ON FULLY COMPLETE SPACES

NOZOMU MOCHIZUKI

(Received Aug. 5, 1961)

In [9], V.Pták discusses open mapping properties of locally convex spaces and shows that the class of B-complete spaces has an essential rôle. Such spaces, which we shall call "*fully complete*" according to [3], seem to share with some kind of mapping properties of Banach spaces. The purpose of the present note is to describe in §1 a few results concerning the range theorems of closed operators in fully complete spaces and in §2 some properties of fully complete spaces. Henceforth, we shall consider locally convex linear topological spaces over the real or complex field and the terminology will refer to [2].

1. Range theorems in locally convex spaces. The following is a consequence of the open mapping theorem ([9]: 4. 7).

THEOREM 1. 1. *Let E be a fully complete space and F a locally convex space. If u is a closed linear operator with domain E_0 in E and range in F and if u is almost open, then $u(E_0)$ is a closed linear subspace of F.*

PROOF. u is open by virtue of the open mapping theorem, and $E/u^{-1}(0)$ is fully complete in the quotient topology. Moreover, since $u^{-1}(0)$ is a subspace of E_0, the quotient topology of E_0 by $u^{-1}(0)$ is identical with the topology induced by $E/u^{-1}(0)$. Now, let v be the induced mapping of u, then $u = v\cdot\varphi_0$ where φ_0 denotes the restriction on E_0 of the canonical mapping of E onto $E/u^{-1}(0)$ and v is one-to-one and open. To prove that v is a closed operator, supposet tha $\{\dot{x}_\alpha \mid \alpha \in A\}$ is a net in $E_0/u^{-1}(0)$ which is convergent to $\dot{x}_0$ in $E/u^{-1}(0)$, and that $v(\dot{x}_x)$ converges to y_0 in F. Then there exists a net $\{x_\alpha \mid \alpha\in A\}$ in E_0 and x_0 in E such that $x_\alpha \in \dot{x}_\alpha$ for all $\alpha \in A, x_0 \in \dot{x}_0$ and $\{x_\alpha\}$ converges to x_0. Therefore we have $v(\dot{x}_\alpha) = u(x_\alpha)\to y_0$, and hence $x_0 \in E_0$ and $y_0 = u(x_0)$, i. e. $x_\alpha \in E_0/u^{-1}(0)$ and $y_0 = v(\dot{x}_0)$.

In the following, we assume that u is one-to-one and $\{y_\alpha \mid \alpha\in A\}$ is a net in $u(E_0)$ such that $y_\alpha \to y_0$ in F. Then $\{x_\alpha \mid \alpha \in A\}$ where $x_\alpha = u^{-1}(y_\alpha)$ is a Cauchy net in E_0, and hence converges to a point x_0 in E. Since u is a closed operator, $x_0 \in E_0$ and $y_0 = u(x_0)$. The proof is completed.

REMARK. Every homomorphic image of a fully complete space is fully complete, but this need not be the case for a closed operator which is open. In fact, if $E=(X, \mathfrak{T})$ is an infinite-dimensional Banach space and if $F=(X,\mathfrak{T}')$ is a normed linear space such that $\mathfrak{T}'$ is strictly finer than $\mathfrak{T}$, then the identity mapping of E onto F is open and has the closed graph in $E \times F$. But F is not complete.

COROLLARY 1. 1. *Let E be a fully complete space, F a locally convex space, and u a closed linear operator with domain E_0 in E and range in F. If $u(E_0)$ is of the second category in F, then $u(E_0) = F$.*

PROOF. The verification is easy from Theorem 1.1 and the arguments in [6].

In the sequel, we shall discuss another applications of Theorem 1. 1 which relate with results given in [1], [5] and [8]. The following is a generalized formulation of the Banach-Hausdorff theorem.

THEOREM 1. 2. *Let E be a fully complete space, F a quasi-barrelled space, and u a closed linear operator with dense domain in E and range in F. Suppose that the adjoint operator ${}^tu$ of u has the inverse which is continuous relative to $\beta(F',F)$ and $\beta(E',E)$. Then u is an open mapping onto F.*

PROOF. We denote by E_0 the domain of u and define H' as the set of y' in F' for which $< u(x), y'>$ is a continuous function of x. Then ${}^tu$ is uniquely defined by $< x, {}^tu(y') > = < u(x), y' >$ for $x \in E_0$ and $y' \in H'$, and ${}^tu(y')$ is an element of E'. Let U be an arbitrary convex and symmetric neighborhood of 0 in E. Then for every $y' \in (u(E_0 \cap U))^{\circ}$, $< u(x), y' >$ is a continuous function of x and hence y' belongs to H'. Therefore we have,

$$\begin{aligned}(u(U \cap E_0))^{\circ} &= (u(U \cap E_0))^{\circ} \cap H' \\ &= {}^tu^{-1}({}^tu(H') \cap (U \cap E_0)^{\circ}).\end{aligned}$$

Since $(U \cap E_0)^{\circ} = U^{\circ}$, $(U \cap E_0)^{\circ}$ is an equicontinuous subset of E' and hence $\beta(E',E)$-bounded. But then, in view of the hypothesis of ${}^tu$, $(u(U \cap E_0))^{\circ}$ is $\beta(F',F)$-bounded in F' and therefore equicontinuous because of the assumption that F is quasi-barrelled. Consequently, there is a neighborhood V of 0 in F such that

$$(u(U \cap E_0))^{\circ\circ} \supset V,$$

hence $$\overline{u(U \cap E_0)} \cap u(E_0) \supset V \cap u(E_0).$$

Namely, u is an almost open mapping from E_0 onto $u(E_0)$. Hence u is open and $u(E_0)$ is closed in F. Thus we have $u(E_0) = ({}^tu^{-1}(0))^{\circ} = F$, which completes the proof.

We shall say that a barrelled space is fully barrelled if and only if every closed linear subspace is also barrelled ([1]). The following relates with Lemma

2.8 in [1] and enables us to deduce directly a result discussed in [5] and [8].

THEOREM 1.3. *Let E be a fully complete space with the dual E' fully barrelled relative to $\beta(E', E)$, F a fully barrelled space with F' fully complete relative to $\beta(F', F)$, and u a closed linear operator with dense domain in E and range in F. Suppose that the range of ${}^t u$ is $\beta(E', E)$-closed. Then, the range of u is closed.*

PROOF. Let E_0 be the domain of u, and $\overline{u(E_0)} = G$. We define a linear mapping u_1 from E_0 into G by $u_1(x) = u(x)$. Then u_1 is a closed operator and u may be written as the composition $j \cdot u_1$ where j denotes the injection mapping of G into F. Let H' be the domain of ${}^t u$, then for $z' \in H'$ and $x \in E_0$, we have $< u(x), z' > = < u_1(x), {}^t j(z') >$. Since $< u(x), z' >$ is a continuous function of x, ${}^t j(z') \in D({}^t u_1)$, and we have

$$< u_1(x), {}^t j(z') > = < x, {}^t u_1 \cdot {}^t j(z') >.$$

Thus, ${}^t u = {}^t u_1 \cdot {}^t j$ on H' and ${}^t j(H') \subset D({}^t u_1)$.

On the other hand, if y' is an element of $D({}^t u_1)$ and z' an extension on F of y', then $y' = {}^t j(z')$ and from $< u(x), z' > = < u_1(x), y' >$ we have $z' \in H'$. Therefore, ${}^t j(H') = D({}^t u_1)$. Consequently, we have $R({}^t u) = R({}^t u_1)$.

Moreover, it is clear that ${}^t u_1$ is one-to-one and both ${}^t u_1$ and ${}^t u$ are closed operators relative to the strong topologies. Since F' is fully complete and ${}^t u(H')$ is barrelled, ${}^t u$ is open relative to $\beta(F', F)$ and $\beta(E', E)$. Therefore ${}^t u_1$ is also an open mapping and Theorem 1.2 implies that $u_1(E_0) = G$. The proof is completed.

COROLLARY 1.2. *Let E and F be Banach spaces, and u a closed linear operator with dense domain in E and range in F. If the range of ${}^t u$ is strongly closed, then the range of u is closed.*

COROLLARY 1.3. *Let E be a Banach space, F a reflexive (F)-space, and u a closed linear operator with dense domain in E and range in F. Suppose that ${}^t u$ has the strongly closed range, then u has also the closed range.*

PROOF. It is sufficient to note that a reflexive (F)-space is fully barrelled and has the fully complete strong dual ([7], [9]).

2. Products of fully complete spaces. A subset M' of E' is *ew^*-closed* if and only if $U^\circ \cap M'$ is $\sigma(E', E)$-closed in U° for every convex and symmetric neighborhood U of 0 in E. Also, the necessary and sufficient condition for E to be fully complete is that every continuous and almost open linear mapping u of E onto F is open for every locally convex space F. We shall show in the sequel that for a product of two fully complete spaces the similar result holds under somewhat strengthened conditions. The following lemmas are due to (4. 4) and (3. 8) in [9].

LEMMA 2. 1. *If u is a continuous and almost open linear mapping of E into F, then ${}^t u(F')$ is ew*-closed in E'.*

LEMMA 2.2. *If a continuous and almost open linear mapping of E onto F is weakly open, i.e. open relative to $\sigma(E, E')$ and $\sigma(F, F')$, then it is open relative to the original topologies.*

Now, let $E = \prod_{i=1}^{n} E_i$ denote a product space of locally convex spaces then $E' = \prod_{i=1}^{n} E_i'$, where $< x, x' > = \sum_{i=1}^{n} < x_i, x'_i >$ for $x = (x_i) \in E$ and $x' = (x'_i) \in E'$. For a continuous linear mapping u of E into F and for $x = (x_i) \in E$, we have $u(x) = \sum_{i=1}^{n} u_i(x_i)$ where each u_i is defined by $u_i(x_i) = u(0, \ldots, 0, x_i, 0, \ldots, 0)$ so that continuous and linear from E_i into F. We put $u_i(E_i) = F_i$ and $H_i = \sum_{j \neq i} F_j,\ i = 1, 2, \ldots, n.$

LEMMA 2. 3. *Under the above hypotheses we have*

$$ {}^t u\Big(\sum_{i=1}^{n} H_i^\circ\Big) = \prod_{i=1}^{n} {}^t u_i(H_i^\circ). $$

If in addition E_i $(i = 1, 2, \ldots, n)$ are fully complete and u is almost open, then $\prod_{i=1}^{n} {}^t u_i(H_i^\circ)$ is $\sigma(E', E)$-closed in E'.

PROOF. If $y'_i \in H_i^\circ$ $(i = 1, \ldots, n)$ and $x = (x_i) \in E$, then we have

$$ \begin{aligned} < x, ({}^t u_i(y'_i)) > &= \sum_{=1}^{n} < u_i(x_i), y'_i > \\ &= < u(x), \sum_{i=1}^{n} y'_i >, \end{aligned} $$

which shows the first assertion.

To prove the second assertion, let U_i be an arbitrary convex and symmetric neighborhood of 0 in E_i and let $\{x'_{i\alpha} | \alpha \in A\}$ be a net in ${}^t u_i(H_i^\circ) \cap U_i^\circ$ which converges to x'_i relative to the $\sigma(E_i', E_i)$-topology in E_i'. Then the net $\{(\delta_{ij} x'_{i\alpha}) \mid \alpha \in A\}$, where δ_{ij} is the Kronecker delta, lies in ${}^t u(F') \cap (E_1 \times \ldots \times E_{i-1} \times U_i \times E_{i+1} \times \ldots \times E_n)^\circ$ and converges to $(\delta_{ij} x'_i)$ relative to the $\sigma(E', E)$-topology. Since by Lemma 2.1 ${}^t u(F')$ is ew*-closed, $(\delta_{ij} x'_i)$ belongs to ${}^tu(F')$. Therefore there exists a $y'_0 \in F'$ such that $(\delta_{ij}x'_i) = {}^tu(y'_0)$ and we have for every $x = (x_i) \in E$,

$$< x_i, x'_i > = < u(x), y'_0 >$$

$$= < u_i(x_i),\ y'_0 > + < \sum_{j \neq i} u_j(x_j),\ y'_0 >.$$

It follows that $x'_i = {}^tu_i(y'_0)$ and $y'_0 \in H_i^\circ$, whence $x'_i \in {}^tu_i(H_i^\circ)$ and ${}^tu_i(H_i^\circ)$ is ew^*-closed. The assumption that E_i is fully complete implies that ${}^tu_i(H_i^\circ)$ is $\sigma(E_i', E_i)$-closed. Thus, $\prod_{i=1}^{n} {}^tu_i(H_i^\circ)$ is $\sigma(E', E)$-closed.

REMARK. If u is a homomorphism, then, since ${}^tu(F')$ is $\sigma(E', E)$-closed, we can see from the above proof that the result of Lemma 2. 3 remains valid without the hypothesis that E_i $(i = 1,2,\dots,n)$ are fully complete.

THEOREM 2. 1. (1) *Suppose that* $E = E_1 \times E_2$ *is a product of fully complete spaces and* u *is a linear, continuous and almost open mapping of* E *onto* F. *If* F_i $(i = 1,2)$ *are closed and* $F_1 \cap F_2 = (0)$, *then* u *is a homomorphism.*

(2) *Suppose that* $E = E_1 \times E_2$ *is a product of* B_r*-complete spaces and* u *is a one-to-one linear mapping of* E *onto* F *which is continuous and almost open. If* F_i $(i = 1, 2)$ *are closed, then* u *is an isomorphism.*

PROOF. (1) From Lemma 2. 3 we have ${}^tu(F_1^\circ + F_2^\circ) = {}^tu_1(F_2^\circ) \times {}^tu_2(F_1^\circ)$, where $F_1^\circ + F_2^\circ$ is $\sigma(F',F)$-closed because ${}^tu_1(F_2^\circ) \times {}^tu_2(F_1^\circ)$ is $\sigma(E',E)$-closed and ${}^tu$ is one-to-one and continuous relative to $\sigma(F', F)$ and $\sigma(E', E)$. But then, $F_1^\circ + F_2^\circ = (F_1 \cap F_2)^\circ = F'$. Therefore ${}^tu(F') = {}^tu_1(F_2^\circ) \times {}^tu_2(F_1^\circ)$, and the $\sigma(E',E)$-closedness of ${}^tu(F')$ and Lemma 2. 2 imply that u is open.

(2) In case u is one-to-one, F is the algebraic direct sum of F_i $(i = 1,2)$. Since ${}^tu(F')$ is ew^*-closed, ${}^tu_1(F_2^\circ)$ is shown to be ew^*-closed in the same way as in Lemma 2. 3. Moreover, if, for an $x_1 \in E_1$, $< x_1, {}^tu_1(F_2^\circ) > = 0$, then $u_1(x_1) \in F_2^{\circ\circ} = F_2$ and therefore $u_1(x_1) \in F_1 \cap F_2 = (0)$, whence $x_1 = 0$. Thus, ${}^tu_1(F_2^\circ)$ is $\sigma(E_1', E_1)$-dense in E_1' and hence ${}^tu_1(F_2^\circ) = E_1'$ because of B_r-completeness of E_1.

Similarly ${}^tu_2(F_1^\circ) = E_2'$, and we have

$${}^tu(F') = {}^tu_1(F_2^\circ) \times {}^tu_2(F_1^\circ) = E'.$$

COROALLRY 2. 1. *If a* B_r*-complete barrelled space is an algebraic direct sum of two closed linear subspaces then it is at the same time the topological direct sum.*

PROOF. Let $F=F_1 \oplus F_2$ be an algebraic direct sum where F is B_r-complete and barrelled and $F_i\,(i=1,2)$ are closed in F. We indicate by φ the canonical mapping of $E=F_1 \times F_2$, the product of F_i, onto F. $F_i\,(i=1,2)$ are B_r-complete and φ is almost open, so that φ is an isomorphism by virtue of Theorem 2.1 (2), which completes the proof.

COROLLARY 2.2. *Let E be a B_r-complete and barrelled space. If there is for every closed linear subspace a complementary closed linear subspace, then E is fully complete.*

PROOF. It is easily seen that for a B_r-complete space to be fully complete it is necessary and sufficient that every quotient space is B_r-complete. Let E_0 be a closed linear subspace of E and E_1 a corresponding closed linear subspace which is complementary to E_0. Then E/E_0 is isomorphic with E_1 which is B_r-complete.

REFERENCES

[1] F. E. BROWDER, Functional analysis and partial differential equations I, Math. Ann., 138(1959), 55-79.
[2] N. BOURBAKI, Espaces vectoriels topologiques, III-V. Paris, (1955).
[3] H. S. COLLINS, Completeness and compactness in linear topological spaces, Trans. Amer. Math. Soc., 79(1955), 256-280.
[4] J. DIEUDONNÉ et L. SCHWARTZ, La dualité dans les espaces (F) et (LF), Ann. Inst., Fourier Grenobles, 1(1950), 61-101.
[5] J. T. JOICHI, On closed operators with closed range, Proc. Amer. Math. Soc., 11(1960), 80-83.
[6] J. L. KELLEY, Hypercomplete linear topological spaces, Michigan Math. Journ., 5(1958), 236-246.
[7] W. ROBERTSON AND A. P. ROBERTSON, On the closed graph theorem, Proc. Glasgow Math. Assoc., 3(1956-1958), 9-12.
[8] G. C. ROTA, Extension theory of differential operators, Comm. Pure Appl. Math., 11(1958), 23-65.
[9] V. PTAK, Completeness and the open mapping theorem, Bull. Soc. Math. France, 86(1958), 41-74.

TÔHOKU UNIVERSITY.