SIMPLE ALGEBRAS OF COMPLETELY CONTINUOUS OPERATORS

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As in the theory of hypercomplex numbers, it may be expected, the study of simple algebras gives a foundation of that of operator algebras, although in the latter each algebra can not be reduced into simple components. As an approach on this line, simple algebras of a special type, i.e. of completely continuous operators. will be discussed in this note.

In such algebras, I. Kaplansky [1] proved, they are always isometrically isomorphic to the full completely continuous operator algebra on a suitable Hilbert space. It shows, the algebraic structure of such algebras is completely determined within the isomorphism. Therefore, it remains the relative structure with respect to the given full completely continuous operator algebra in question. The present note concerns chiefly on it.

In § 1, a simple algebra A of completely continuous operators on a Hilbert space H will be discussed in a connection with the reducibility of the space. The space H will be considered as an A-module, and it will be decomposed into direct irreducible summands with a restriction on A (Theorem 2). Incidentally, Kaplansky's theorem, which is fundamental for our investigation, will be given an alternative proof (Theorem 1).

In § 2, the commutors of such algebra will be discussed. The commutor is taken with respect to the full completely continuous operator algebra. A simple argument will show that the commutor coincides with the annihilator if A is infinite dimensional (Proposition 2). This hinders the program, which traces an analogue of the classical theory of algebras.

Pseudo-commutor, which will be defined in § 3, will be used to avoid the conflict. It gives a satisfactory dual object for our purpose. The pseudo-commutor of a simple algebra of completely continuous operators is the full matrix algebra embedded in the full operator algebra, and conversely (Theorem 3). This solution is closedly connected with (and implied by) the direct factorization theorem of Murray-von Neumann [3]. It reduces, in some sense, the problems of the relative structure of such algebras into that of hypercomplex numbers of operators.

Throughout the note, we may use some abbreviations and special use of symbols for the convenience. We shall list them as follows:

$B(H)$: the algebra of all operators on H;

$C(H)$: the algebra of all completely continuous operators on H;

nos: normalized orthogonal system;

cnos: complete normalized orthogonal system;

pio: partial isometric operator;

cocop: completely continuous operator;

e, p, q: projections with the range E, P, Q, respectively.
The term "subspace" will be used instead of closed linear manifold. An algebra is called "simple" if it has no proper two-sided closed ideal.

1. Let A be a C^*-algebra, i.e., a self-adjoint uniformly closed algebra of cocops on an infinite dimensional (not necessarily separable) Hilbert space H. By the spectral theorem, the following proposition is clear:

PROPOSITION 1. *A C^*-algebra of completely continuous operators is generated by the set of all (finite dimensional) projections belonging to the algebra.*

Proposition 1 implies that a C^*-algebra of cocops satisfies the condition (B) of I. Kaplansky [2].

Let e be a primitive projection of A in the sense that $0 < q \leqq e$ and $q \in A$ imply $q = e$. Then exe is a scalar multiple of e for all x in A by Proposition 1. If φ and ψ are mutually orthogonal vectors in the range E of e, then

$$(\varphi x, \psi y) = (\varphi ex, \psi ey) = (\varphi exy^* e, \psi) = \lambda(\varphi, \psi) = 0$$

implies

LEMMA 1. *If φ and ψ are mutually orthogonal vectors in the range of a primitive projection e of A, then φA and ψA are mutually orthogonal.*

Now, we shall discuss the "simple" case.

LEMMA 2. *Let A be a simple C^*-algebra of cocops, and let $\{\varphi_1, \dots, \varphi_n\}$ be a cnos of the range E of a primitive projection e of A. Then the closure H_i of $\varphi_i A$ is irreducible under A.*

PROOF. If H_i is reducible, then there exists a projection p commuting with A and $H_i = H_i p + H_i(1-p)$. Since A is simple, and since $H_i p$ is reduced by A, A is represented on $H_i p$ faithfully. Hence, there exists $\varphi e = \varphi \neq 0$ with $\varphi \in H_i p$. Similarly $H_i(1-p)$ has φ' with $\varphi' e = \varphi' \neq 0$. Therefore

$$\{\varphi_1, \dots, \varphi_{i-1}, \varphi, \varphi', \varphi_{i+1}, \dots, \varphi_n\}$$

is a nos in E, and this contradicts to the completeness of $\{\varphi_1, \dots, \varphi_n\}$.

LEMMA 3. *In the previous Lemma, let*

$$M = H_1 + H_2 + \dots + H_n$$

and N be the orthocomplement of M. Then M and N are reduced by A. Moreover, N is annihilated by A, i.e., $NA = 0$.

PROOF. The first half is obvious. To prove the remainder, by Proposition 1, it is sufficient to show that each projection p of A annihilates each vector φ of N. If the contrary is true, then φA spans a subspace F on which A is represented faithfully, since A is simple. Therefore, the primitive projection e of the previous Lemma has a proper vector in N.

This is a contradiction.

It is to be noted that N is the set of all elements ξ of H with $\xi x = 0$ for all x of A. For, since such ξ is annihilated by each projection of A, ξ is orthogonal to every proper subspace of the primitive idempotent of A, and A is generated by such primitive projections. N will be called the *nilspace* of A. If $N = 0$, A will be called *semi-normal on H.*

To proceed further, now we shall give a proof of a theorem of I. Kaplansky, which is described in Introduction.

THEOREM 1 (KAPLANSKY). *If a simple C*-algebra of completely continuous operators is represented normally on H in the sense that φA spans H, and if H is irreducible, then A is the algebra of all completely continuous operators on H and $H = \varphi A$.*

PROOF. Firstly, we shall show that there exists a one-dimensional projection e with $\varphi e = \varphi$ for some φ. If not, each primitive projection of A is at least two-dimensional, and by Lemma 1 H is reducible.

Next, we shall show that each ψ with the norm unity such as $\psi = \varphi x$ for some $x \in A$ has the one-dimensional projection p such that $\psi p = \psi$. Let $v = ex$. Then v is a pio with initial domain $\{\lambda\varphi\}$ and with final range $\{\lambda\psi\}$. Hence $v = \psi \times \varphi \in A$ in the sense that $\xi\ (\psi \times \varphi) = (\xi, \varphi)\psi$. Then $v^* = \psi \times \varphi$ and $v^* \in A$. Therefore

$$v^*v = (\psi \times \varphi)(\varphi \times \psi) = (\varphi,\ \varphi)(\psi \times \psi) = \psi \times \psi$$

belongs to A.

Finally, we shall show, that each ϕ with $\|\phi\| = 1$ allows the expression $\phi = \varphi x$. Let $\{\psi_n\}$ be a sequence of norm unity converging to ϕ, and let $\psi_n = \varphi v_n$ where v_n is one-dimensional pio. Then $\| v_n - v_m \| = \| \varphi v_n - \varphi v_m \| \to 0$, whence there exists v in A with $\varphi v = \phi$.

THEOREM 2. *If a simple C*-algebra of completely continuous operators acts on a Hilbert space, then the space is decomposed into the direct sum of the nilspace and the finite number of mutually orthogonal irreducible subspaces each of which is operator-isomorphic. The algebra is isomorphic with the full completely continuous operator algebra of each irreducible summand.*

PROOF. By Lemma 3 and Kaplansky's theorem, it is sufficient to show the operator isomorphism of H_i. Since Theorem 1 gives $H_i = \varphi_i A$, the isomorphism is given by $\varphi_i x \leftrightarrow \varphi_j x$, then clearly H_i and H_j are operator-isomorphic considering as A-modulo. This proves the theorem.

2. Let A be a simple C*-aglebra of cocops. We shall denote

$$A^c = \{x \in C(H);\ a \in A \to ax = xa\}.$$

A^c will be called the *commutor* of A. Clearly, $A^c = A' \cap C(H)$ where A' is the commutor of A in $B(H)$ in the sense of Murray-von Neumann [3], and A^c is a C*-algebra, whence by Proposition 1 it is generated by the projections belonging to it. Therefore, A^c will be determined by the projections of A^c.

LEMMA 4. *If A is a simple C^*-algebra of cocops and if p is a projection of A^c, then $x \to xp$ defines a representation of A on Hp. Moreover, if A is infinite dimensional, then $xp = 0$ for all $x \in A$.*

PROOF. The first half is obvious. The second half follows from the fact that A can not allow non-trivial representation on finite dimensional space.

LEMMA 5. *If A is a simple C^*-algebra of cocops and semi-normal on H, then A^c vanishes.*

PROOF. Since each $p \in A^c$ has $px = 0$ for all $x \in A$, the proper space P of p is a part of N. Since $N = 0$ by the semi-normality, $p = 0$. This shows $A^c = 0$.

PROPOSITION 2. *If A is a infinite dimensional simple C^*-algebra of cocops, then A^c is the set of all x with $Ax = xA = 0$, i.e., the annihilator of A in $C(H)$. A^c is isomorphic to $C(N)$, whence it is simple.*

PROOF. The first half follows from Proposition 1 and Lemma 4: since an annihilator x is contained in A^c, and since A^c is generated by its projections. If a projection q having its proper space in N is chosen, then q annihilates A, whence q belongs to A^c. Therefore, a cocop x having its domain and range in N annihilates A by Proposition 1, whence the algebra B of such cocops, which is naturally isomorphic to $C(N)$, is contained in A^c. On the other hand, B contains each p of Lemma 4, and so B contains A^c. This shows the second half.

In the classical theory of simple algebras, the commutor of a simple subalgebra A of an algebra C(with a suitable condition) satisfies (i) A^c is simple, (ii) $A^{cc} = A$, and (iii) $C = A \times A^c$. Clearly, (i) is true for our A^c. Whereas (ii) and (iii) fail. For an example, let A be semi-normal and H be reducible, then $A^c = 0$ and so $A^{cc} = C(H) > A$, whence (ii) is not true.

In the theory of factorization of Murray—von Neumann [3], more satisfactory analogue of the classical theory is given: If A is a factor of type (I), then its commutor A' is a factor of type (I) and (ii)-(iii) are satisfied. In the factor case, (i) fails. Because, I_∞-factor is not simple if H is infinite dimensional (J. von Neumann [4]).

3. In this section, we shall introduce a notion modifying the notion of commutor as follows;

DEFINITION 1. A simple C^*-algebra A on a Hilbert space H is called (i) an algebra of *type* (0) if A is finite dimensional and contains 1, (ii) an algebra of *type* (1) if A consists of cocops and is semi-normal on H.

DEFINITION 2. The *pseudo-commutor* A^π of A is the set of all elements x such that x commutes with each element a of A and

(i) x is a cocop if A is of type (0),

(ii) x is linear if A is of type (1).

By the Wedderburn Theorem, A is isomorphic with the full matrix algebra K_n if A is of type (0). In both cases, the pseudo-commutor A^π is a C*-algebra.

LEMMA 6. *If A is an algebra of type* (0) (*or* (1)), *then A^π is of type* (1) (*or* (0)). *Hence two notions are mutually conjugate by the pseudo-commutor operation.*

PROOF FOR TYPE (0): Let p_k be the projection of A defined by the matrix (α_{ij}), for $\alpha_{ij}=1$ if $(i, j)=(k,k)$ and $\alpha_{ij}=0$ if $(i, j)\neq(k, k)$. Put $P_k=Hp_k$. Obviously, P_k's are mutually orthogonal, and so $H=\sum_i P_i$. Therefore, H is represented by the set of "vectors" $\xi=(\xi_1,\ldots,\xi_n)$, $\xi_i\in H_i$. Since in K_n each p_i is unitarily equivalent to p_1, P_i is unitarily equivalent to P_1. Identifying with the equivalence, each matrix (β_{ij}) acts on H as an operator defined as usually by

$$(3.1)\qquad (\xi_1\cdots,\xi_n)\,(\beta_{ij})=(\xi'_1,\ldots,\xi'_n);\ \ \xi'_i=\sum_{j=1}^{n}\beta_{ji}\xi_j.$$

Since p_i is reduced by A^π, $\xi x\in P_i$ if $\xi\in P_i$ and $x\in A^\pi$, whence

$$(3.2)\qquad (\xi_1,\ldots,\xi_n)x=(\xi_1x,\ldots,\xi_nx).$$

That is, A^π can be considered as a certain C*-subalgebra of $C(P_1)$. On the other hand, since P_i's are mutually isomorphic, identifying each operator of $C(P_i)$, the above equation gives a cocop x on H and x commutes with A. Thus A^π is restricted to $C(P_1)$ on P_1. This shows A^π is of type (1).

PROOF FOR TYPE (1): By Theorem 2, H is reduced into the direct sum of mutually orthogonal irreducible subspaces P_i, each of which is mutually operator-isomorphic. This shows, $\xi=(\xi_1,\cdots,\xi_n)$, $\xi_i\in P_i$, if $\xi=\xi_1+\cdots+\xi_n$, is a representation in a vector form, and (3.2) gives a representation of x of A. It is sufficient to show that A^π consists of all operators of the form (3.1), since such operators are clearly contained in A^π.

Let $\{\varphi_1,\ldots,\varphi_n\}$ be a cnos of the proper space E of a primitive idempotent e as in Lemma 2. Then $\varphi_i A=P_i$ by Lemma 2. Since A^π reduces e, $c\in A^\pi$ gives a matrix (γ_{ij}) on E by

$$(\alpha_1\varphi_1,\cdots,\alpha_n\varphi_n)c=(\alpha'_1\varphi_1,\cdots,\alpha'_n\varphi_n):\ \alpha'_i=\sum_{j=1}^{n}\alpha_j\gamma_{ji}.$$

Hence, we have

$$\begin{aligned}(\xi_1,\cdots,\xi_n)c&=(\varphi_1x,\cdots,\varphi_nx)c=(\varphi_1,\cdots\varphi_n)xc=(\varphi_1,\cdots,\varphi_n)cx\\&=(\beta_1\varphi_1,\cdots.\beta_n\varphi_n)x=(\beta_1\varphi_1x,\cdots,\beta_n\varphi_nx)=(\beta_1\xi_1,\cdots,\beta_n\xi_n)\end{aligned}$$

for $x\in A$, where $\beta_i=\sum_j\gamma_{ji}$. This shows

$$c=\sum_{i,j=1}^{n}\gamma_{ji}p_j,$$

and proves the statement.

Now, by the previous lemma and its proof, the following theorem follows immediately:

THEOREM 3. *If A is a simple C^*-algebra of completly continuous operators which is semi-normal on H, then the pseudo-commutor A^π of A satisfies*:
(i) *A^π is simple,* (ii) $A^{\pi\pi} = A$. *and* (iii) $C(H) = A \times A^\pi$.

The finial statement follows from the previous lemma and a result of a previous paper [5] of one of the authors.

COROLLARY. *If A and B are simple C^*-algebras of completely continuous operators and semi-normal on H, then A and B are unitary equivalent if and only if their psedo-commutors A^π and B^π are isomorphic, or what is the same, they have the same ranks.*

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