THE RADON-NIKODYM THEOREM OF TRACES FOR A CERTAIN OPERATOR ALGEBRA

MASAHIRO NAKAMURA AND ZIRÔ TAKEDA

(Received August 6, 1952)

The present note contains a specialization of a theorem due to H. A. Dye [4]. He shows, among others, that a theorem of the Radon-Nikodym type is true for some states on a certain W^*-algebra of the finite type. In the present note, the states may be restricted to the traces. Thus, our theorem will be much restricted than that of Dye. However, it will be proved, under a suitable condition, the resulting Radon-Nikodym derivative shall be contained in the center of the algebra. And, it will be used in the proof the classical Radon-Nikodym theorem. More precisely, the classical Rodon-Nikodym theorem will be proved under a special circumstance. This method of the proof may clarify and justify the generalized formulation of Dye in the connection of the classical theorem.

For the proof, an excellent tactic of K. Yosida [16] and M. K. Fort [5] will be employed. The existence of the centering in a W^*-algebra of finite type, which is due to J. Dixmier [3], becomes our principal weapon. The notion of order-continuity, originally due to G. Birkhoff [2], will be used to avoid the separability restriction of the algebra.

The body of the present note contains the materials of the following order: 1. *Definitions and the notation*, 2. *Complete additivity of traces*, 3. *Order-continuity of traces*, 4. *Statement of the theorem*, 5. *Reduction to the center*, 6. *Proof of the heorem*, 7. *A comment to unbounded case, and the references.*

1. Let R be a ring of operators in the sense of F. J. Murray and J. von Neumann [11] or a W^*-algebra in the sense of I. E. Segal. We may assume throughout this note R is of *finite type* in the sense of J. Dixmier [3] and acts on a separable Hilbert space H. (The separability of H is not necessary for our proof. Alternatively, the σ-finiteness of R in the sense of Dye is sufficient. Even, these enumerability assumption can be avoided with the use of order-continuity of traces, cf. Section 3). The most remarkable property of a W^*-algebra of finite type is the existence of the *centering*: An operation defined on the algebra taking the center elements as its value, $x \to x^\natural$ satisfies the following relations; it is additive, homogeneous, positive, and

(1.1.1) $$(xy)^\natural = (yx)^\natural,$$

(1.1.2) $$(x^\natural y)^\natural = x^\natural y^\natural,$$

(1.1.3) $$x \in Z \text{ implies } x^\natural = x,$$

where Z denotes the center of the algebra. These are established by

J. Dixmier [3].

A *trace* τ is a linear functional on R(as a Banach space, the operator bound as its norm), which is positive, normalized and central in the sense:

(1.2.1) $$\tau(xx^*) \geqq 0,$$

(1.2.2) $$\tau(1) = 1,$$

(1.2,3) $$\tau(xy) = \tau(yx).$$

A useful characterization of traces is (cf. R. Godement [6], Y. Misonou and M. Nakamura [10]), in the connection with the centering,

(1.2.4) $$\tau(x) = \tau(x^\natural),$$

that is, the values of a trace are determined on the center completely. A trace will be called a *character* of the algebra if it can not be described as a non-trivial convex combination of traces. The collection X of all characters of the algebra will be called the *character space* with its weak* topology as functionals on the algebra. It is known, the character space is compact and homeomorphic to the *structure space* or the *spectrum* of the center (i.e., the maximal ideals with the Stone topology). The correspondence will be given as follows:

(1.3) $$M = \{x; \chi(xx^*) = 0\}.$$

It is also known, that the above correspondence give a homeomorphism of the character space and the *spectrum* of the algebra (i.e., the set of all maximal ideals with the Stone topology). These are contained in [3] or [10]. The character will be characterized as a trace having

(1.4) $$\chi(x^\natural y^\natural) = \chi(x^\natural)\chi(y^\natural),$$

i.e., it is multiplicative on the center, and conversely.

As an application of the Gelfand-Neumark-Arens theorem (cf. e.g, Arens[1]), the center Z of the algebra is isometrically isomorphic with the space of all continuous functions on the character space X. Hereafter, we may identify them. We may also use, for the convenience, the notation $\chi(x)$ instead of $x(\chi)$.

Since a trace τ acts on the center Z as a linear functional, by the well-known theorem of Riesz-Markhoff-Kakutani (cf. S. Kakutani [7]), there exists a regular measure $d\tau$ with total measure one on X allowing the integral representation:

(1.5) $$\tau(x) = \int \chi(x) \, d\tau(\chi),$$

where the integration ranges over the whole X. (We may assume this except the contrary stated.) By the help of (1.2.4), the relaiton (1.5) can be extended without the restriction on x, i.e., (1.5) holds for all x in R.

Let L be the lattice of all projections of the algebra. L forms a complemented and also ortho-complemented modular lattice (Moreover, L forms a continuous geometry in the sense of J. von Neumann [13]. However, the continuity of the lattice operations is not necessary for us). By a theorem of I. Kaplansky [9], it is easy to verify that a trace τ is a modular function (in the sense of G. Birkhoff [2]), i. e., for any p and q in L,

(1.6) $$\tau(p \vee q) + \tau(p \wedge q) = \tau(p) + \tau(q).$$

Furthermore, let K be the elements of L which are belonging also to the center of the algebra. The elements of K will be often called *central projections*. By a theorem of J. von Neumann [13], K is the center of L and forms a complete Boolean algebra. By the function representation on X, the central projections correspond to the characteristic functions of open-closed sets of X and conversely. By the help of (1.5), (1.2.4) and (1.1.2),

(1.7) $$\tau(ex) = \int_E \chi(x)\, d\tau(\chi),$$

where E is an open-closed set in X having e as its characteristic function.

The separability of the underlying space will be used in the following form:

(1.8) A set of mutually orthogonal projections of R is at most countable. This property is termed by H. A. Dye as *σ-finiteness.* We shall assume this through the sections, except the contrary explicitly stated.

2. Following to H. A. Dye, a trace τ will be called *completely additive* provided that

(2.1) $$\tau\left(\sum_{i=1}^{\infty} p\right) = \sum_{i=1}^{\infty} \tau(p_i),$$

whenever p_i is a set of mutually orthogonal elements of L. Although $d\tau$ is completely additive on all Borel sets of X, it is not true on the lattice of all open-closed sets. For example, it is not hard to verify the following situation: Let (m) be the Banach algebra of all bounded sequences of complex numbers. (m) is the space of all bounded continuous functions of all natural numbers N. (cf. S. Kakutani and M. Nakamura [8]). The character space $X(m)$ of (m) is the Čech compacting M of N. It is known that (m) is a W^*-algebra. Then an ideal point χ of M(i. e., $\chi \in M - N$) is not completely additive. For, let e_n be the characteristic functions of the set $\{1, 2, \ldots, n\}$, then $1 = \vee_i e_i$, and $\chi(e_i) = 0$ for all i. More generally, it holds:

PROPOSITION 1. *A character of an abelian W^*-algebra is completely additive if and only if it is an isolated point of the character space.*

PROOF. Sufficiency is obvious. To prove the converse, suppose the contrary. Let E be the complement of χ in X. By (1.8), E contains at most enumerable disjoint open-closed sets. Let $\{e_i\}$ be the characteristic functions of a maximal collection of such sets. Since χ is not isolated, $\Sigma_i e_i = 1$. On the other hand, $\chi(e_i)$ vanishes for each i.

COROLLARY. *If all traces of a W^*-algebra of finite type is completely additive, then it is the direct sum of finite number of finite factors.*

An example of a completely additive trace is the wave function:

(2.2) $$\omega(x) = (\varphi x, \varphi), \qquad \|\varphi\| = 1,$$

if it was already a trace. More generally, a sequentially strongly continuous trace is completely additive. Conversely, Dye [4; p. 248] showed that a completely additive state (hence, trace) is sequentially strongly continuous:

(2.3) $$x_n \to x \text{ (strongly) implies } \tau(x_n) \to \tau(x).$$

However, it does not need, in the below, such stronger property. It needs in the following weaker statement:

PROPOSITION 2. *Let z_n be a positive elements of the center with $z_n \leqq z_{n+1}$. If $\|z_n\|$ is bounded by a constant, then it converges strongly to an element z of the center. And, if τ is completely additive, then $\tau(z_n)$ converges to $\tau(z)$.*

In the other words, τ is *sequentially order-continuous* on the center. We shall prove this in the following sequence of lemmas, one of which is a specialization of a lemma of Dye, another due to Y. Misonou, and the first of them is a consequence of results due to I. Kaplansky [9]. In the next proposition we do not need the separability. To make it clear, we shall state them in the terms of AW^*-algebra of I. Kaplansky.

PROPOSITION 3. *A uniformly closed ideal of an AW^*-algebra is generated by the projections which belong to the ideal.*

PROOF. Let I be an ideal which is uniformly closed. In a C^*-algebra, by a theorem due to Segal [14], an ideal is generated by its positive elements, we shall show that each positive x can be uniformly approximated by a finite linear combination of projections of I. Consider a maximal abelian subalgebra A including x. A is an AW^*-algebra, too. A is represented as the space of all continuous functions on a compact Y, in which the lattice of all open-closed sets is complete. Hence the closure E_1 of the set $\{t;\ x(t) > \varepsilon\}$ is open-closed. $x(t) \geqq \varepsilon$ on it and $x(t) \leqq \varepsilon$ otherwise. Let E_n be the closure of $\{t;\ x(t) > n\varepsilon\}$ and let e_n be its characteristic function. Then, it is easy to see, that

$$\|x - \Sigma_i\, n\, e_i\| \leqq \varepsilon.$$

Since each e_i is a multiple of x, e_i belongs to I. This is the required.

Following proposition is communicated by Y. Misonou in conversation. His proof differs from ours.

PROPOSITION 4 (MISONOU). *If a trace τ defined on a W^*-algebra R is completely additive, then the kernel I of τ:*

(2.4) $$I = \{x;\ \tau(xx^*) = 0\}$$

is a principal ideal generated by a central projection $1-e$, i.e., $I = (1-e)R$. Hence I is weakly closed.

PROOF. Let J be the annihilator of I. By a theorem of I. Kaplansky, J is an ideal with $J = Re$ where e is a central projection. Let $\{p_i\}$ be a maximal family of mutually orthogonal projections of I, and $p = \sum_i p_i \in I$. If q is a projection which is orthogonal to p, by the above Proposition 3, q belongs to J, i.e., $q \leq e$. Hence $p = 1 - e$. This and Proposition 3 show the statement.

For the above e, by its definition, we have

(2.5) $$\tau(p) = 0 \text{ implies } p(1-e) = p.$$

Such e will be called the *carrier projection* of τ in the sense of Dye [4; p. 246]. Hence, we have

PROPOSITION 5. *The carrier projection of a completely additive trace is central.*

It is to be noted, that Proposition 5 can be proved more simply and directly from a theorem of lattice theory due to G. Birkhoff [2; p. 73]. Since τ defines a modular functional on L, the vanishing points of τ form a neutral ideal of L. Then (1.8) and complete additivity imply that the ideal is principal. Hence the generating element is central.

LEMMA 1. *Let A be an abelian W^*-algebra and τ be a completely additive trace of A. Then every first category set of character space X of A is a null-set with respect to $d\tau$.*

PROOF. To prove this, it needs to prove that $\tau(S)$ vanishes where S is a nondense closed set. By (1.8), the exists a sequence $\{E_n\}$ of closed-open sets whose join is $X - S$ and assuming that $\{E_n\}$ is mutually disjoint. Let e_n be the coresponding projection of E_n. Then $1 = \sum e_i$. Hence $\tau(X - S) = 1$ and $\tau(S) = 0$.

PROOF OF PROPOSITION 2. The first half of the statement is a known fact about operators (cf. Sz. Nagy [15]). To prove the remainder, we may assume this, i.e., z_n converges strongly to z. For each n, let

$$F_n = \bigcap_m \{\chi;\ z_m(\chi) \leqq z(\chi) - 1/n\},$$

then F_n is closed and contains no non-void open set, whence it is non-dense. Hence $F = \bigcup_n F_n$ is a set of first category and equals to $\{\chi; \lim_n \chi(z_n) \neq \chi(z)\}$. Therefore, by Lemma 1, z_n converges to z on X almost everywhere with respect to $d\tau$. Consequently, it implies, by Fatou's lemma, $\tau(z_n)$ converges to $\tau(z)$.

3. Since the separability of the underlying space and the property (1.8) is too restrictive, it will be hoped to find a condition which makes to generalize the Radon-Nikodym Theorem for an arbitrary W^*-algebra of finite type. This is possible if the traces receive somewhat more stronger restriction. In this section, we wish to discuss it briefly.

A trace τ of a W^*-algebra R (without separability assumption) is said to be *order-continuous* if

(3.1) $$0 \leqq x_\alpha \uparrow x \text{ implies } \tau(x_\alpha) \rightarrow \tau(x),$$

where $x_\alpha \uparrow x$ means the order convergence of the phalanx x_α defined on a directed set D: i.e., $\{x_\alpha\}$ is an abelian family of non-negative elements of R with $x_\alpha \leqq x_\beta$ for $\alpha \leqq \beta$ (of D), and x is their least upper bound. (It is sufficient for our purpose to assume that (3.1) holds in the center only). Obviously, an order-continuous trace is completely additive, and it holds Proposition 2 automatically. Moreover, Proposition 4 and 5 are still true for order-continuous case. Proofs may be carried out with a few verbal change (It is to be mentioned, that Proposition 5 is a consequence of the Hahn-Birkhoff decomposition theorem of vector lattices (cf., G. Birkhoff [2; 72], M. Nakamura [12]).

4. THEOREM. *Let R be a W^*-algebra of finite type acting on a Hilbert space H, and let τ be a trace of R. Suppose that R and τ satisfy one of the following three conditions*:

(4.1) *R is σ-finite and τ is completely additive*;

(4.2) *H is separable and τ is completely additive*;

(4.3) *τ is order-continuous.*

If another trace σ of R satisfies

(4.4) $$0 \leqq \sigma \leqq \kappa\tau,$$

where κ is a constant. Then there exists essentially unique positive element a in the center Z of the algebra satisfying the following relations:

(4.5) $$0 \leqq a \leqq \kappa,$$

and for an arbitrary element x in R,

(4.6) $$\sigma(x) = \tau(xa).$$

Moreover, let X be the character space of R, σ allows the integral representation:

(4.7) $$\sigma(x) = \int \chi(x)\chi(a)d\tau(\chi),$$

where the integration ranges over X and $d\tau$ is the regular measure on X induced by the trace τ. Resulting a will be called the Radon-Nikodym derivative *of σ with respct to τ, and will be denoted by $a = d\sigma/d\tau$.*

5. We shall examine the theorem under the condition (4.2), since the other cases can be proved with a few modification. First at all, it is easy to deduce that (4.4) implies the complete additivity of σ. Therefore, by the classical Radon-Nikodym Theorem, we have

(5.1) $$\sigma(e) = \int_E f(\chi)d\tau(\chi) = \int_X \epsilon(\chi)f(\chi)d\tau(\chi),$$

where e is a central projection and f is the (classical) Radon-Nikodym derivative. It is easy to deduce from (4.4), $f(\chi)$ is essentially bounded by

κ.

In (5.1), if f is continuous, then there exists an element a of Z which satisfies (4.5) and

$$(5.2) \qquad \sigma(e) = \int \chi(e)\chi(a)\,d\tau(\chi) = \int \chi(ea)\,d\tau(\chi) = \tau(ea).$$

From this, we have, by (1.7), (1.1.2) and (1.2.4), that the computation

$$\sigma(x) = \int \chi(x)d\sigma(\chi) = \int \chi(x)\chi(a)d\tau(\chi) = \int \chi(x^{\natural} a)d\tau(\chi) = \int \chi(xa)d\tau(\chi) = \tau(xa)$$

shows (4.6) and (4.7). Therefore, the key point of the proof of the theorem is to show that the Radon-Nikodym derivative f is unique and continuous on X (except the equivalence). We show this in the next section, following the line of K. Yoshida and M. K. Fort.

6. Proof is divided into some lemmas.

LEMMA 2 *Suppose that*

$$(6.1) \qquad D = \{z;\ 0 \leqq z \in Z,\ e \in K \text{ implies } \tau(ez) \leqq \sigma(e)\}.$$

Then D is (i) *a directed set,* (ii) *inductively ordered, and* (iii) *there exists a maximum element z in D (except equivalence).*

PROOF. (i) is clear. Ad (ii), let C be a simply ordered subset of D. Let μ be $\sup\{\tau(z);z\in C\}$. $\mu \leqq \kappa$ by (4.4). Choose $\{z_i\}$ such that

$$(6.2) \qquad z_n \leqq z_{n+1} \in C, \quad \tau(z_i) \to \mu.$$

z_n converges strongly to c of Z. By Proposition 2 (or order-continuity), $\tau(ez_n)$ converges to $\tau(ec)$. Hence $\tau(c) = \mu$ and $\tau(ec) \leqq \sigma(e)$, or $c \in D$. Let b be an arbitrary element of C. Since we can assume $z_i \leqq b$ for all i, we can also assume $c \leqq b$, whence by $\mu = \tau(c) \leqq \tau(b) \leqq \mu$ we have $\tau(b) = \mu$. Therefore $0 \leqq \tau(b-a) = \mu - \mu = 0$ shows $b = c$ almost everywhere. This shows (ii). Now, (iii) is an immediate consequence of (i) and (ii).

LEMMA 3. *If $0 < \pi \leqq \kappa\tau$, then there exist an e of K and a positive α such that*

$$(6.3) \qquad \pi(p) \geqq \alpha\tau(p)$$

for all p of K with $p \leqq e$.

PROOF. Let $\rho(p) = \pi(p) - \alpha\tau(p)$, then the Hahn-decomposition yields

$$(6.4) \qquad \rho(p) \geqq 0, \text{ i.e., } \sigma(p) \geqq \alpha\tau(p) \qquad \text{for } e \geqq p \in K.$$

Hence, there exist a positive α and non-zero e, since otherwise π vanishes identically.

LEMMA 4. (5.2) *is satisfied by a of Lemma* 1.

If $\tau(pa) \neq \sigma(p)$ for some p of K, put $\pi(p) = \sigma(p) - \tau(pa)$. By Lemma 1 and (4.4),

$$0 \leqq \pi(p) \leqq \sigma(p) \leqq \kappa\tau(p).$$

Let e be the element of K described in Lemma 2. Then $p \leqq e$ implies

$$\sigma(p) - \tau(pa) = \pi(p) \gtrless \alpha\tau(p) \geqq \tau(\alpha p).$$

If $p \leqq 1 - e$, $\tau(pe) = 0$. Hence αe belongs to D with $\tau(\alpha e) > 0$. Thus,

$$\sigma(e) \geqq \tau(ea) + \tau(\alpha e) = \tau(ea + \alpha e) > \tau(ea).$$

Hence $a + z > a$ and contradicts to the maximality of a.

It is not hard, without using the integral computation, (5.2) implies (4.6). Since each x of Z is approximated uniformly with linear combinations of central projections.

It is to be remarked, that the proof is purely algebraical. No spacial restriction is used except the separablity. Our method essentially bases on the following facts: (i) Existence of the centering, (ii) lattice completeness of the center, and (iii) order-continuity of traces. Hence, it is not impossible to extend the theorem to a certain hypercentral C^*-algebra (with restrictions on the center). But, it is uncertain how useful this remark is.

7. To extend the theorem, without assuming (4.4), it requires some other concepts. For an analogy to the classical theorem of Radon-Nikodym, we shall introduce the absolute continuity of traces following to H. A. Dye: A trace σ is *absolutely continuous* with respect to a trace τ if

$$p \in L \text{ and } \tau(p) = 0 \quad \text{imply} \quad \sigma(p) = 0. \tag{7.1}$$

In this case, as expected, resulting derivative a will be unbound, and so $\tau(ea)$ will lose its mean. In this section, we shall outline this case.

LEMMA 5. *If $f(X)$ is a non-negative function defined for almost every points of X and measurable with respect to a spectral measure $de(X)$, then the set M of all elements ξ with*

$$\int |f(X)|^2 d\|\xi e(X)\|^2 < +\infty \tag{7.2}$$

is a dense linear set, and then there exists a non-negative operator a satisfying

$$(\xi a, \eta) = \int f(X)\, d(\xi e(X), \eta) \tag{7.3}$$

for every ξ and η in M.

Since the proof is similar to that of usual case, we shall omit it. (cf. B. v. Sz. Nagy [15; p. 44]). We shall write such a as usual

$$a = \int f(X)\, de(X). \tag{7.4}$$

If σ is absolutely continuous with respect to τ, if both σ and τ are completely additive, and if R is of finite type, then the classical Radon-Nikodym Theorem yields

$$\sigma(e) = \int f(\mathcal{X})e(\mathcal{X})\, d\tau(\mathcal{X}), \tag{7.5}$$

where f is the Radon-Nikodym derivative. Within the equivalence, we can choose, f is continuous (allowing infinity), non-negative, integrable, and vanishes whenever $\mathcal{X}$ lies in an open-closed set with τ-measure zero. Let $de(\mathcal{X})$ be the spectral measure defined by the central projections of R. Then f satisfies the conditions of Lemma 5. Therefore a of (7.5) exists. For this a, if we define

$$\tau(xa) = \int \mathcal{X}(x) f(\mathcal{X})\, d\tau(\mathcal{X}), \tag{7.6}$$

then $\tau(xa)$ exists for all x of R, and it is linear on R. Using this, we can also prove (4.6) and (4.7) in this case. Thus,

PROPOSITION 6. *If σ and τ are completely additive traces of W^*-algebra of finite type, and if σ is absolutely continuous with respect to τ, then* (4.6) *is true in the sense defined above.*

This is, clearly, a specialization of Dye's theorem.

REFERENCES

[1] R. ARENS, On theorem of Gelfand and Neumark, Proc. Nat. Acad. Sci. U. S. A., 32(1946), 237-239.

[2] G. BIRKHOFF, Lattice Theory, 1st. ed, New York, 1940.

[3] J. DIXMIER, Les anneaux d'opérateurs de classe finie, Ann. Ecole Norm. Sup., 66 (1949), 209-261.

[4] H. A. DYE, The Radon-Nikodym Theorem for finite rings of operators, Trans Amer. Math. Soc., 72(1952), 243-280.

[5] M. K. FORT, A specialization of Zorn's Lemma, Duke Math. J., 15(1948), 763-765.

[6] R. GODEMENT, Mémoire sur la théorie des caractères dans les groupes localement compacts unimodulaires, J. de Math., 30(1951), 1-110.

[7] S. KAKUTANI, Concrete representation of abstract (L)-space and the mean ergodic theorem, Ann. of Math., 42(1941), 523-537.

[8] S. KAKUTANI AND M. NAKAMURA, Banach limits and the Čech compactification of a countable discrete set, Proc. Imp. Acad. Tokyo, 19(1943). 224-229.

[9] I. KAPLANSKY, Projections in Banach algebras, Ann. of Math., 53(1951), 235-249.

[10] Y. MISONOU AND M. NAKAMURA, Centering of an operator algebra, Tôhoku M. J., 3(1951), 243-248.

[11] F. J. MURRAY AND J. VON NEUMANN, On rings of operators, Ann of Math., 37 (1936), 116-229.

[12] M. NAKAMURA, A proof of the Hahn-Birkoff Theorem, Proc Jap. Acad., 26(1950).

[13] J. VON NEUMANN, Continuous Geometry, Princeton, 1937.

[14] I. E. SEGAL, Two-sided ideals of operator algebras, Ann. of Math., 50(1949) 856-865.

[15] B. VON SZ. NAGY, Spektraldarstellung linearer Transformationen des Hilbertschen Raumes, Berlin, 1942.

[16] K. YOSIDA, Vector lattices and additive set functions, Proc. Imp. Acad. Tokyo, 17 (1941), 228-242.

OSAKA NORMAL COLLEGE AND TÔHOKU UNIVERSITY