ON EXCEPTIONAL VALUES OF ENTIRE AND MEROMORPHIC FUNCTIONS

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1. Let $F(z)$ be a meromorphic function and let $T(r, F)$ be its Nevanlinna characteristic function. Let $N(r, a) = N(r, F - a)$; $N(r, F) = N(r, \infty)$ have the usual meaning in the Nevanlinna theory.

Define

$$\delta(a) = 1 - \limsup_{r\to\infty} \frac{N(r, a)}{T(r, F)},$$

$$\Delta(a) = 1 - \liminf_{r\to\infty} \frac{N(r, a)}{T(r, F)}.$$

If $\delta(a) > 0$ we say that a is an exceptional value for $F(z)$ in the sense of Nevanlinna (e. v. N); and if $\Delta(a) > 0$ we call a as an e. v. V (exceptional value in the sense of Valiron).

2. Let $f(z)$ be an entire function and let

$$\mu(r, f) = \mu(r) = \operatorname*{Min}_{|z| = r} |f(z)|.$$

It is clear that if 0 is an asymptotic value for $f(z)$ then $\mu(r) \to 0$ as $r \to \infty$. We show that the converse is not true. We prove:

THEOREM 1. *For an entire function $f(z)$, the minimum modulus $\mu(r)$ tending to zero does not imply that 0 is an asymptotic value.*

LEMMA *If 0 is an e. v. N for the entire function $f(z)$ then $\mu(r) \to 0$ as $r \to \infty$.*

PROOF. In the terminology of Nevanlinna

$$m\left(r, \frac{1}{f}\right) = m(r, 0) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{1}{f(re^{i\theta})} \right| d\theta.$$

Hence
$$m(r, 0) \leqq \log^+ \frac{1}{\mu(r)}.$$

But
$$\liminf_{r\to\infty} \frac{m(r, 0)}{T(r, f)} > 0,$$

because 0 is an e. v. N, so

$$T(r,f) \leqq A\log^+ \frac{1}{\mu(r)} \qquad \text{for } r \geqq r_0,$$

and the lemma follows because

$$T(r,f) \to \infty \text{ as } r \to \infty.$$

To complete the proof of Theorem 1, let

$$f_1(z) = \prod_{\nu=1}^{\infty} \left(1 + \frac{z^2}{\nu^{4/3}}\right),$$

$$f_2(z) = \prod_{n=1}^{\infty} \phi_n[(-1)^n z]$$

where $$\phi_n(z) = \left(1 + \frac{iz}{r_n}\right)^{\lambda_n} \exp\left(\frac{-\lambda_n i z_n}{r_n}\right)$$

$$r_n = 2^{2.8^{n-1}},\ \lambda_n = 8r_n^{3/2}.$$

Define $$f(z) = f_1(z) f_2(z).$$

Then $f(z)$ is an entire function of order $3/2$ for which $\delta(0) > 0$ see A. A. Goldberg [1].
Thus 0 is an e.v.N for $f(z)$, so $\mu(r,f) \to 0$ by the lemma. But 0 is not an asymptotic value for $f(z)$.

3. THEOREM 2. *Let $F(z)$ be a meromorphic function of order ρ $(0 < \rho < \infty)$; and let $\rho(r)$ be Lindelöf proximate order relative to $T(r, F)$. Let $n(r, a_i)$ be the number of zeros of $F(z) - a_i$ in $|z| \leqq r$; all the a_i being different $(0 \leqq |a_i| \leqq \infty$; in case $a_i = \infty$, $n(r, a_i) = n(r, \infty)$ is the number of poles). Then*

$$\limsup_{r\to\infty} \sum_{i=1}^{q} \frac{n(r, a_i)}{r^{\rho(r)}} \geqq (q-2)\rho$$

where q is an integer $\geqq 3$.

PROOF. Let $$\limsup_{r\to\infty} \sum_{i=1}^{q} \frac{n(r, a_i)}{r^{\rho(r)}} = k$$

and if possible let $k < (q - 2)\rho$, then

$$\sum_{i=1}^{q} n(r, a_i) < (k + \varepsilon) r^{\rho(r)} \qquad \text{for } r \geqq r_0.$$

Hence $$\sum_{i=1}^{q} N(r, a_i) + O(\log r) < (k + \varepsilon) \int_{r_0}^{r} t^{\rho(t)-1}\, dt \sim \frac{k + \varepsilon}{\rho} r^{\rho(r)}$$

$$= \frac{k + \varepsilon}{\rho} T(r, F)$$

for a sequence of values of r.
Further from the second theorem of Nevanlinna

$$(q-2) T(r, F) < \sum_{i=1}^{q} N(r, a_i) + O(\log r).$$

Hence for an infinity of values of r we have
$(q - 2)\, T(r, F) < \dfrac{k + \varepsilon}{\rho} T(r, F) + O\ (\log r)$, and since ε is arbitrarily small $(q - 2)\rho \leqq k$. This gives a contradiction. Hence the result follows.

4. Let $f(z)$ be an entire function and let $\rho_1(a)$ be the exponent of convergence of the zeros of $f(z) - a$. If $\rho_1(a) < \rho$ we say that a is an e.v.B for $f(z)$.

If $$\liminf_{r \to \infty} \frac{T(r, f)}{n(r, a)\phi(r)} > 0$$

for a positive non-decreasing function $\phi(x)$ such that

$$\int_{A}^{\infty} \frac{dx}{x\phi(x)} < \infty,$$

then a is defined to be an e·v·E, see S.M.Shah [2]. Let $f(z)$ be an entire function of order ρ $(0 < \rho < \infty)$ and let $\rho(r)$ be proximate order relative to $\log M(r, f)$, that is,

$$\rho(r) \to \rho \text{ as } r \to \infty,$$

$$r\rho'(r) \log r \to 0 \text{ as } r \to \infty,$$

$$\log M(r, f) \leqq r^{\rho(r)} \text{ for } r \geqq r_0$$

and $\log M(r, f) = r^{\rho(r)}$ for a sequence of values of r. Valiron has proved that for a class of entire functions of finite non-zero order if

$$\frac{n(r, a)}{r^{\rho(r)}} \to 0 \text{ as } r \to \infty,$$

then $\frac{n(r, x)}{r^{\rho(r)}}$ lies between two positive constants for every $x \neq a$.
Hence it is reasonable to define a as an exceptional value in some sense if

$$\lim_{r\to\infty} \frac{n(r, a)}{r^{\rho(r)}} = 0. \tag{1}$$

We shall call a as an e.v.L(in the sense of Lindelöf) if (1) holds.

THEOREN 3. (i) *If a is an e.v.B then a is an e.v.L also but the converse is not true.*

(ii) *If a is an e.v.L then a is e.v.V with $\Delta(a) = 1$.*

(iii) *If a is e.v.E then a is e.v.L also but the converse is not true.*

PROOF. (i) Let a be e.v.B then

$$n(r, a) = O(r^c) \quad c < \rho$$

so
$$\frac{n(r, a)}{r^{\rho(r)}} < Ar^{c-\rho(r)} \quad \text{for } r \geqq r_0.$$

Further
$$\rho(r) > \frac{\rho + c}{2} \quad \text{for } r \geqq r_0.$$

Hence
$$\frac{n(r, a)}{r^{\rho(r)}} \to 0 \text{ as } r \to \infty$$

so a is e.v.L.
That the converse is not true can be seen from the function

$$f(z) = \prod_2^\infty \left(1 + \frac{z}{n(\log n)^2}\right).$$

Here
$$M(r, f) \sim \frac{r}{\log r}, \quad n(r, 0) \sim \frac{r}{(\log r)^2}.$$

Set
$$\rho(r) = 1 - \frac{\log\log r}{\log r}.$$

Then it can easily be seen that $\rho(r)$ is a proximate order relative to log $M(r,f)$.

Here $$\rho = \rho_1(0) = 1,$$

so 0 is not e.v.B.
But

$$\frac{n(r, 0)}{r^{\rho(r)}} \to 0 \text{ as } r \to \infty$$

and $\dfrac{n(r, x)}{r^{\rho(r)}}$ lies between two positive constants for all $x \neq 0$ and thus 0 is e.v.L.

(ii) Let a be e.v.L then

$$n(r, a) < \varepsilon r^{\rho(r)} \text{ for } r \geqq r_0,$$

$$N(r, a) < \varepsilon \int_{r_0}^{r} t^{\rho(t)-1}\,dt \sim \frac{\varepsilon r^{\rho(r)}}{\rho},$$

$$\begin{aligned} N(2r, a) &< \frac{\varepsilon}{\rho}(2r)^{\rho(2r)} \sim \frac{\varepsilon}{\rho} 2^{\rho} r^{\rho(r)} \\ &= \frac{\varepsilon}{\rho} 2^{\rho} \log M(r, f) \quad \text{for a sequence of values of } r \\ &< \frac{\varepsilon}{\rho} 2^{\rho}\, T(2r, f). \end{aligned}$$

Hence $$\liminf_{r\to\infty} \frac{N(r, a)}{T(r, f)} = 0.$$

So a is e·v·V with $\Delta(a) = 1$.
We omit the proof of the first part of (iii). That the converse is not true can again be seen from the same example

$$f(z) = \prod_{2}^{\infty}\left(1 + \frac{z}{n(\log n)^2}\right).$$

Here $$\frac{\log M(r, f)}{n(r, 0)\phi(r)} \sim \frac{\dfrac{r}{\log r}}{\dfrac{r}{(\log r)^2}\phi(r)} = \frac{\log r}{\phi(r)}.$$

But since
$$\int_A^\infty \frac{dx}{x\phi(x)} < \infty$$

so $\log x = o(\phi(x))$ when $x \to \infty$.

Hence
$$\lim_{r\to\infty} \frac{\log M(r,f)}{n(r,0)\phi(r)} = 0.$$

Hence a fortiori

$$\lim_{r\to\infty} \frac{T(r,f)}{n(r,0)\phi(r)} = 0.$$

So 0 is not e.v.E though it is e.v.L.

5. Nevanlinna [3] has proved that if $F(z)$ is a meromorphic function of order $\rho < 1/2$ for which

$$\lim_{r\to\infty} \frac{N(r,a)}{T(r,F)} = 0 \tag{2}$$

then
$$\limsup_{r\to\infty} \frac{N(r,x)}{T(r,F)} = 1 \tag{3}$$

for every $x \neq a$ $(0 \leqq |x| \leqq \infty)$.
Of course (3) is not true for every meromorphic function of order $< 1/2$ for every x. For instance if $f(z)$ is an entire function of order $< 1/2$ then $F(z) = f(z)/(z-a)$ will be a meromorphic function of order $< 1/2$ for which

$$\frac{N(r,\infty)}{T(r,F)} \to 0 \text{ as } r \to \infty.$$

We give a method to construct a class of meromorphic functions of any given order for which (3) holds for every x $(0 \leqq |x| \leqq \infty)$. We prove:

THEOREM 4. *Given any ρ $(0 < \rho < \infty)$, there exists a meromorphic function of order ρ for which* (3) *holds for every* x $(0 \leqq |x| \leqq \infty)$.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function of order ρ for which

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_{n+1} - \lambda_n} < \infty,$$

and let as usual

$$\mu(r,f) = \operatorname*{Min}_{|z|=r} |f(z)|,$$

then $\mu(r,f) \to \infty$ as $r \to \infty$ through a sequence $r = r_n$; see A. J. Macintyre and P. Erdös [4].

Set
$$F(z) = \frac{1}{f(z)} + z, \tag{4}$$

then $T(r, F) = T(r,f) + O(\log r)$.
Let a be any number such that $0 \leqq |a| < \infty$. Then for $|z| = r_n$ we have uniformly as $n \to \infty$

$$F(z) - a = z - a + o\,(1).$$

Hence

$$m\left(r, \frac{1}{F-a}\right) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{1}{F-a} \right| d\theta = 0$$

for $r = r_n$ and $n > n_0$.

Hence
$$\limsup_{r\to\infty} \frac{N(r, F-a)}{T(r, F)} = 1.$$

Also $m\,(r, F) = \log r + o\,(1) = o\,(T(r, F))$ as $r \to \infty$ through the sequence $r = r_n$. Hence

$$\limsup_{r\to\infty} \frac{N(r, F)}{T(r, F)} = 1.$$

This proves (3) for every finite or infinite x.

REMARK. If the meromorphic function is of order $< 1/2$, the construction is still easier, since in (4) any entire function $f(z)$ of order $< 1/2$ will serve the purpose, because by a well known theorem of Wiman for such an entire function $\limsup_{r\to\infty} \mu(r,f) = \infty$. We also remark that by choosing a suitable entire function

$f(z)$ for which the sequence r_n is sufficiently dense we could have even achieved

$$\lim_{r\to\infty} \frac{N(r,x)}{T(r,F)} = 1$$

for every x $(0 \leqq |x| \leqq \infty)$.
Finally we prove:

THEOREM 5. *For every meromorphic function $F(z)$ of order ρ $(0 \leqq \rho < \infty)$,*

$$\limsup_{r\to\infty} \frac{n(r,a)}{T(r,F)} \geqq \rho$$

provided that a is not e.v. V for $F(z)$.

We omit the proof.

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