ON MEROMORPHIC FUNCTIONS WITH ESSENTIAL SINGULARITIES OF LOGARITHMIC CAPACITY ZERO

MASATSUGU TSUJI

(Received August 5, 1949; in revised form, August 4, 1950)

Let M be a bounded closed set on the z-plane, which is of logarithmic capacity zero and $w = w(z)$ be one-valued and meromorphic outside M, every point of which is an essential singularity of $w(z)$. Let ω be a transcendental singularity of the inverse function $z = z(w)$ of $w = w(z)$, whose projection be a_0. Let K: $|w - a_0| < \rho$ be a disc and F_ρ be a connected piece of the Riemann surface of $z(w)$, which has ω on its boundary and lies above K. Let F_ρ be mapped on a domain Δ on the z-plane. Δ is multiply connected in general and the boundary of Δ consists of Jordan curves $\{\Gamma_\nu\}$, Jordan curves or arcs $\{\Gamma'_\nu\}$ and a closed sub-set M_0 of M, where there are no points of M on and inside of Γ_ν and if Γ'_ν is a Jordan curve, then Γ'_ν contains points of M in its inside and if Γ'_ν is a Jordan arc, then Γ'_ν ends at two points of M_0.

Γ_ν is the boundary of a hole in Δ. We add the insides $[\Gamma_\nu]$ of Γ_ν to Δ and let

$$\tilde{\Delta} = \Delta + \sum_\nu [\Gamma_\nu]. \tag{1}$$

Then the following theorem holds.

THEOREM. (i) *F_ρ covers any point of K infinitely often, except a set of logarithmic capacity zero.*

(ii) *If $\tilde{\Delta}$ is of finite connectivity, then F_ρ covers any point of K infinitely often, with two possible exceptions.*

(iii) *If Δ is of finite connectivity, then F_ρ covers any point of K infinitely often, with one possible exception.*

(i) is proved by the author[1], (ii) and (iii) are due to K. Noshiro[2]. When M consists of only one point, then $\tilde{\Delta}$ is simply connected, so that (ii) contains Kunugui's theorem[3], that if M consists of only one point, then F_ρ covers any point of K infinitely often, with two possible exceptions. I will give a simple proof in the following lines.

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1) M. Tsuji: Theory of meromorphic functions in the neighbourhood of a closed set of capacity zero. Jap. Jour. Math. 19 (1948).

2) K. Noshiro: Note on the cluster sets of analytic functions. The Jour. Math. Soc. Japan. vol. 1. No. 4. (1950).
K. Noshiro : Contribution to the theory of the singularities of analytic functions. Jap. Jour. Math. 19 (1948).

3) K. Kunugui: Une généralisation des théorèmes de MM. Picard-Nevanlinna sur les fonctions méromorphes. Proc. Imp. Acad. 17 (1941).

PROOF. (i). (a) First we will prove that F_ρ covers any point of K at least once, except a set of logarithmic capacity zero.

Since M_0 is a sub-set of M, we have $\operatorname{cap} M_0 = 0$, where $\operatorname{cap} M_0$ means the logarithmic capacity of M_0.

We map Δ on $|\zeta| < 1$ by $z = z(\zeta)$, then M_0 is mapped on a null set e_0 on $|\zeta| = 1$. Let $v(\zeta) = w(z(\zeta))$, then $v(\zeta)$ is regular and $|v(\zeta) - a_0| < \rho$ in $|\zeta| < 1$ and $|v(\zeta) - a_0| = \rho$ on the complementary arcs of e_0 on $|\zeta| = 1$, so that $v(\zeta)$ belongs to the U-class in Seidel's sense. Hence by Frostman's theorem[4], F_ρ covers any point of $K : |w - a_0| < \rho$, except a set of logarithmic capacity zero.

(b) To prove that F_ρ covers any point of K infinitely often, except a set of logarithmic capacity zero, we will first prove the following lemma.

LEMMA. *If a disc K_0 contained in K is covered exactly n-times by F_ρ, then K is covered exactly n-times by F_ρ, so that $z(w)$ has only algebraic singularities in K.*

PROOF. Let D be a connected domain, which contains K_0, such that every point of which is covered exactly n-times by F_ρ and let E be its boundary which lies in K. Suppose that D does not coincide with K, then $E \neq 0$ and let w_0 be a boundary point of D, which lies in K.

From the definition, w_0 is covered at most n-times by F_ρ. If w_0 is covered n-times, then the part of F_ρ above a small disc K_1 about w_0 contains n discs $F_1, \cdots, F_n$, consisting of only inner points, where a piece of the Riemann surface of $(w - w_0)^{1/k}$ is considered as k discs.

If there is no connected piece of F_ρ above K_1 other than $F_1, \cdots, F_n$, then K_1 is covered n-times by F_ρ, so that K_1 belongs to D, which contradicts the hypothesis, that w_0 is a boundary point of D. Hence there is another connected piece F_0 above K_1, other than $F_1, \cdots, F_n$, then F_0 does not cover the common part D_0 of D and K_1. This contradicts (a), since $\operatorname{cap} D_0 > 0$. Hence every point of E is covered at most $(n-1)$-times by F_ρ. We will prove that $\operatorname{cap} E = 0$. Let E_k be the sub-set of E, which is covered k-times by F_ρ, then $E = \sum_{k=0}^{n-1} E_k$. Suppose that $\operatorname{cap} E > 0$, then for some k, $\operatorname{cap} E_k > 0$ $(0 \leqq k \leqq n-1)$. Since $\operatorname{cap} E_0 = 0$ by (a), we have $1 \leqq k \leqq n-1$. Let E_k^0 be a closed sub-set of E_k, such that $\operatorname{cap} E_k^0 > 0$. Then there is a point $w_0 \in E_k^0$, such that $\operatorname{cap} E_k^0(K_1) > 0$, for any small disc K_1 about w_0, where $E_k^0(K_1)$ is the part of E_k^0 contained in K_1.

Since $w_0 \in E_k$, w_0 is covered k-times by F_ρ. Hence the part of F_ρ above K_1 contains k discs $F_1, \cdots, F_k$ consisting of only inner points. Since $1 \leqq k \leqq n-1$, there is another connected piece F_0 above K_1 other than $F_1, \cdots, F_k$. Since $E_k^0(K_1)$ is covered k-times in $F_1, \cdots, F_k$, F_0 does not cover $E_k^0(K_1)$, which contradicts (a), since $\operatorname{cap} E_k^0(K_1) > 0$. Hence $\operatorname{cap} E = 0$.

Let $z_1(w), \cdots, z_n(w)$ be n branches of $z(w)$ in D and w_0 be any point

4) O. Frostman: Potentiel déquilibre et capacité des ensembles. Lund (1935).

of E. We may suppose that Δ is a bounded domain, so that $z_i(w)$ are bounded in a neighbourhood U of w_0. We put

$$\prod_{i=1}^{n}(z - z_i(w)) = z^n + a_1(w)\, z^{n-1} + \cdots + a_n(w) = 0,$$

then $a_i(w)$ are one-valued and regular and bounded in U. Since cap $E = 0$, $a_i(w)$ are regular at w_0, so that U is covered n-times by F_ρ, hence w_0 belongs to D, which is absurd. Hence D has no boundary point in K, so that D coincides with K, q. e. d.

(c) By this lemma, we will prove that F_ρ covers any point of K infinitely often, except a set of logarithmic capacity zero. Let E be the set of points in K, which is covered finite times by F_ρ, then $E = \sum_{k=0}^{\infty} E_k$, where E_k is a sub-set of E, which is covered k-times by F_ρ.

Suppose that cap $E > 0$, then for some k, cap $E_k > 0$. Since cap $E_0 = 0$ by (a), we have $1 \leqq k < \infty$. Let E_k^0 be a closed sub-set of E_k, such that cap $E_k^0 > 0$. Then there is a point $w_0 \in E_k^0$, such that cap $E_k^0(K_1) > 0$ for any small disc K_1 about w_0. Since $w_0 \in E_k$, w_0 is covered k-times by F_ρ, hence the part of F_ρ above K_1 contains k discs $F_1, \cdots, F_k$ consisting of only inner points. Since by the hypothesis, $z(w)$ has a transcendental singularity at a_0, we see, by the lemma, that there is another connected piece F_0 above K_1 other than $F_1, \cdots, F_k$. Since $E_k^0\,(K_1)$ is covered k-times in $F_1, \cdots, F_k$, F_0 does not cover $E_k^0(K_1)$, which contradicts (a), since cap $E_k^0(K_1) > 0$. Hence we have cap $E = 0$, so that F_ρ covers any point of K infinitely often, except a set of logarithmic capacity zero.

(ii) Next suppose that $\tilde{\Delta}$ is of finite connectivity.
Since M_0 is a bounded closed set of logarithmic capacity zero, by Evans' theorem[5], we can distribute a positive mass $d\mu(a)$ on M_0 of total mass 1, such that

$$(2) \qquad u(z) = \int_{M_0} \log \frac{1}{|z-a|}\, d\mu(a), \qquad \left(\int_{M_0} d\mu(a) = 1\right)$$

tends to $+\infty$, when z tends to any point of M_0.

We put

$$(3) \qquad \begin{aligned} &\int_{M_0} \log\frac{1}{z-a}\; d\mu(a) = u(z) + i\theta(z), \\ &t = e^{u+i\theta} = r(z)e^{i\theta(z)}, \quad (r(z) = e^{u(z)}), \\ &w(z) = F(t). \end{aligned}$$

Let C_r be the niveau curve $r(z) = \text{const.} = r$, then C_r consists of a finite number of Jordan curves, which cluster to M_0 as $r \to \infty$. Since the total mass is 1, by integrating on the negative sense on C_r, we have

$$(4) \qquad \int_{C_r} d\theta(z) = \int_{C_r} \frac{\partial\theta}{\partial s}\, ds = \int_{C_r} \frac{\partial u}{\partial \nu} ds = 2\pi,$$

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5) G. C. Evans: Potentials and positively infinite singularities of harmonic functions. Monatshefte für Math. u. Phys. 43 (1936).

where ds is the arc element and ν is the inner normal of C_r, so that $\frac{\partial\theta}{\partial s} > 0$. We write r, θ instead of $r(z)$, $\theta(z)$.

Let $C_r(\Delta)$ be the part of C_r, which lies in Δ, then by (4) and $\frac{\partial\theta}{\partial s} > 0$,

$$\int_{C_r(\Delta)} d\theta(z) \leqq 2\pi. \tag{5}$$

Let Δ_r be the part of Δ, which lies outside C_r.

Suppose that F_ρ covers three points α, β, γ of K finite times and $z_1, \cdots, z_N$ be zero points of $(w(z)-\alpha)(w(z)-\beta)(w(z)-\gamma)$. We take off these points from Δ_r and let Δ_r^0 be the remaining domain. Then $w(z) \neq \alpha, \neq \beta, \neq \gamma$ in Δ_r^0. Let F_r be the image of Δ_r^0 on the w-plane, then F_r is the covering surface of the basic domain F_0, which is obtained from K by taking off three points α, β, γ. Let $|F_r|$ be the area of F_r and $L(r)$ be the length of the image of $C_r(\Delta)$ and $S(r) = |F_r|/\pi\rho^2$, then since the number of closed Γ'_ν is finite, we have for a suitable $A > 0$, $r_0 > 0$,

$$|F_r| = A + \int_{r_0}^{r} dr \int_{C_r(\Delta)} |F'(t)|^2\, r\, d\theta,$$

$$L(r) = \int_{C_r(\Delta)} |F'(t)|\, r\, d\theta, \qquad (r \geqq r_0), \tag{6}$$

so that by (5),

$$[L(r)]^2 \leqq \int_{C_r(\Delta)} r\,d\theta \int_{C_r(\Delta)} |F'(t)|^2 r\,d\theta \leqq 2\pi r \int_{C_r(\Delta)} |F'(t)|^2 r\,d\theta$$

$$= 2\pi r \frac{d|F_r|}{dr} = 2\pi^2\rho^2 r \frac{dS(r)}{dr}. \tag{7}$$

Since by (i), F_ρ covers any point of K infinitely often, except a set of logarithmic capacity zero, we have

$$\lim_{r\to\infty} S(r) = \infty. \tag{8}$$

Suppose that $L(r) \geqq [S(r)]^{3/4}$ in a set of intervals $I_\nu = [r_\nu, r'_\nu]$ ($\nu = 1, 2, \cdots\cdots$), then we have by (7),

$$\sum_\nu \int_{I_\nu} d\log r \leqq 2\pi^2\rho^2 \sum_\nu \int_{I_\nu} \frac{dS(r)}{[S(r)]^{3/2}} \leqq 2\pi^2\rho^2 \int^\infty \frac{dt}{t^{3/2}} < \infty,$$

so that there exists $r_1 < r_2 < \cdots\cdots < r_n \to \infty$, such that $L(r_n) \leqq [S(r_n)]^{3/4}$, hence by (8),

$$\lim_{n\to\infty} \frac{L(r_n)}{S(r_n)} = 0. \tag{9}$$

Let ρ be the Euler's chracteristic, then by Ahlfors' fundamental theorem on covering surfaces[6],

$$\overset{+}{\rho}(F_r) \geqq \rho(F_0)S(r) - hL(r), \quad \overset{+}{\rho} = \mathrm{Max}(\rho,\ 0), \tag{10}$$

where h is a constant depending on F_0 only. Since $\rho(F_0) = 2$ and

6) L. Ahlfors: Zur Theorie der Überlagerungsflächen. Acta Math. 65 (1935).

(11) $$\overset{+}{\rho}(F_r) \leqq \lambda(r) + N,$$
where $\lambda(r)$ is the number of holes in Δ_r, we have
(12) $$\lambda(r) + N \geqq 2S(r) - hL(r).$$
Next we will evaluate $\lambda(r)$.

Let $\Lambda(r)$ be the total length of the image of the boundary of holes of Δ_r, then by Ahlfors' covering theorem,
$$|S(r) - \Lambda(r)/2\pi\rho| \leqq h\,L(r), \text{ or}$$
$$\Lambda(r)/2\pi\rho \leqq S(r) + h\,L(r).$$
Since $\lambda(r) \leqq \Lambda(r)/2\pi\rho$, we have
$$\lambda(r) \leqq S(r) + h\,L(r),$$
so that from (12),
$$S(r) \leqq 2h\ L(r) + N,$$
which contradicts (9). Hence F_ρ covers any point of K infinitely often, with two possible exceptions.

(iii) Next suppose that Δ is of finite connectivity and suppose that F_ρ covers two points α, β of K finite times and let N be the number of zero points of $(w(z)-\alpha)(w(z)-\beta)$ in Δ. If we take α, β instead of α, β, γ in the proof of (ii), then $\rho(F_0) = 1$ and
$$\overset{+}{\rho}(F_r) \leqq \overset{+}{\rho}(\Delta) + N,$$
so that by (10),
$$\overset{+}{\rho}(\Delta) + N \geqq S(r) - hL(r), \text{ or}$$
$$S(r) \leqq hL(r) + \overset{+}{\rho}(\Delta) + N,$$
which contradicts (9). Hence F_ρ covers any point of K infinitely often, with one possible exception.

Hence the theorem is completely proved.

We remark that by modifying the proof slightly, the same result holds, when $w(z)$ is one-valued and meromorphic in a neighbourhood of M.

REMARK. Let $G(x, y)$ be an integral function of x and y and $y(x)$ be an analytic function defined by $G(x, y) = 0$ and F be its Riemann surface spread over the x-plane. Let E be the set of x, which is not covered by F. Then Julia[7] proved that E does not contain a continuum. Generalizing this Julia's theorem, I have proved that *if $y(x)$ is not an algebroid function, then F covers any point of the x-plane infinitely often, except a set of logarithmic capacity zero.*[8] This can be deduced from the following theorem by means of a lemma analogous to the lemma proved in (i) (*b*).

THEOREM. *Let K: $|x - a_0| < \rho$ be a disc and F_ρ be a connected piece of F, which lies above K. Then F_ρ covers any point of K at least once, except a set of logarithmic capacity zero.*

PROOF. We map F_ρ on $|z| < 1$ by $x = x(z)$. Then by Fatou's theorem,

7) G. Julia: Sur le domaine d'existence d'une fonction implicite définie par une relation entière $G(x, y) = 0$. Bull. Soc. Math. (1926).
8) M. Tsuji, l. c. (1).

$\lim_{z \to e^{i\theta}} x(z) = x_0$ exists almost everywhere on $|z| = 1$, when $z \to e^{i\theta}$ non-tangentially to $|z| = 1$. If $|x_0 - a_0| < \rho$, then x_0 is an accessible boundary point of F. Since, as Julia proved, if x tends to an accessible boundary point x_0 of F, then $\lim_{x \to x_0} y(x) = \infty$, we have

$$\lim_{z \to e^{i\theta}} y(x(z)) = \infty,$$

so that if we denote the set of such $e^{i\theta}$ by E, then by Lusin-Privaloff's theorem,[9] $mE = 0$, hence almost all points of $|z| = 1$ are mapped on $|x - a_0| = \rho$, so that $x(z)$ belongs to the U-class in Seidel's sense, hence by Frostman's theorem, F_ρ covers any point of K at least once, except a set of logarithmic capacity zero.

MATHEMATICAL INSTITUTE, TOKYO UNIVERSITY.

9) Lusin-Privaloff: Sur l'unicité et multiplicité des fonctions analytiques. Ann. Sci. Nor. Sup. 42(1925).