varphi]$ is a partially ordered system.

THEOREM 2. *$[\varphi] > [\varphi']$ if and only if $\varphi \vee \varphi' \sim \varphi$ (or $\varphi \wedge \varphi' \sim \varphi'$).*

PROOF. If $[\varphi] > [\varphi']$, then $A^{\varphi} < A^{\varphi'}$ for all A in 2^S. Hence, by Lemma 1, $A^{\varphi\vee\varphi'} = A^{\varphi} \wedge A^{\varphi'} = A^{\varphi}$ for all A in 2^S. Therefore we have $\varphi \vee \varphi' \sim \varphi$. Conversely let $\varphi \vee \varphi' \sim \varphi$, then $A^{\varphi} \wedge A^{\varphi'} = A^{\varphi\vee\varphi'} = A^{\varphi}$ for all A in 2^S. Consequently we have $[\varphi] > [\varphi']$.

In a similar way, we can prove that $[\varphi] > [\varphi']$ is equivalent to $\varphi \wedge \varphi' \sim \varphi'$.

LEMMA 2. $[\varphi] \vee [\varphi'] = [\varphi \vee \varphi']$, $[\varphi] \wedge [\varphi'] = [\varphi \wedge \varphi']$.

PROOF. By Lemma 1, we have $A^{\varphi\vee\varphi'} = A^{\varphi} \wedge A^{\varphi'}$ for all A in 2^S. Let φ'' be an element of Φ such that $[\varphi] < [\varphi'']$, $[\varphi'] < [\varphi'']$, then $A^{\varphi''} < A^{\varphi}$, $A^{\varphi''} < A^{\varphi'}$. Hence $A^{\varphi''} < A^{\varphi} \wedge A^{\varphi'}$. On the other hand, $[\varphi] \vee [\varphi']$ is the infimum of such $[\varphi'']$, that is, the equivalence class to the exponent φ of the supremum of $A^{\varphi''}$ such that $A^{\varphi''} < A^{\varphi} \wedge A^{\varphi'}$. Consequently $[\varphi] \vee [\varphi'] = [\varphi \vee \varphi']$.

As $[\varphi] \wedge [\varphi'] = [\varphi \wedge \varphi']$ will be proved in a same manner, we omit the proof.

LEMMA 3. *If $A^{\varphi_n} \downarrow A^{\varphi}(0)$ (or $A^{\varphi_n} \uparrow A^{\varphi}(0)$) for all A in 2^S. then $[\varphi_n] \uparrow [\varphi](0)$ (or $[\varphi_n] \downarrow [\varphi](0)$).*

PROOF. Since $A^{\varphi_n} \downarrow A^{\varphi}(0)$ for all A in 2^S, by use of Lemma 1 we get $A^{\varphi} = \bigwedge_n A^{\varphi} = A^{\bigvee_n \varphi_n}$ for all A. In a way similar to the proof of Lemma 2, we can verify that $\bigvee_n [\varphi_n] = [\bigvee_n \varphi_n]$. Consequently $[\varphi] = [\bigvee_n \varphi_n] = \bigvee_n [\varphi_n]$, hence $[\varphi_n] \uparrow [\varphi]$ (0).

As an obvious consequence of Lemma 2 and Lemma 3, we get the following

THEOREM 3. *The system $[\varphi]$ forms a lattice and moreover if $A^{\varphi_n} \to A^{\varphi}(0)$ for all A in 2^S, then $[\varphi_n] \to [\varphi](0)$.*

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5) This notation is due to J. W. TUKEY, Convergence and uniformity in topology, p. 4.