ON THE STRONG LAW OF LARGE NUMBERS

SHIGERU TSURUMI

(Received September 12, 1955)

1. The purpose of this paper is to state an extension of Kolmogorov's theorem [3] which provides a necessary and sufficient condition for the validity of the strong law of large numbers for a sequence of independent, identically distributed random variables.

We consider the probability space $(\mathbf{X}, P)$ such that $\mathbf{X}$ is a space whose points are denoted by t and P is a probability measure. Then our extension is stated as follows.

THEOREM. *Let $\{X_n(t)\}$ be a sequence of independent random variables satisfying that*
(1.1) *there exists a positive constant K such that, for any positive integer m and for any extended real numbers*[1] $a_1, b_1, \dots, a_m, b_m$,

$$\limsup_n \frac{1}{n}\sum_{i=0}^{n-1} P\{t\,;\ a_1 \leqq X_{1+i}(t) < b_1, \dots, \ a_m \leqq X_{m+i}(t) < b_m\}$$
$$\leqq K\cdot P\{t; a_1 \leqq X_1(t) < b_1, \dots, \ a_m \leqq X_m(t) < b_m\}.$$

Then the following (1.2) *and* (1.3) *are equivalent.*

$$\sum_{n=1}^{\infty} \int_{|X_n(t)|\in A_n} |X_n(t)|\, dP < \infty \tag{1.2}$$

for some Borel sets $A_1, A_2, \dots$ satisfying

$$A_i \cap A_j = 0 \quad (i \neq j), \qquad \bigcup_{n=1}^{\infty} A_n = [0, \infty),$$

where some of A_n's may be empty.

$$P\left\{t\,;\ \lim_n \frac{1}{n}\sum_{i=1}^{n} X_i(t) = c\right\} = 1 \tag{1.3}$$

for some constant c.

The proof appears in § 2.

If $\{X_n(t)\}$ is identically distributed, (1.1) holds trivially and the sum in (1.2) is equal to the first absolute moment $E(|X_1|)$ common for all X_n's, so that the theorem is reduced to Kolmogorov's.

2. To prove the theorem we need a lemma. Before stating this we must prepare several definitions and notations.

---

1) By an extended real number we mean either a usual real number or one of the symbols $+\infty$ and $-\infty$. In what follows we make the convention that when $a = -\infty$, "$a \leqq$" is replaced by "$a <$".

Let Ω be an infinite product space of real lines Ω_n $(n = 1, 2, \ldots.)$, that is, Ω be the set consists of all infinite sequences $\omega = (x_1, x_2, \ldots.)$ with $x_n \in \Omega_n$ $(n = 1, 2, \ldots.)$. If we put for any positive integer m and for any extended real numbers a_i, b_i $(i = 1, 2, \ldots, m)$

$$\Lambda = \{(x_1, x_2, \ldots.) ; a_1 \leqq x_1 < b_1, \ldots, a_m \leqq x_m < b_m\},$$

then Λ is a subset of Ω. Let $\mathfrak{C}$ be the class consists of all sets of the form above, $\mathfrak{B}_0$ be the field consists of all finite unions of sets in $\mathfrak{C}$ and $\mathfrak{B}$ be the Borel field generated by $\mathfrak{B}_0$. Further let T be the shift transformation of Ω onto itself defined by

$$T(x_1, x_2, \ldots.) = (x_2, x_3, \ldots.)$$

for every $(x_1, x_2, \ldots.) \in \Omega$. Let $\mathfrak{M}$ be the class consists of all sets Λ's satisfying $T^{-1}\Lambda = \Lambda$, $\Lambda \in \mathfrak{B}$.

Then we have

LEMMA.[2] *Let α be a probability measure on $\mathfrak{B}$ satisfying that*
(2.1) *there exists a positive constant K such that, for every $\Lambda \in \mathfrak{B}_0$,*

$$\limsup_n \frac{1}{n} \sum_{i=0}^{n-1} \alpha(T^{-i}\Lambda) \leqq K \cdot \alpha(\Lambda).$$

Then there exists at least one probability measure γ on $\mathfrak{B}$ such that

$$\gamma(\Lambda) \leqq K \cdot \alpha(\Lambda) \qquad \textit{for every } \Lambda \in \mathfrak{B}, \tag{2.2}$$

$$\gamma(T^{-1}\Lambda) = \gamma(\Lambda) \qquad \textit{for every } \Lambda \in \mathfrak{B}, \tag{2.3}$$

$$\gamma(\Lambda) = \alpha(\Lambda) \qquad \textit{for every } \Lambda \in \mathfrak{M}, \tag{2.4}$$

$$\gamma(\Lambda) \geqq \liminf_n \frac{1}{n} \sum_{i=1}^{n-1} \alpha(T^{-i}\Lambda) \qquad \textit{for every } \Lambda \in \mathfrak{B}_0. \tag{2.5}$$

Further with respect to such a measure γ it holds that, for every γ-integrable function $f(\omega)$, $\lim_n \frac{1}{n} \sum_{i=0}^{n-1} f(T^i\omega)$ exists and is finite for α-almost every ω.

PROOF. With respect to the constant K in (2.1) we put

$$\mathfrak{F} = \left\{ \Lambda ; \limsup_n \frac{1}{n} \sum_{i=0}^{n-1} \alpha(T^{-i}\Lambda) \leqq K \cdot \alpha(\Lambda), \ \Lambda \in \mathfrak{B} \right\}.$$

We have clearly $\mathfrak{B}_0 \cup \mathfrak{M} \subset \mathfrak{F}$. Since $\left\{ \frac{1}{n} \sum_{i=0}^{n-1} \alpha(T^{-i}\Lambda) \right\}$ is a bounded sequence for every $\Lambda \in \mathfrak{B}$, we can put

$$\beta(\Lambda) = \operatorname*{Lim}_n \frac{1}{n} \sum_{i=0}^{n-1} \alpha(T^{-i}\Lambda)$$

for every $\Lambda \in \mathfrak{F}$, where $\operatorname*{Lim}_n$ denotes the Mazur-Banach limit [1: pp. 33–34]. Then by virtue of the properties of the Mazur-Banach limit we have that
(2.6) *β is non-negative, finitely additive on $\mathfrak{F}$,*

2) This lemma is reminiscent of C. Ryll-Nardzewski's result ([4], [5]) and shown essentially in the proof of Theorem 2 in [6] under the stronger assumption that the inequality in (2.1) holds for every set in a certain class wider than $\mathfrak{B}_0$.

$$(2.7)\qquad \beta(\Lambda) \leqq K \cdot \alpha(\Lambda) \qquad \textit{for every } \Lambda \in \mathfrak{F},$$
$$(2.8)\qquad \beta(T^{-1}\Lambda) = \beta(\Lambda) \qquad \textit{for every } \Lambda \in \mathfrak{F},$$
$$(2.9)\qquad \beta(\Lambda) = \alpha(\Lambda) \qquad \textit{for every } \Lambda \in \mathfrak{M},$$
$$(2.10)\qquad \beta(\Lambda) \geqq \liminf_n \frac{1}{n} \sum_{i=0}^{n-1} \alpha(T^{-i}\Lambda) \qquad \textit{for every } \Lambda \in \mathfrak{F}.$$

Thus, by (2.6) and (2.7), β is non-negative, countably additive on $\mathfrak{F} \supset \mathfrak{B}_0$, so that there exists an unique extension γ such that γ is a probability measure on $\mathfrak{B}$ and $\gamma(\Lambda) = \beta(\Lambda)$ for every $\Lambda \in \mathfrak{F}$. Then from (2.9) and (2.10) it follows that γ satisfies (2.4) and (2.5). If we now put $\mathfrak{G} = \{\Lambda\,;\ \gamma(T^{-1}\Lambda) = \gamma(\Lambda),\ \Lambda \in \mathfrak{B}\}$, it follows from (2.8) that $\mathfrak{B}_0 \subset \mathfrak{F} \subset \mathfrak{G}$. Let $\{\Lambda_n\}$ be any monotone sequence of sets in $\mathfrak{G}$. Then $\gamma(T^{-1}\Lambda_n) = \gamma(\Lambda_n)$ for every n and hence $\gamma(T^{-1}(\lim_n \Lambda_n)) = \gamma(\lim_n \Lambda_n)$, so that $\lim_n \Lambda_n \in \mathfrak{G}$. By the monotone class theorem (see, for example [2: p. 599, Theorem 1.2]) it follows that $\mathfrak{G} \supset$ (*Minimal Borel extension of* $\mathfrak{B}_0$) $= \mathfrak{B}$. This concludes that γ satisfies (2.3). It follows similarly by (2.7) that γ satisfies (2.2). Thus the first assertion is proved.

Let $f(\omega)$ now be any γ-integrable function. Then upon applying the individual ergodic theorem by virtue of (2.3) we deduce that $\lim_n \frac{1}{n} \sum_{i=0}^{n-1} f(T^i\omega)$ exists and is finite for γ-almost every ω. Since the set of all points at each of which the limit above does not exist belongs to $\mathfrak{M}$, we obtain the second assertion by virtue of (2.4). Thus the proof of the lemma is terminated.

PROOF OF THE THEOREM. We define Ω, $\mathfrak{C}$, $\mathfrak{B}_0$ and $\mathfrak{B}$ as in the preceding and further define a transformation φ of $\mathbf{X}$ into Ω by

$$\varphi t = (X_1(t), X_2(t), \ldots)$$

for every $t \in \mathbf{X}$. Let us now put

$$\alpha(\Lambda) = P(\varphi^{-1}\Lambda)$$

for every $\Lambda \in \mathfrak{B}_0$. Then α is a probability measure on $\mathfrak{B}$. By the assumption (1.1) α satisfies that the inequality in (2.1) of the lemma holds for every $\Lambda \in \mathfrak{C}$, while every set in $\mathfrak{B}_0$ can be described as a finite union of disjoint sets in $\mathfrak{C}$, so that (2.1) holds. Thus by the lemma there exists a probability measure γ on $\mathfrak{B}$ satisfying (2.2)~(2.5).

For each positive integer n, let $x_n(\omega)$ be the coordinate function defined by

$$x_n(\omega) = x_n$$

for every $\omega = (x_1, x_2, \ldots) \in \Omega$. Then it is easy to see that if, for an arbitrary but fixed positive integer m, Φ is any Borel function in m-dimensional space, then

$$(2.11)\qquad \begin{aligned} &\alpha\{\omega\,,\ a \leqq \Phi(x_1(\omega), \ldots, x_m(\omega)) < b\} \\ &= P\{t\,;\ a \leqq \Phi(X_1(t), \ldots, X_m(t)) < b\} \end{aligned}$$

for any extended real numbers a, b (cf. [2: pp. 12–15]).

We are now in a position to prove the implication: (1.2) $\rightarrow$ (1.3). Let us put $\Lambda_n = \{\omega ; |x_n(\omega)| \in A_n\}$ for each A_n of (1.2). Since $x_n(\omega) = x_1(T^{n-1}\omega)$ for every n and $\omega \in \Omega$, it holds that

$$T^{i-1}\Lambda_i \cap T^{j-1}\Lambda_j = 0 \quad (i \neq j), \qquad \bigcup_{n=1}^{\infty} T^{n-1}\Lambda_n = \Omega.$$

By (2.3), (2.2), (2.11) and (1.2) we have

$$\int_{\Omega} |x_1(\omega)| d\gamma = \sum_{n=1}^{\infty} \int_{T^{n-1}\Lambda_n} |x_1(\omega)| \, d\gamma = \sum_{n=1}^{\infty} \int_{\Lambda_n} |x_n(\omega)| \, d\gamma$$

$$\leqq K \sum_{n=1}^{\infty} \int_{\Lambda_n} |x_n(\omega)| d\alpha = K \sum_{n=1}^{\infty} \int_{|X_n(t)| \in A_n} |X_n(t)| dP < \infty,$$

so that $x_1(\omega)$ is γ-integrable. Since $x_i(\omega) = x_1(T^{i-1}\omega)$ for every i and $\omega \in \Omega$, by the lemma it holds that $\lim_n \frac{1}{n} \sum_{i=1}^{n} x_i(\omega)$ exists and is finite for α-almost every ω. By (2.11) it follows that $\lim_n \frac{1}{n} \sum_{i=1}^{n} X_i(t)$ exists and is finite with probability 1. Since $\{X_n(t)\}$ is a sequence of independent random variables, we obtain (1.3) by virtue of the zero-one law.

Next we shall prove the implication: (1.3) $\rightarrow$ (1.2). For each positive integer m, we put $\Lambda_m = \{\omega ; m-1 \leqq |x_1(\omega)| < m\}$, $\Gamma_0 = \Omega, \Gamma_m = \{\omega ; m \leqq |x_m(\omega)|\}$ and $\mathbf{C}_m = \{t ; m \leqq |X_m(t)|\}$. Then by (1.3) $X_m/m \rightarrow 0$ with probability 1 and hence $P(\limsup_m \mathbf{C}_m) = 0$ so that $\sum_{m=1}^{\infty} P(\mathbf{C}_m) < \infty$ on account of the Borel-Cantelli lemma. Since $\Lambda_m \in \mathfrak{B}_0$ for every m, we have by (2.5)

$$\gamma(\Lambda_m) \geqq \liminf_n \frac{1}{n} \sum_{i=0}^{n-1} \alpha(T^{-i}\Lambda_m)$$

$$= \liminf_n \frac{1}{n} \sum_{i=1}^{n} \alpha\{\omega ; m-1 \leqq |x_i(\omega)| < m\},$$

so that to each m there correspnds a positive integer $p(m)$ such that

$$\alpha\{\omega ; m-1 \leqq |x_{p(m)}(\omega)| < m\} < \gamma(\Lambda_m) + \frac{1}{2^m \cdot m}. \tag{2.12}$$

For each positive integer n we put

$$A_n = \bigcup_{p(m)=n} [m-1, m),$$

where $\bigcup_{p(m)=n}$ means that the union runs over all m's with $p(m) = n$. Here we note that some of A_n's may be empty. Then every A_n is a Borel set and further

$$A_i \cap A_j = 0 \quad (i \neq j), \qquad \bigcup_{n=1}^{\infty} A_n = [0, \infty).$$

Since, by (2. 3), $\gamma(\Lambda_m) = \gamma\{\omega\ ;\ m-1 \leqq |x_m(\omega)| < m\}$ and $\gamma\{\omega\ ;\ m-1 \leqq |x_m(\omega)|\} = \gamma(\Gamma_{m-1})$ for every m, we have by (2. 11), (2. 12) and (2. 2) that

$$\sum_{n=1}^{\infty} \int_{|X_n(t)| \in A_n} |X_n(t)|\, dP = \sum_{m=1}^{\infty} \int_{m-1 \leqq |X_{p(m)}(t)| < m} |X_{p(m)}(t)|\, dP$$

$$= \sum_{m=1}^{\infty} \int_{m-1 \leqq |x_{p(m)}(\omega)| < m} |x_{p(m)}(\omega)|\, d\alpha \leqq \sum_{m=1}^{\infty} m \cdot \alpha\{\omega\ ;\ m-1 \leqq |x_{p(m)}(\omega)| < m\}$$

$$< \sum_{m=1}^{\infty} m \cdot \left[\gamma(\Lambda_m) + \frac{1}{2^m \cdot m} \right] = 1 + \sum_{m=1}^{\infty} m \cdot \gamma\{\omega\ ;\ m-1 \leqq |x_m(\omega)| < m\}$$

$$= 1 + \sum_{m=1}^{\infty} m \cdot [\gamma\{\omega\ ;\ m-1 \leqq |x_m(\omega)|\} - \gamma\{\omega;\ m \leqq |x_m(\omega)|\}]$$

$$= 1 + \sum_{m=1}^{\infty} m \cdot [\gamma(\Gamma_{m-1}) - \gamma(\Gamma_m)] \leqq 2 + \sum_{m=1}^{\infty} \gamma(\Gamma_m)$$

$$\leqq 2 + K \sum_{m=1}^{\infty} \alpha(\Gamma_m) = 2 + K \sum_{m=1}^{\infty} P(\mathrm{C}_m) < \infty,$$

as was to be proved. Thus the theorem is completely proved.

REFERENCES

[1] BANACH, S., Théorie des opérationes linéaires, Warszawa 1932.
[2] DOOB, J. L., Stochastic processes, New-York 1953.
[3] KOLMOGOROV, A., Grundbegriffe der Wahrscheinlichkeitsrechnung, Berlin 1933.
[4] RYLL-NARDZEWSKI, C., On the ergodic theorems (I) (Generalized ergodic theorems), Studia Math., 12(1951) 65–73.
[5] TSURUMI, S., On the ergodic theorem, This Jour., 2nd ser., 6(1954) 53–68.
[6] TSURUMI, S., On general ergodic theorems, This Jour., 2nd ser., 6(1954) 264–273.

MATHEMATICAL INSTITUTE, TOKYO METROPOLITAN UNIVERSITY.